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Book Review

Adrian Brown
SETI Institute
E-mail: abrown@seti.org

Abstract. Phillip M. Morse and Herman Feshbach, Professors of Physics at the MIT, published their biblical-sized textbook 'Methods of Theoretical Physics' with McGraw-Hill in May 1953. At 1978 pages and published in two books, it is an intimidating twin tome that should still be atop the reading lists or the bookshelves of every mathematical physicist.

What material is covered in this book? In the most concise of terms, this book is devoted to the study of differential equations and associated boundary conditions that describe physical fields. The thirteen chapters address what circumstances warrant the use of which differential equations, and most often addresses the question of coordinate system transformations, for example, how do Green's functions for Laplace's Equation transform under different coordinate systems? Under what circumstances the solutions can be expected to be separable? Many examples are covered to illustrate these points.

Why is this book relevant to Software Programmers? This book is part of the background that any scientific programmer is likely to need in dealing with physical fields. This book was written before personal computers became ubiquitous, however it is still an outstanding effort to tie the methods of solving differential equations governing fields together in one book.

The book never received a second edition, however, it was reprinted to an outstanding standard by Feshbach Publishing since 2004, run by the children of Herman Feshbach. Their website is feshbachpublishing.com.

The majority of this review is a mini-commentary of the book showing what is covered in a very terse fashion, which may be useful as a summary even for those who have already read the full text. I then give a brief analysis of the approach to mathematical physics taken by the book. Finally, I will discuss who will benefit from reading this magnificent treatise, nearly 60 years after it was first published.

Methods of Theoretical Physics, by Philip McCord Morse and Herman Feshbach, ISBN 0976202123.

1. Outline of the book

One approach of classifying the different sections of this text is to run through the table of contents. However, reading through the book, I had the strong sense that the consumable unit for Morse and Feshbach is not the Chapter, but is the second subunit, by which I mean 'Section 1.2' or 'Section 3.4'. These are logical separation points within the book that contain the key ideas, and are often written as standalone essays one could extract from the book whole. For that reason, I will concentrate on the book outline at the subunit level.

Chapter 1 is an interesting chapter. I do not consider it to be part of the core of the book, and as an introduction it is extremely difficult for less experienced physical mathematicians to access. Reading through later chapters, which often do have excellent introductions, I often thought to myself that Chapter 1 was erected by Morse and Feshbach as a kind of warning about what might lie ahead in the book. 'Beware, you who have just wandering over to peek in this book – it is not for the faint of heart'.

Section 1.1 deals with differential equations scalar fields and introduces the Laplacian. *Section 1.2* introduces PDE's of Vector fields, and introduces lines of flow, surface and line integrals. Singularities (a big topic in later chapters) are mentioned. It is in *Section 1.3* that the reader is given curvilinear coordinates to tackle, as we encounter scale factors, curvature, volume element scaling, rotation and transformation of vectors. Contravariant and Covariant vectors are introduced. This material is then re-used throughout the book. *Section 1.4* is a fairly standard introduction to the ∇ operator, covering gradient, divergence, Gauss' Theorem and introduces Poisson's Equation, Curl and Vorticity, Stokes Theorem. It is at this point that the book steps up several levels somewhat unnecessarily. *Section 1.5* covers covariant and contravariant vectors, axial vectors, Christoffel symbols, Covariant Derivatives, Divergence and Curl Tensor notation and other selected differential operators, all in 10 pages. This is clearly no more than a reprinting of the equations – no meaningful introductory remarks are given here. *Section 1.6* is similar in style, covering Dyadics, stressstrain relationships, complex quaternions, abstract vector spaces operators in quantum theory, Hermitian operators, transformation of operators, quantum mechanical operators, and spin operators in 38 pages. This is probably the most excruciating section of the book, and lacks development and cohesion. Particularly disappointing is the introduction to Abstract Vector Spaces, which are treated through the book, but poorly introduced here amongst other material with a lack of cohesion. *Section 1.7* covers the Lorentz Transform, four vectors and spinors in a more cohesive, but still rather cursory fashion, in only 14 pages. Given the grab-bag of topics Morse and Feshbach threw into Sections 1.5 and 1.6 I would advise the reader not to get disheartened when ploughing through this section.

