Erratum

Correction to "Pythagorean fuzzy interaction aggregation operators and their application to multiple attribute decision making"

Guiwu Wei*
School of Business, Sichuan Normal University, Chengdu, P.R. China


1. Introduction

More recently, Pythagorean fuzzy set (PFS) [1, 2] has emerged as an effective tool for depicting uncertainty of the MADM problems. The PFS is also characterized by the membership degree and the non-membership degree, whose sum of squares is less than or equal to 1, the PFS is more general than the IFS. In some cases, the PFS can solve the problems that the IFS cannot, for example, if a DM gives the membership degree and the non-membership degree as 0.8 and 0.6, respectively, then it is only valid for the PFS. In other words, all the intuitionistic fuzzy degrees are a part of the Pythagorean fuzzy degrees, which indicates that the PFS is more powerful to handle the uncertain problems. Zhang and Xu [3] provided the detailed mathematical expression for PFS and introduced the concept of Pythagorean fuzzy number (PFN). Meanwhile, they also developed a Pythagorean fuzzy TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) for handling the MCDM problem within PFNs. Wei [4] utilized arithmetic and geometric operations to develop some Pythagorean fuzzy interaction aggregation operators: Pythagorean fuzzy interaction weighted average (PFIWA) operator, Pythagorean fuzzy interaction weighted geometric (PFIWG) operator, Pythagorean fuzzy interaction ordered weighted average (PFIOWA) operator, Pythagorean fuzzy interaction ordered weighted geometric (PFIOWG) operator, Pythagorean fuzzy interaction hybrid average (PFIHA) operator and Pythagorean fuzzy interaction hybrid geometric (PFIHG) operator and studied the prominent characteristics of these proposed operators.

The aim of this paper is to point out some errors to the Theorems 1, Theorem 10 and some equations in Wei [4] and we propose the revised theorems, equations and their proof.

2. Preliminaries

The basic concepts of PFNs [1, 2] are briefly reviewed in this section.
**Definition 1.** [1, 2] Let $X$ be a fix set. A PFS is an object having the form

$$P = \{ (x, (\mu_P(x), v_P(x))) \mid x \in X \}$$

where the function $\mu_P : X \to [0, 1]$ defines the degree of membership and the function $v_P : X \to [0, 1]$ defines the degree of non-membership of the element $x \in X$ to $P$, respectively, and, for every $x \in X$, it holds that

$$(\mu_P(x))^2 + (v_P(x))^2 \leq 1.$$  

For convenience, Zhang and Xu [3] called $\tilde{a} = (\mu, v)$ a Pythagorean fuzzy number (PFN).

**Definition 2.** [4] Let $\tilde{a} = (\mu, v)$ be a Pythagorean fuzzy number, a score function $S$ of a Pythagorean fuzzy number can be represented as follows:

$$S(\tilde{a}) = \frac{1}{2} \left( 1 + \mu^2 - v^2 \right), \ S(\tilde{a}) \in [0, 1].$$

**Definition 3.** [5] Let $\tilde{a} = (\mu, v)$ be a Pythagorean fuzzy number, an accuracy function $H$ of a Pythagorean fuzzy number can be represented as follows:

$$H(\tilde{a}) = \mu^2 + v^2, \ H(\tilde{a}) \in [0, 1].$$

to evaluate the degree of accuracy of the Pythagorean fuzzy number $\tilde{a} = (\mu, v)$, where $H(\tilde{a}) \in [0, 1]$. The larger the value of $H(\tilde{a})$, the more the degree of accuracy of the Pythagorean fuzzy number $\tilde{a}$.

