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Limitations of Efficient Reducibility to the Kolmogorov Random Strings

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Abstract. We show the following results for polynomial-time reducibility to R_C, the set of Kolmogorov random strings.

1. If $\mathrm{P} \neq \mathrm{NP}$, then SAT does not dtt-reduce to R_C.
2. If PH does not collapse, then SAT does not n^α--reduce to R_C for any $\alpha < 1$.
3. If PH does not collapse, then SAT does not n^α-T-reduce to R_C for any $\alpha < \frac{1}{2}$.
4. There is a problem in E that does not dtt-reduce to R_C.
5. There is a problem in E that does not n^α--reduce to R_C, for any $\alpha < 1$.
6. There is a problem in E that does not n^α-T-reduce to R_C, for any $\alpha < \frac{1}{2}$.

These results hold for both the plain and prefix-free variants of Kolmogorov complexity and are also independent of the choice of the universal machine.

Keywords: Kolmogorov random strings, polynominal-time reducibility, Turing reduction, universal machine

1. Introduction

Because the Kolmogorov complexity function $C(x)$ is noncomputable, the set

$$R_C = \{x \mid C(x) > |x|\}$$

of Kolmogorov random strings is undecidable. In fact, R_C has no infinite computably enumerable subset. From this and the fact that the complement $\overline{R_C}$ is computably enumerable, Arslanov's completeness criterion implies that R_C is hard for the c.e. sets under Turing reductions. Kummer [7] showed a stronger result: $\overline{H} \leq_{\mathrm{dtt}} R_C$, where $\overline{H}$ is the complement of the halting problem and $\leq_{\mathrm{dtt}}$ denotes a disjunctive truth-table reduction. Neither of these reductions from the halting problem to R_C is efficient. This raises the question [1]: what can be efficiently reduced to R_C?

Recall that the Kolmogorov complexity [9] of a binary string x is the length of a shortest program that prints x on a universal Turing machine U:

$$C_U(x) = \min\{|p| \;\mid\; U(p) \text{ prints } x\}.$$

For the most part, the theory of Kolmogorov complexity does not depend on the choice of the universal machine U: for any two universal machines U and V, C_U and C_V are within an additive constant of each other. As usual, we fix a universal machine U and omit it from the notation, writing $C(x)$ instead of $C_U(x)$. There are, however, situations when the choice of universal machine matters and then we will be explicit with the subscript. We use the notation $\mathrm{P}_\tau(A)$ to denote the class of problems that reduce to A by $\leq_\tau^{\mathrm{p}}$-reductions.

Kummer's result [7] implies there is a computable time bound $t(n)$ such that for every decidable A, $A \leq_{\mathrm{dtt}}^{t(n)} R_C$. Kummer's proof is nonconstructive and does not yield any information about the function $t(n)$. In fact, Allender et al. [1] show that some uncertainty about the time bound $t(n)$ is inevitable. They show that the $t(n)$ in Kummer's theorem

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may be arbitrarily large, depending on the choice of the universal machine U. Formally, for every computable time bound $t(n)$, there exists a universal machine U and a decidable set A such that A does not $\leq_{\mathrm{dtt}}^{t(n)}$-reduce to R_{C_U}. On the other hand, independent of U, there exist decidable sets with arbitrarily high time complexity that reduce to R_{C_U} via a polynomial-time dtt-reduction: for every computable time bound $t(n)$ and every universal machine U, there is a set $A \in \mathrm{DEC} - \mathrm{DTIME}(t(n))$ such that $A \leq_{\mathrm{dtt}}^{\mathrm{p}} R_{C_U}$. While this result shows $\mathrm{P}_{\mathrm{dtt}}(R_C)$ contains sets of high time complexity, the set A in this theorem is constructed via padding, which makes A very sparse. Thus while A has high time complexity, A is very simple in other terms. We show that this simplicity is inherent: any such A is highly predictable in the sense of polynomial-time dimension. From this it follows that R_C is not hard for E under $\leq_{\mathrm{dtt}}^{\mathrm{p}}$-reductions. This holds for every universal machine, i.e. $\mathrm{E} \not\subseteq \mathrm{P}_{\mathrm{dtt}}(R_{C_U})$ for every U. We also show that R_C is not polynomial-time dtt-hard for NP unless P = NP. Both of these results follow from showing that if a decidable set $\leq_{\mathrm{dtt}}^{\mathrm{p}}$-reduces to R_C, then the set $\leq_{\mathrm{dtt}}^{\mathrm{p}}$-reduces to a tally set. These results complement the result of Allender et al. [1] that

$$\mathrm{P} = \mathrm{DEC} \cap \bigcap_U \mathrm{P}_{\mathrm{dtt}}(R_{C_U}),$$

where the intersection is over all universal machines. While the class $\mathrm{DEC} \cap \mathrm{P}_{\mathrm{dtt}}(R_{C_U})$ contains arbitrarily complex sets, it is intuitively "close" to P for every U, in that it has small dimension and cannot contain NP unless P = NP.