Chapter 2 is entitled 'Equations Governing Fields'. *Section 2.1* is a far more reasonable introduction to the topic of flexible string mathematics. One wonders how much the inventors of String Theory were inspired by the well-explained introduction to harmonic motion, delta function on a string, wave energy and flow, wave impedance (a big topic through the book due to the Moore's interest in Acoustics and Sound), friction and diffusion, and boundary conditions for forced motion and elasticity. This 26 page section is the first of the separable essays in the book. *Section 2.2* deals with transverse and longitudinal waves in an elastic medium in a similarly rewarding fashion. *Section 2.3* is another fascinating essay on the Motion of Fluids, addressing the continuity equation, stresses in incompressible fluids, Bernoulli's equation and Mach-inspired sub and supersonic flow with shock waves. *Section 2.4* is an outstanding essay on diffusive flow, introducing the diffusion equation, porosity, pressure, mean free-path and scattering cross section, and has a large section on the diffusion of light, which will recur throughout the book. This section no doubt inspired Monte Carlo modelers for the rest of the 20th century. *Section 2.5* deals with the Electromagnetic Field in particular, Maxwell's Equations, Lorentz Invariance, Gauge Transformations, moving charges, surfaces as boundaries, and again the wave transmission and impedance. *Section 2.6* is an essay on Quantum Mechanics, discussing photons vs. waves, introducing the uncertainty principle, Poisson's Brackets, Fundamental Equation of Quantum Mechanics, transformation to momentum space, Hamiltonian Function and Schroedinger Equation, Harmonic Oscillator, and the introduction of time into the picture. The Klein Gordon and Dirac equations are then derived.

Chapter 3 covers the basics of a method that features throughout the book – the Variational Principle. *Section 3.1* is a little gem of an essay on the Variational Integral and Euler Equations. *Section 3.2* beautifully uncovers classical dynamics and Hamilton's Principle, working up to examples of a charged particle in an EM field, then relativistic and dissipative systems. *Section 3.3* covers scalar fields, deriving the Lagrangian and Hamiltonian for the flexible string, wave equations, velocity potential, wave impedance, diffusion, Schroedinger and Klein Gordon equations. *Section 3.4* derives the Lagrangian density dyadic for isotropic elastic media, plane waves, and the EM Field. A very useful list of Variational equations discussed in Chapter 3 is provided for easy reference.

Chapter 4 is a fantastic introduction to Complex Analysis. *Section 4.1* is an outstanding entree to contour integrals and Cauchy's Theorem. *Section 4.2* follows this up with an amazing discussion of the Cauchy–Riemann conditions, conformal representations in the complex plane, Cauchy's Principle and Cauchy's Integral Formula. The essay finishes with a discussion of Hilbert transforms, impedance and Poisson's Formula. *Section 4.3* covers isolated singularities, poles and multipoles and analytical regions of the Taylor and Laurent series, Liouville's theorem and meromorphic functions. It ends with an in-depth discussion of analytic continuation, branch points, and mirror, contour and recurrence techniques for analytical continuation. *Section 4.4* extends the discussion to multivalued functions and provides a worked example along with a rare use of a plot to show the results. *Section 4.5* covers the calculus of residues to actually evaluate integrals using complex analysis, and concludes with a link from the complex plane to gamma and elliptic functions. *Section 4.6* discusses the method of steepest descent, particularly as used on asymptotic series. A particularly useful worked example is covered for a class of series that link in with the gamma function. *Section 4.7* covers conformal mapping in detail, with the major result being the Schwarz–Christoffel transformation of the inside of a polygon to the upper half of the w plane. *Section 4.8* is an essay on Fourier, Laplace and Mellin transforms, seen through the prism of Lebesgue Integrals and including a good discussion of regions of analyticity.