Based on the score function $S$ and the accuracy function $H$, Wei [4] gave an order relation between two Pythagorean fuzzy numbers, which is defined as follows:

**Definition 4.** [4] Let $\tilde{a}_1 = (\mu_1, v_1)$ and $\tilde{a}_2 = (\mu_2, v_2)$ be two Pythagorean fuzzy number, $s(\tilde{a}_1)$ and $s(\tilde{a}_2)$ be the scores of $\tilde{a}$ and $\tilde{b}$, respectively, and let $H(\tilde{a}_1)$ and $H(\tilde{a}_2)$ be the accuracy degrees of $\tilde{a}$ and $\tilde{b}$, respectively, then if $S(\tilde{a}) < S(\tilde{b})$, then $\tilde{a}$ is smaller than $\tilde{b}$, denoted by $\tilde{a} < \tilde{b}$; if $S(\tilde{a}) = S(\tilde{b})$, then

(1) if $H(\tilde{a}) = H(\tilde{b})$, then $\tilde{a}$ and $\tilde{b}$ represent the same information, denoted by $\tilde{a} = \tilde{b}$; (2) if $H(\tilde{a}) < H(\tilde{b})$, $\tilde{a}$ is smaller than $\tilde{b}$, denoted by $\tilde{a} < \tilde{b}$.

**Definition 5.** [3] Let $\tilde{a}_1 = (\mu_1, v_1), \tilde{a}_2 = (\mu_2, v_2)$, and $\tilde{a} = (\mu, v)$ be three Pythagorean fuzzy numbers, and the basic operations on them are defined as follows:

(1) $\tilde{a}_1 \oplus \tilde{a}_2 = \left( \sqrt{(\mu_1)^2 + (\mu_2)^2 - (\mu_1)(\mu_2)^2}, v_1v_2 \right)$;
(2) $\tilde{a}_1 \otimes \tilde{a}_2 = \left( \mu_1\mu_2, \sqrt{(v_1)^2 + (v_2)^2 - (v_1)(v_2)^2} \right)$;
(3) $\lambda \tilde{a} = \left( \sqrt{1 - (1 - \mu^2)^\lambda}, v^\lambda \right), \lambda > 0$;
(4) $(\tilde{a})^\lambda = \left( \mu^\lambda, \sqrt{1 - (1 - v^2)^\lambda} \right), \lambda > 0$;
(5) $\tilde{a}^c = (v, \mu)$.

### 3. Pythagorean fuzzy interaction aggregation operators

In this section, we shall give the corrected Theorem 1 and its proof and the corrected Equations (6 and 7).

**Theorem 1.** Let $\tilde{a}_j = (\mu_j, v_j)$ $(j = 1, 2, \ldots, n)$ be a collection of Pythagorean fuzzy numbers, then their aggregated value by using the PFIWA operator is also a PFN, and

$$\text{PFIWA}_ω(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \bigoplus_{j=1}^{n} (ω_j\tilde{a}_j)$$

(5)

\[
\begin{align*}
\text{PFIWA}_ω(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) &= \left( \frac{1 - \prod_{j=1}^{n} \left( 1 - (\mu_j)^2 \right)^{ω_j}}{\prod_{j=1}^{n} \left( 1 - (\mu_j)^2 \right)^{ω_j} - \prod_{j=1}^{n} \left( 1 - (\mu_j)^2 + (v_j)^2 \right)^{ω_j}} \right) \\
&= \left( \prod_{j=1}^{n} \left( 1 - (\mu_j)^2 \right)^{ω_j} \right) \left( \prod_{j=1}^{n} \left( 1 - (\mu_j)^2 + (v_j)^2 \right)^{ω_j} \right)
\end{align*}
\]

where $ω = (ω_1, ω_2, \ldots, ω_n)^T$ be the weight vector of $\tilde{a}_j (j = 1, 2, \ldots, n)$, and $ω_j > 0$, $\sum_{j=1}^{n} ω_j = 1$.