Allender et al. [2] showed that R_C is hard for PSPACE under polynomial-time Turing reductions: $\mathrm{PSPACE} \subseteq \mathrm{P}_{\mathrm{T}}(R_C)$. Buhrman et al. [3] showed that R_C is hard for BPP under polynomial-time truth-table reductions: $\mathrm{BPP} \subseteq \mathrm{P}_{\mathrm{tt}}(R_C)$. We consider bounded query Turing and truth-table reductions. Based on the Winnow algorithm [10] and polynomial-time dimension [6], we show that R_C is not $\leq_{n^\alpha\text{-tt}}^{\mathrm{p}}$-hard for E, for any $\alpha < 1$. This is an improvement of a result in [1] which obtained the same consequence for EE. Also, we use the techniques of [4, 5] to show that R_C is not $\leq_{n^\alpha\text{-tt}}^{\mathrm{p}}$-hard for NP unless $\mathrm{NP} \subseteq \mathrm{coNP/poly}$ and the polynomial-time hierarchy collapses by Yap's theorem [13]. Finally, we obtain the same consequences for $\leq_{n^\alpha\text{-T}}^{\mathrm{p}}$-reductions, for all $\alpha < \frac{1}{2}$.

2. Preliminaries

We use standard notions of polynomial-time reducibilities [8]. We also need the following two notions of reducibility.

Definition 2.1. Let $\mathcal{B} = (B_n \mid n \geq 0)$ be a family of subsets of Σ^*. We say that A NP-*reduces to* $\mathcal{B}$ if there is an NPMV function N such that for all n, for all $x \in \Sigma^n$, $x \in A$ iff at least one output of $N(x)$ is in B_n.

Definition 2.2. Let $\mathcal{B} = (B_n \mid n \geq 0)$ be a family of subsets of Σ^*. We say that A *disjunctively reduces to* $\mathcal{B}$ *in* $t(n)$ *time* if there is an algorithm M such that for all n, for all $x \in \Sigma^n$, $M(x)$ outputs a list of strings in $t(n)$ time and $x \in A$ iff at least one output of $M(x)$ is in B_n.

The following lemma is from [4], based on a technique of [5]. An AND-function (of order 1) for a set A is a polynomial-time computable function g such that for all strings $x_1, x_2, \ldots, x_n$, $|g(x_1, \ldots, x_n)| = O\big(\sum_{i=1}^n |x_i|\big)$ and $g(x_1, x_2, \ldots, x_n) \in A$ iff $x_i \in A$ for all i.

Lemma 2.3. *Let A have an AND-function and let $\alpha < 1$. Let $\mathcal{B} = (B_n \mid n \geq 0)$ be a family of sets with $|B_n| \leq 2^{n^\alpha}$ for sufficiently large n. If A NP-reduces to $\mathcal{B}$, then $A \in \mathrm{NP/poly}$.*

The p-dimension [11] of a complexity class is a real number in $[0, 1]$. The p-dimension of P is 0 and the p-dimension of E is 1. For this paper, we do not need the full details of p-dimension; all we require is the fact that a p-dimension 0 class cannot contain E and the following lemma. The proof of this lemma relies on the Winnow online learning algorithm [10] and is straightforward to prove using the approach of [6].

Lemma 2.4. *Let $\alpha < 1$ and let $c \geq 1$. Let X be the class of all A for which there exists a family $\mathcal{B} = (B_n \mid n \geq 0)$ with $|B_n| \leq 2^{n^\alpha}$ such that A disjunctively reduces to $\mathcal{B}$ in 2^{cn} time. Then X has* p-*dimension 0. In particular, X does not contain* E.

3. Disjunctive Reductions

Theorem 3.1. *If A is decidable and $A \leq^{\mathrm{p}}_{\mathrm{dtt}} R_C$, then $A \leq^{\mathrm{p}}_{\mathrm{dtt}} B$ for some $B \in$ TALLY.*

Proof. We use the proof technique from [1] that A is decidable and $A \leq^{\mathrm{p}}_{\mathrm{mtt}} R_C$ (monotone truth-table) implies $A \in \mathrm{P/poly}$, observing that we can encode in a tally set to obtain the stronger result.