Chapter 5 is entitled 'Ordinary Differential Equations' and the emphasis here is upon the separation solution theme that is utilized throughout the last part of the book. *Section 5.1* deals with separable coordinates in two dimensions including boundary conditions for Laplace, Wave and coordinate systems (rectangular, parabolic, polar, elliptic). Scale factors and separation constants are discussed. The third dimension is then introduced, alongs with the Staeckel determinant, quadric surfaces and the range of coordinate systems is expanded. The conditions under which solutions are separable is discussed in depth at this point. *Section 5.2* discusses separable series solutions. The Wronskian is introduced, the matter of independence of solutions is discussed before delving into series solutions around ordinary and singular points and multiple ordinary and singular points. The Hypergeometric equation and Gegenbauer functions act as a lead in for Legendre and Tschebyscheff (Chebyshev) functions. Asymptoptic Series, Continued Fractions and Mathieu functions are discussed and the section ends with a summation of recursion formulas for the separable solutions presented. *Section 5.3* discusses Integral Representations of ODEs, in particular the Euler transform of the Hypergeometric Function, and this leads into discussions of the contours for integral representations of Gegenbauer/Legendre (inc. associated and second kind) functions. The Laplace transform uses a different kernel than the Euler transform and this is again used to examine the Hypergeometric functions. Stokes' Phenomenon (where one term of the solution regularly disappears over part of the contour) is discussed. Another confluent hypergeometric function, the Bessel function, is introduced, along with Hankel and Neumann functions. Approximations for large order and Mathieu functions are discussed at the end of the section. A table and diagrams of separable coordinates systems in three dimensions is also given.

Chapter 6 introduces treatment of boundary conditions and Eigenfunctions. *Section 6.1* is an excellent discussion of the genera of differential equations, Characteristic Curves and Dirichlet, Neumann and Cauchy conditions, and then Elliptic and Parabolic equations complete the section. *Section 6.2* discusses discretization into difference equations on nets and the effect on boundary conditions. Green's Functions are then introduced for the first time. Satisfactory boundary conditions for hyperbolic, elliptic and parabolic equations are summarized at the end of the section. *Section 6.3* introduces Eigenfunctions and how they may be utilized in order to satisfy competing boundary conditions along coordinate surfaces. The Sturm–Liouville Equation and its factorization into linear PDE's is discussed, the variation principle and completeness are mentioned. The Gibbs Phenomenon for discontinuous curves is discussed. Generating Functions are introduced and applied to Legendre Polynomials. Eigenfunctions for PDEs are mentioned, along with a discussion of the density of eigenfunctions. This is a very well written chapter, although the inclusion of the section on Abstract Vector Spaces at the end was perhaps questionable. A table discusses using the Schmidt method to generate Eigenfunctions for the Sturm–Liouville Equation, generating Gegenbauer (Legendre, Tschebyscheff), Laguerre and Hermite Polynomials, their Generating Functions and recurrence formulas.

Chapter 7 is devoted to Green's functions. *Section 7.1* very nicely discusses generating Green's functions for source points and boundary points, and the relation between volume and surface Green's functions. *Section 7.2* discusses Green's Functions for steady waves, covering the Helmholtz Equation, Inhomogenous Equations, and the effect of boundary conditions. The Method of Images is introduced and expansions of the Green's function in Eigenfunctions and on the Infinite Domain and in Polar Coordinates are discussed. A very useful general technique for the production of steady-wave Green's functions for the Sturm Liouville operator (and hence steady state D.E.s in general) is presented. *Section 7.3* discusses the Green's function for the Scalar Wave Equation. The very useful Reciprocity Relation $G(r|r_0) = G(r_0|r)$ is introduced and discussed. Two and One dimensional Green's functions for the wave equation are given. The mathematical expression of Huygens' Principle is demonstrated. Boundaries and Eigenfunction expansions are mentioned and then the example of transient motion of a circular membrane is given. Finally, the Green's function for the Klein–Gordon equation is addressed. *Section 7.4* discusses Green's functions for Diffusion, discussing causality and reciprocity, inhomogeneous boundary conditions, and infinite and finite boundaries. *Section 7.5* addresses the Green's function in Abstract Operator form, extending the applicability of the aforementioned equations to any linear equation in physics. To achieve this, adjoint operators in differential and integral form are introduced. The chapter closes with development of the Green's operator for adjoint, conjugate and Hermitian and non-Hermitian (biorthogonal) cases. A table of Green's functions is provided.