**Proof.** We prove Equation (5) by mathematical induction on $n$.

(1) When $n = 2$, we have

\[
\text{PFIWA}_ω(\tilde{a}_1, \tilde{a}_2) = ω_1\tilde{a}_1 \oplus ω_2\tilde{a}_2
\]
By Theorem 1, we can see that both $\omega_1 \alpha_1$ and $\omega_2 \alpha_2$ are PFNs, and the value of $\omega_1 \alpha_1 \oplus \omega_2 \alpha_2$ is also a PFN. From the operational laws of Pythagorean fuzzy number, we have

$$\omega_1 \tilde{\alpha}_1 = \left( \sqrt{1 - (1 - (\mu_1)^2)^{\omega_1}}, \sqrt{1 - (1 - (\mu_1)^2)^{\omega_1} - (1 - ((\mu_1)^2 + (v_1)^2))^\omega_1} \right);$$

$$\omega_2 \tilde{\alpha}_2 = \left( \sqrt{1 - (1 - (\mu_2)^2)^{\omega_2}}, \sqrt{1 - (1 - (\mu_2)^2)^{\omega_2} - (1 - ((\mu_2)^2 + (v_2)^2))^\omega_2} \right).$$

Then

$$\text{PFIWA}_{\omega}(\tilde{\alpha}_1, \tilde{\alpha}_2) = \omega_1 \tilde{\alpha}_1 \oplus \omega_2 \tilde{\alpha}_2$$

$$= \left( \sqrt{1 - (1 - (\mu_1)^2)^{\omega_1}}, \sqrt{1 - (1 - (\mu_2)^2)^{\omega_2}} \right) \oplus \left( \sqrt{1 - (1 - (\mu_2)^2)^{\omega_2}}, \sqrt{1 - (1 - (\mu_2)^2)^{\omega_2} - (1 - ((\mu_2)^2 + (v_2)^2))^\omega_2} \right)$$

$$= \left( \sqrt{1 - (1 - (\mu_1)^2)^{\omega_1} (1 - (\mu_2)^2)^{\omega_2}}, \right.$$  

$$\left. \sqrt{1 - (1 - ((\mu_1)^2 + (v_1)^2))^\omega_1 [1 - ((\mu_2)^2 + (v_2)^2)]^\omega_2} - \frac{2}{\prod_{j=1}^{2} (1 - (\mu_j)^2)^{\omega_j}} - \frac{2}{\prod_{j=1}^{2} (1 - ((\mu_j)^2 + (v_j)^2))^\omega_j} \right)$$

$$= \left( \sqrt{1 - \prod_{j=1}^{2} (1 - (\mu_j)^2)^{\omega_j}}, \right.$$  

$$\left. \sqrt{\prod_{j=1}^{k} (1 - (\mu_j)^2)^{\omega_j} - \prod_{j=1}^{k} (1 - ((\mu_j)^2 + (v_j)^2))^\omega_j} \right).$$

Suppose that $n = k$, Equation (5) holds, i.e.,

$$\text{PFIWA}_{\omega}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_k) = \omega_1 \tilde{\alpha}_1 \oplus \omega_2 \tilde{\alpha}_2 \oplus \cdots \omega_k \tilde{\alpha}_k$$

And the aggregated value is a PFN, Then when $n = k + 1$, by the operational laws of Pythagorean fuzzy number, we have
by which aggregated value is also a PFN. Therefore, when $n = k + 1$, Equation (5) holds.

Thus, by ① and ②, we know that Equation (5) holds for all $n$. The proof is completed.

Furthermore, Equations (6 and 7) can be corrected as follows:

$$
\text{PFIWA}_{\omega}(\check{\alpha}_1, \check{\alpha}_2, \ldots, \check{\alpha}_{k+1}) = \omega_1 \check{\alpha}_1 \oplus \omega_2 \check{\alpha}_2 \oplus \cdots \omega_k \check{\alpha}_k \oplus \omega_{k+1} \check{\alpha}_{k+1}
$$

$$
= \left( \sqrt{1 - \prod_{j=1}^{k} \left( 1 - (\mu_j)^2 \right)^{\omega_j}} \right) \oplus \left( \sqrt{1 - (\mu_{k+1})^2}\right)^{\omega_{k+1}}
$$

$$
= \left( \prod_{j=1}^{k+1} \left( 1 - (\mu_j)^2 \right)^{\omega_j} - \prod_{j=1}^{k} \left( 1 - \left( (\mu_j)^2 + (\nu_j)^2 \right) \right)^{\omega_j} \right)
$$