Suppose A is decidable and $A \leq^{\mathrm{p}}_{\mathrm{dtt}} R_C$ via a reduction computable in time n^d. Let the queries on input x be denoted by $Q(x)$. For some constant c, we claim only the queries of length at most $l(n) = c \log n$ "matter."

For any x, we have $x \in A$ iff $Q(x) \cap R_C \neq \emptyset$. Define $Q'(x) = Q(x) \cap \Sigma^{\leq l(n)}$, where $n = |x|$. We claim that for each $x \in A$, there is some $q \in Q'(x)$ such that for all y with $|y| = |x|$, $q \in Q'(y)$ implies $y \in A$.

Suppose the claim is false. Then given n, we can find the first string x of length n such that $x \in A$ and each query $q \in Q'(x)$ belongs to $Q'(y)$ for some $y \notin A$. This implies that $Q'(x) \cap R_C = \emptyset$. Since $x \in A$, it follows that $Q(x) - Q'(x)$ contains a string $r \in R_C$. This string r has $C(r) > l(n)$ because $r \notin Q'(x)$. We can describe r by describing n and the index of r in $Q(x)$. Since $|Q(x)| \leq n^d$, this takes at most $(d+3)\log n$ bits, a contradiction if we choose $c = d + 4$.

Let $\{w_1, \ldots, w_N\}$ be an enumeration of $\Sigma^{\leq l(n)}$. Let I_n be the collection of all i where for all y of length n, $w_i \in Q(y)$ implies $y \in A$. Our desired tally set is $\{0^{\langle n,i \rangle} \mid n \geq 0 \text{ and } i \in I_n\}$, where $\langle \cdot, \cdot \rangle$ is a pairing function on the natural numbers. □

Corollary 3.2. *If* $\mathrm{P} \neq \mathrm{NP}$*, then* $\mathrm{NP} \not\subseteq \mathrm{P_{dtt}}(R_C)$.

Proof. Suppose that $\mathrm{NP} \subseteq \mathrm{P_{dtt}}(R_C)$. By Theorem 3.1, $\mathrm{SAT} \leq^{\mathrm{p}}_{\mathrm{dtt}} B$ for a tally set B. Then $\overline{\mathrm{SAT}} \leq^{\mathrm{p}}_{\mathrm{ctt}} \overline{B} \cap 0^*$. Ukkonen [12] showed that $\mathrm{P} = \mathrm{NP}$ if coNP has a sparse $\leq^{\mathrm{p}}_{\mathrm{ctt}}$-hard set. □

Corollary 3.3. *The class* $\mathrm{P_{dtt}}(R_C) \cap \mathrm{DEC}$ *has* p*-dimension 0.*

Proof. Theorem 3.1 implies $\mathrm{P_{dtt}}(R_C) \cap \mathrm{DEC} \subseteq \mathrm{P_{dtt}}(\mathrm{TALLY}) \subseteq \mathrm{P_{dtt}}(\mathrm{SPARSE})$. This last class was shown to have p-dimension 0 in [6]. □

Corollary 3.4. $\mathrm{E} \not\subseteq \mathrm{P_{dtt}}(R_C)$.

Proof. This follows from Corollary 3.3 because E has p-dimension 1. □

4. Truth-Table Reductions

Theorem 4.1. *Let $\alpha < 1$.*

1. *If A is decidable, A has an AND-function, and $A \leq^{\mathrm{p}}_{n^\alpha\text{-tt}} R_C$, then $A \in \mathrm{NP/poly}$.*
2. *The class* $\mathrm{P}_{n^\alpha - \mathrm{tt}}(R_C) \cap \mathrm{DEC}$ *has* p*-dimension 0.*

Proof. The main idea of the proof is from [1]. We expound the argument here and show how to apply Lemmas 2.3 and 2.4.

Let A be decidable such that $A \leq^{\mathrm{p}}_{n^\alpha\text{-tt}} R_C$. Write $Q(x)$ for the truth-table reduction's queries on input x and $Z_x \subseteq \Sigma^{n^\alpha}$ for the query answer sequences that cause the reduction to accept x. That is, if $Q(x) = \{q_1, \ldots, q_{n^\alpha}\}$ in lexicographic order, then $x \in A$ if and only if $R_C[q_1] \cdots R_C[q_{n^\alpha}] \in Z_x$.