Chapter 8 is on Integral Equations (as opposed to differential equations) to describe fields. *Section 8.1* opens by classifying the Integral Equations of transport theory, acoustics and wave mechanics physics into homogenous and inhomogenous Fredholm and Volterra equations of the first and second kind, with positive/negative definite kernels. *Section 8.2* discusses the general properties of Integral Equations based on their kernel. Properties of positive definite, semi-definite, indefinite, non-real-non-definite and singular kernels are discussed. *Section 8.3* discusses the exact solutions of the Fredholm Equation of the first kind using series solutions and Schmidt orthogonalization. Biorthogonal, Gegenbauer Polynomials, and the Moment Problem round out the section. *Section 8.4* discusses exact solutions for Integral Equations of the second kind – the same series are used as in Section 8.3 though the manipulation of the equation is different. *Section 8.5* explores Fourier Transforms, Hankel Transforms, and displaced kernels such as $v(x - x_0)$, Laplace Transforms and discusses the Method of Weiner and Hopf where Integral Equation boundaries are semi-infinite rather than infinite. It illustrates the method, and then ends with the Weiner–Hopf solution for the radiative transfer in a homogenous medium with isotropic scattering (Milne) problem. A table is given at the end of the chapter summarizing all types of integral equations.

Chapter 9 delves into approximate solutions (primarily perturbational and variational) for differential equations. *Section 9.1* is an excellent essay describing the perturbational method, which is best used when a solution is sought that is close to a known exact solution. The improvement from iterative–perturbative to Feenberg to Fredholm to the Variation–Iteration to modified iterative–perturbative type schemes is illustrated through using a Mathieu function example. Finally the use of non-orthogonal basis functions is discussed. *Section 9.2* discusses perturbations of the boundary conditions and the perturbation of the boundary shape, using successive approximations (including improved iteration–perturbation and Feenberg) and secular determinant types (Fredholm and variation–perturbation). The scalar Helmholtz equation is used as an exemplar. *Section 9.3* discusses perturbation methods for scattering and diffraction. This excellent essay discusses a first improvement to the Born approximation (Kirchoff approximation) which is the equivalent to the first iteration–perturbation solution, and then higher order Born approximations. It then discusses Fredholm series improvements, and long wavelength approximations. The short wavelength WKBJ approximation is then discussed in one and three dimensions for potential barrier and bound systems. *Section 9.4* is a long (52 page) essay on Variational methods (suitable for larger variations than perturbative theory). This method inserts trial functions into a Lagrangian that satisfy the boundary conditions and then attempts to constrain their coefficients. An example of a circular membrane is used and phase shifts and nonlinear coefficients, used on potential barrier and surface perturbation and radiation problems are featured. The Variation–Iteration method is then re-introduced to enable an estimation of the accuracy of the results, the circular membrane is used as an example. The section ends with a discussion of the Method of Minimized Iterations. A useful summary table of approximate methods is given at the end of the chapter.

Chapter 10 moves into a different mode that will be followed in the remainder of the book. The general study of fields and their behavior is over, from this point forward, the techniques introduced are applied to specific physical problems.

Chapter 10 is intended to cover solutions of Laplace's and Poisson's Equations, $\nabla^2\psi = 0$ and $\nabla^2\psi = -4\pi\rho$. *Section 10.1* discusses solutions in 2 dimensions, first in Cartesian coordinates (*prism heated on a side*), and then polar coordinates (*flow of viscous liquids, internal heating of cylinders, potential near a slotted cylinder*) then elliptic coordinates (*viscous flow through a slit, elliptic cylinders in uniform fields, potential inside cylinder with a narrow slot*), parabolic coordinates, bipolar coordinates (*two cylinders in a uniform field*). In each case the Green's Function is presented. *Section 10.2* discusses complex variables and the 2D Laplace Equation, which is a special case that allows use of a powerful technique utilizing the complex variable and analytic functions. Examples of *lift and circulation during non-viscous flow, distributions of line sources, grids, and linear arrays, periodic distributions of images, potentials around prisms, parallel plate and variable condensers* and *other rectangular shapes* are tackled. *Section 10.3* discusses Poisson and Laplace equation solutions for three dimensions, covering rectangular coordinates, circular cylindrical coordinates, spherical coordinates (*field of charged disks, currents in wire loops, charged spherical caps, dipoles, quadrupoles and multipoles, spherical shell with a hole*), prolate cylindrical coordinates (*oblate spheroids, flow through an orifice*). In each of these cases integral representation and Green's functions are presented. Parabolic, bispherical, toroidal, ellipsoidal coordinates are also discussed at the end of the chapter. A table of Trigonometric, Hyperbolic, Bessel and Legendre Functions is also given.