$$
\text{PFIOWA}_{w}(\check{\alpha}_1, \check{\alpha}_2, \ldots, \check{\alpha}_n) = \oplus_{j=1}^{n} (w_j \check{\alpha}_{\sigma(j)})
$$

$$
= \left( \sqrt{1 - \prod_{j=1}^{n} \left( 1 - (\mu_{\sigma(j)})^2 \right)^{w_j}} \right)
$$

$$
\text{PFIHA}_{w,\omega}(\check{\alpha}_1, \check{\alpha}_2, \ldots, \check{\alpha}_n) = \oplus_{j=1}^{n} (w_j \check{\alpha}_{\sigma(j)})
$$

$$
= \left( \prod_{j=1}^{n} \left( 1 - \left( (\mu_{\sigma(j)})^2 \right)^{w_j} \right) - \prod_{j=1}^{n} \left( 1 - \left( (\mu_{\sigma(j)})^2 + (\nu_{\sigma(j)})^2 \right)^{w_j} \right) \right)
$$
4. Pythagorean fuzzy interaction geometric aggregation operators

In this section, we shall give the corrected Theorem 10 and its proof and the corrected Equations (9 and 10).

**Theorem 10.** The aggregated value by using PFIWG operator is also a PFN, where

\[
PFIWG_\omega (\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n)
= \left( \prod_{j=1}^{k} \left( 1 - (v_j)^2 \right)^{\omega_j} - \prod_{j=1}^{k} \left( 1 - (\mu_j + v_j)^2 \right)^{\omega_j} \right) \sqrt{1 - \prod_{j=1}^{k} \left( 1 - (v_j)^2 \right)^{\omega_j}}
\]

where \( \omega = (\omega_1, \omega_2, \ldots, \omega_n)^T \) be the weight vector of \( \alpha_j \) \((j = 1, 2, \ldots, n)\), and \( \omega_j > 0, \sum_{j=1}^{k} \omega_j = 1 \).

**Proof.** We prove Equation (8) by mathematical induction on \( n \).

1. When \( n = 2 \), we have

\[
PFIWG_\omega (\tilde{\alpha}_1, \tilde{\alpha}_2) = (\tilde{\alpha}_1)^{\omega_1} \otimes (\tilde{\alpha}_2)^{\omega_2}
\]

By Theorem 1, we can see that both \((\tilde{\alpha}_1)^{\omega_1}\) and \((\tilde{\alpha}_2)^{\omega_2}\) are PFNs, and the value of \((\tilde{\alpha}_1)^{\omega_1} \otimes (\tilde{\alpha}_2)^{\omega_2}\) is also a PFN. From the operational laws of Pythagorean fuzzy number, we have

\[
(\tilde{\alpha}_1)^{\omega_1}
= \left( \sqrt{1 - (v_1)^2}^{\omega_1} - (1 - (\mu_1 + v_1)^2)^{\omega_1}, \sqrt{1 - (1 - (v_1)^2)^{\omega_1}} \right);
\]

\[
(\tilde{\alpha}_2)^{\omega_2}
= \left( \sqrt{1 - (v_2)^2}^{\omega_2} - (1 - (\mu_2 + v_2)^2)^{\omega_2}, \sqrt{1 - (1 - (v_2)^2)^{\omega_2}} \right).
\]