Chapter 11 introduces time by examining the Wave Equation. *Section 11.1* is on the Wave Equation in one dimension – first looking at Fourier and Laplace Transform techniques and then covering several examples with Green's functions (*string with friction,* *string with elastic support and non-rigid supports, reflection from a frictional support, sound waves in a tube, tube with varying cross-section, acoustic circuit elements, free waves* and *moveable supports*). *Section 11.2* moves to waves in two dimensions, starting with a great discussion on the link between Green's functions and Fourier transforms, then rectangular coordinates (*variable boundary admittance*), polar coordinates (*waves in a circular boundary, radiation from circular boundary, scattering of a plane wave from a cylinder and a knife edge, Fresnel diffraction from a knife edge and from a cylinder with a slit*), parabolic coordinates (*waves outside parabolic boundaries*), elliptical coordinates (*waves inside an elliptic boundary, radiation from a vibrating strip, radiation from a current-carrying strip, scattering of waves from strips, diffraction through a slit – Babinet's principle*) are discussed. *Section 11.3* moves to into a discussion of three dimensional scenarios. Starting with rectangular coordinates (*distortion of standing wave by strip, transmission through ducts, acoustic transients in a rectangular duct, constriction of a duct, wave transmission around corner, membrane in circular tube, radiation from tube termination, transmission in elastic tubes*) spherical coordinates and the radial functions (*waves inside a sphere, vibrations in a hollow, flexible sphere, vibrating string in a sphere, radiation from a sphere, radiation from a piston in a sphere, scattering of a plane wave from sphere, scattering from a Helmholtz resonator, scattering from an ensemble of scatterers, scattering of sound from air bubbles*) spheroidal and oblate spheroidal coordinates are all mentioned. *Section 11.4* addresses Integral and Variational Techniques. Examples include *iris diaphragm in pipe, hole in an infinite plane, reflection in a lined duct, radiation from end of circular pipe* and the Fourier transform is used to examine *radiation from vibrating source*. The Variational principle is then used to examine *the scattering of waves* and *scattering from a strip*). Tables of Cylindrical Bessel Functions, Weber Functions, Mathieu Functions, Spherical Bessel Functions and Spheroidal Functions and a short table of Laplace Functions of common functions are given.

Chapter 12 is a slightly different chapter – it tackles diffusion and quantum wave mechanics. *Section 12.1* addresses solutions of the diffusion equation (*transient surface heating of a slab, radiative heating, transient internal heating of a slab*). A long discussion of diffusion and absorption of particles in a nuclear reactor is given. *Heating of a sphere* completes the section. *Section 12.2* examines the distribution functions for Milne and Chandresekhar's radiation problems, including uniform, forward scattering approximations, diffuse scattering and emission. The Variational Calculation of Density, loss of energy on collision, uniform space density and Age Theory – all methods suitable for Monte Carlo transport problems are discussed. Departing from classical physics, *Section 12.3* delves into solutions of the Schroedinger equation (*The Harmonic Oscillator, perturbation theory, bound and free states, reflection and transmission, potential barriers, central force fields and angular momentum, inverse cube force and Coulomb field, Rutherford Scattering, perturbations of degenerate systems, the Stark Effect, scattering form central fields, Ramsauer and other resonance effects, slow incident particles, the Born approximation and structure factors, Variation–Iteration method, Variation and Variation–Iteration Methods for Scattering, two particles-one dimension, coupled harmonic oscillators, several particles, Inversion and Parity, Symmetrizing two-particle systems*). Finally, the *scattering of an Electron from Hydrogen Atom* is discussed (*elastic or inelastic and the possible exchange of the particles*). A table of Jacobi functions and Semicylindrical Functions is given.

Chapter 13 discusses Vector Fields. Many of the examples are extensions of those in Chapter 11. *Section 13.1* discusses Vector Boundary Conditions, Eigenfunctions and Green's functions. The vector Helmholtz equation is used to generate examples of these functions (including the Green's function which is actually a Green's dyadic). The same is done for the vector Laplace equation. *Section 13.2* looks at static and steady-state solutions, first in two dimensions with polar coordinates, then circular cylindrical coordinates and finally spherical coordinates. *Viscous flow around a sphere* is examined, as well as *elastic deformation of a sphere*. *Section 13.3* discusses vector wave solutions (*reflection of plane waves from a plane, waves in a duct – 1. including generation of waves by a wire 2. reflection of waves from the end of the tube 3. change of duct size and reflection from a wire across the tube*, then *elastic waves along a bar, torsional forced motions of a rod, nonstationary viscous flow in tube, electromagnetic resonators including 1. driving currents, 2. wave guides and 3. klystron cavities,* then *scattering from cylinders, spherical waves, radiation from dipoles and multipoles, standing waves in a spherical enclosure, vibrations of an elastic sphere, radiation from a vibrating current distribution, radiation from a half-wave antenna and current loop, scattering from a sphere, distortion of a field by small object*).

Many numerical tables for functions in different coordinate systems are given in the final appendices of the book. These will likely not be of interest to the modern reader.

2. Pedagogical approach

Morse and Feshbach wrote the longest book written to date on theoretical physics – it is a complex and all-encompassing tome. These days one is far more likely to see shorter works that take on just part of the task – books on Electromagnetic Theory, Vector Analysis, Partial Differential Equations, Complex Analysis or Linear Analysis.

This book was written before personal computers became readily available to scientists, so it is amazing that is has managed to maintain its relevance.

Morse and Feshbach write several times that they are not interested in the physics of the problems they tackle – they are interested in the mathematical approach to finding a solution. They stick to this mantra throughout the book, and it is this discipline that allows the book to stay relevant. When one reaches for Morse and Feshbach, one knows what to expect.

The authors clearly agonized over what to leave in and out of the book. They said as much in the Preface:

> A discussion of the physical concepts and experimental procedures in all the branches of physics which use fields for their description would itself result in an overlong shelf, duplicating the subject matter of many excellent texts, and by its prolixity, disguising the fundamental unity of the subject.

I think that is the essence of way Morse and Feshbach remains a crucial read – because of its breadth and scope, it is able to bring together almost all of the techniques for mathematical analysis of Fields in Physics, and demonstrate to the reader their interrelations. For example – how a two dimensional solution of the Poisson equation can be extended to three dimensions and remain separable in certain coordinate systems.

The standard of the problems at the end of the chapters is extremely demanding – the problems will provide plenty of work for the motivated student. No answers are provided to the problems.

Morse and Feshbach provide plenty of tables for reference at the end of relevant chapters and at the end of the book. Most of these are unnecessary in today's computer-driven environment.

3. Who will this book appeal to and why?

This book is definitely not an introductory book, and wastes little time in exacting explanations of how to step through the derivations presented. Presumably this was done because the perceived audience of the book were science or engineering professionals with strong mathematical backgrounds, and not college students.

It would be easy to list the number of physical fields that have developed, matured, or simply appeared since Morse and Feshbach was published. The book contains the fundamentals for Monte Carlo approaches and finite difference and finite element theory, which have received significant attention with the advent of computing. Quantum Field Theory is of course absent, and group theory and chaos theory do not make an appearance.

Even so, in 2011, almost 60 years after first publication, it is still hard to imagine a practicing applied mathematician or applied physics professor who does not have a copy of this book nearby. The incredible breadth and sheer number of problems covered make it an invaluable resource that is still unmatched in scope.

Texts that are more suitable for the beginning student are now available (those by Haberman or Farlow for example) however Morse and Feshbach provide solutions where others fear to tread. As I have noted above, Morse and Feshbach sought to include enough mathematical material to enable the reader to grasp the connections between the disparate theoretical methods of physics. No other book can claim that to this day.