Then

\[
PFIWG_\omega (\tilde{\alpha}_1, \tilde{\alpha}_2)
= \left( \prod_{j=1}^{2} \left( 1 - (v_j)^2 \right)^{\omega_j} - \prod_{j=1}^{2} \left( 1 - (\mu_j + v_j)^2 \right)^{\omega_j} \right) \sqrt{1 - \prod_{j=1}^{2} \left( 1 - (v_j)^2 \right)^{\omega_j}}
\]
Suppose that \( n = k \), Equation (5) holds, i.e.,

\[
PFIWG_\omega(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_k) = (\tilde{\alpha}_1)^{\omega_1} \otimes (\tilde{\alpha}_2)^{\omega_2} \otimes \cdots (\tilde{\alpha}_k)^{\omega_k}
\]

\[
= \left( \prod_{j=1}^{k} \left( 1 - (v_{j})^2 \right)^{\omega_j} - \prod_{j=1}^{k} \left( 1 - (\mu_{j} + v_{j})^2 \right)^{\omega_j} \right) \sqrt{1 - \prod_{j=1}^{k} \left( 1 - (v_{j})^2 \right)^{\omega_j}}.
\]

And the aggregated value is also a PFN. Therefore, when \( n = k + 1 \), the operational laws of Pythagorean fuzzy number, we have

\[
PFIWG_\omega(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_{k + 1}) = (\tilde{\alpha}_1)^{\omega_1} \otimes (\tilde{\alpha}_2)^{\omega_2} \otimes \cdots (\tilde{\alpha}_k)^{\omega_k} \otimes (\tilde{\alpha}_{k + 1})^{\omega_{k + 1}}
\]

\[
= \left( \prod_{j=1}^{k+1} \left( 1 - (v_{j})^2 \right)^{\omega_j} - \prod_{j=1}^{k+1} \left( 1 - (\mu_{j} + v_{j})^2 \right)^{\omega_j} \right) \sqrt{1 - \prod_{j=1}^{k+1} \left( 1 - (v_{j})^2 \right)^{\omega_j}}.
\]

by which aggregated value is also a PFN, Therefore, when \( n = k + 1 \), Equation (8) holds.

Thus, by (6) and (7), we know that Equation (8) holds for all \( n \). The proof is completed.

Furthermore, Equations (9 and 10) can be corrected as follows:

\[
PFIOWG_\omega(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \bigotimes_{j=1}^{n} (\tilde{\alpha}_{\sigma(j)})^{w_j}
\]

\[
= \left( \prod_{j=1}^{n} \left( 1 - (\nu_{\sigma(j)})^2 \right)^{w_j} - \prod_{j=1}^{n} \left( 1 - (\mu_{\sigma(j)} + \nu_{\sigma(j)})^2 \right)^{w_j} \right) \sqrt{1 - \prod_{j=1}^{n} \left( 1 - (\nu_{\sigma(j)})^2 \right)^{w_j}}, \tag{9}
\]

\[
PFIHG_{\omega, \omega}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \bigotimes_{j=1}^{n} (\tilde{\alpha}_{\sigma(j)})^{w_j}
\]

\[
= \left( \prod_{j=1}^{n} \left( 1 - (\nu_{\sigma(j)})^2 \right)^{w_j} - \prod_{j=1}^{n} \left( 1 - (\mu_{\sigma(j)} + \nu_{\sigma(j)})^2 \right)^{w_j} \right) \sqrt{1 - \prod_{j=1}^{n} \left( 1 - (\nu_{\sigma(j)})^2 \right)^{w_j}}, \tag{10}
\]
5. An approach to multiple attribute decision making with Pythagorean fuzzy information

In this section, we shall give the corrected Equations (11 and 12).

\[ \tilde{r}_i = (\mu_i, \nu_i) = \text{PFIWA}_\omega (\tilde{r}_{i1}, \tilde{r}_{i2}, \cdots, \tilde{r}_{in}) \]
\[ = \bigoplus_{j=1}^{n} (\omega_j \tilde{r}_{ij}) \]
\[ = \left( \sqrt{1 - \prod_{j=1}^{n} \left(1 - (\mu_{ij})^2 \right)^{\omega_j}} \right) \]
\[ = \left( \sqrt{\prod_{j=1}^{n} \left(1 - (\mu_{ij})^2 \right)^{\omega_j} - \prod_{j=1}^{n} \left(1 - \left((\mu_{ij})^2 + (\nu_{ij})^2 \right) \right)^{\omega_j}} \right), \quad i = 1, 2, \cdots, m. \] (11)

\[ \tilde{r}_i = (\mu_i, \nu_i) = \text{PFIWG}_\omega (\tilde{r}_{i1}, \tilde{r}_{i2}, \cdots, \tilde{r}_{in}) \]
\[ = \bigotimes_{j=1}^{n} (\tilde{r}_{ij})^{\omega_j} \]
\[ = \left( \sqrt{\prod_{j=1}^{n} \left(1 - (\nu_{ij})^2 \right)^{\omega_j} - \prod_{j=1}^{n} \left(1 - \left((\mu_{ij})^2 + (\nu_{ij})^2 \right) \right)^{\omega_j}} \right), \quad i = 1, 2, \cdots, m. \] (12)

6. Illustrative example and comparative analysis

In this section, with corrected Equations (11 and 12), the Tables 1–3 are corrected as follows:

Step 1. According to Table 1, aggregate all Pythagorean fuzzy numbers \( \tilde{r}_{ij} (j = 1, 2, \cdots, n) \) by using the PFIWA (PFIWG) operator to derive the overall Pythagorean fuzzy numbers \( \alpha_i (i = 1, 2, 3, 4, 5) \) of the alternative \( A_i \). The aggregating results are shown in Table 1.

Step 2. According to the aggregating results shown in Table 1 and the score functions of the ERP systems are shown in Table 2.

---

### Table 1
The aggregating results of the ERP systems by the PFIWA (PFIWG) operators

<table>
<thead>
<tr>
<th></th>
<th>PFIWA</th>
<th>PFIWG</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>(0.727, 0.324)</td>
<td>(0.660, 0.446)</td>
</tr>
<tr>
<td>A2</td>
<td>(0.458, 0.583)</td>
<td>(0.355, 0.651)</td>
</tr>
<tr>
<td>A3</td>
<td>(0.407, 0.538)</td>
<td>(0.341, 0.579)</td>
</tr>
<tr>
<td>A4</td>
<td>(0.663, 0.511)</td>
<td>(0.495, 0.674)</td>
</tr>
<tr>
<td>A5</td>
<td>(0.667, 0.474)</td>
<td>(0.520, 0.632)</td>
</tr>
</tbody>
</table>

### Table 2
The score functions of the ERP systems

<table>
<thead>
<tr>
<th></th>
<th>PFIWA</th>
<th>PFIWG</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>0.712</td>
<td>0.619</td>
</tr>
<tr>
<td>A2</td>
<td>0.435</td>
<td>0.351</td>
</tr>
<tr>
<td>A3</td>
<td>0.438</td>
<td>0.391</td>
</tr>
<tr>
<td>A4</td>
<td>0.589</td>
<td>0.396</td>
</tr>
<tr>
<td>A5</td>
<td>0.610</td>
<td>0.435</td>
</tr>
</tbody>
</table>

### Table 3
Ordering of the ERP systems

<table>
<thead>
<tr>
<th></th>
<th>Ordering</th>
</tr>
</thead>
<tbody>
<tr>
<td>PFIWA</td>
<td>A1 &gt; A5 &gt; A4 &gt; A3 &gt; A2</td>
</tr>
<tr>
<td>PFIWG</td>
<td>A1 &gt; A5 &gt; A4 &gt; A3 &gt; A2</td>
</tr>
</tbody>
</table>
functions, the ordering of the ERP systems are shown in Table 3. Note that “>” means “preferred to”. As we can see, depending on the aggregation operators used, the ordering of the ERP systems is some and the best ERP system is A1.

7. Conclusion

In this study we utilize arithmetic and geometric operations to investigate some Pythagorean fuzzy interaction aggregation operators in detail, and point out that Theorems 1 and 10 in Wei [4] are incorrect by the Pythagorean fuzzy operational laws. Finally we propose the modifications of these theorems and equations. In the future, we shall continue working in the extension and application of the developed operators to other domains and fuzzy setting, such as picture fuzzy sets, dual hesitant fuzzy sets, and so on [6–22].

References