Let $l(n) = n^\epsilon$, where $0 < \epsilon < 1 - \alpha$. We claim that the truth-table reduction is still correct if we only use the queries of length at most $l(n)$. Formally, let $Q'(x) = Q(x) \cap \Sigma^{\leq l(n)}$ and let Z'_x be the restriction of Z_x with bits corresponding to strings in $Q(x) - Q'(x)$ removed.

Call two strings x and y of the same length *equivalent* if $Q'(x) = Q'(y)$. We claim that for each $x \in A$, there is some $z_x \in Z'_x$ such that for all y equivalent to x, $z_x \in Z'_y$ iff $y \in A$.

Suppose the claim is false. We can find the least $x \in A$ such that for all $z \in Z'_x$, there is some y_z equivalent to x such that $z \in Z'_y$ iff $y_z \notin A$. Let v be the correct answer sequence for $Q'(x) \cap R_C$ and let r be the number of 1's in v;

that is, $r = |Q'(x) \cap R_C|$. Given x and r, we can enumerate $\overline{R_C}$ to compute $Q'(x) \cap R_C$ and obtain v. Then we can compute y_v such that query answers v are incorrect for y_v. This means that $Q(y_v) - Q'(y_v)$ must contain a string in R_C with length $> l(n)$. However, we can describe this string by describing n, r, and its index in $Q(y_v)$, which takes $O(\log n)$ bits, a contradiction.

We define a family of sets $\mathcal{B} = (B_n \mid n \geq 0)$ as follows. For each equivalence class $[x]$ with queries $Q'(x) = \{w_1, \ldots, w_{n^\alpha}\}$ and $z_x \in Z'_x$ the answer sequence that is correct for all strings in the equivalence class, we put the tuple $\langle w_1, \ldots, w_{n^\alpha}, z_x\rangle$ in B_n. Note that $|B_n| < 2^{n^\gamma}$ where $\alpha + \epsilon < \gamma < 1$. By the claim, A NP-reduces to $\mathcal{B}$. It follows from Lemma 2.3 that $A \in \mathrm{NP/poly}$ if A has an AND-function.

We also have that A is disjunctively reducible in 2^n time to $\mathcal{B}$. Therefore Lemma 2.4 applies to show $\mathrm{P}_{n^\alpha-\mathrm{tt}}(R_C) \cap$ DEC has p-dimension 0. □

Corollary 4.2. *If* $\mathrm{NP} \subseteq \mathrm{P}_{n^\alpha-\mathrm{tt}}(R_C)$ *for some* $\alpha < 1$, *then* $\mathrm{NP} \subseteq \mathrm{coNP/poly}$.

Proof. This follows from Theorem 4.1 because the hypothesis implies $\overline{\mathrm{SAT}} \leq^{\mathrm{p}}_{n^\alpha\text{-tt}} R_C$ and $\overline{\mathrm{SAT}}$ has an AND-function. □

Corollary 4.3. *If the polynomial-time hierarchy does not collapse, then* $\mathrm{NP} \not\subseteq \mathrm{P}_{n^\alpha-\mathrm{tt}}(R_C)$ *for any* $\alpha < 1$.

Proof. This is immediate from Corollary 4.2 and Yap's theorem [13] that $\mathrm{NP} \subseteq \mathrm{coNP/poly}$ implies the polynomial-time hierarchy collapses to its third level. □

Corollary 4.4. *For any* $\alpha < 1$, $\mathrm{E} \not\subseteq \mathrm{P}_{n^\alpha-\mathrm{tt}}(R_C)$.

Proof. This follows from Theorem 4.1 because E has p-dimension 1. □

5. Turing Reductions

Theorem 5.1. *Let* $\alpha < \frac{1}{2}$.

1. *If A is decidable, A has an AND-function, and* $A \leq^{\mathrm{p}}_{n^\alpha\text{-T}} R_C$, *then* $A \in \mathrm{NP/poly}$.
2. *The class* $\mathrm{P}_{n^\alpha-\mathrm{T}}(R_C) \cap \mathrm{DEC}$ *has* p*-dimension 0.*

Proof. Let $\alpha < \beta < \frac{1}{2}$. Suppose that $A \in \mathrm{DEC}$ and $A \leq^{\mathrm{p}}_{n^\alpha\text{-T}} R_C$ via M. Let M' be the Turing machine that simulates M and whenever M makes a query of length at least n^β, M' makes no query and proceeds as if the answer to the query were no. We use the following concepts:

- An *advice* is a tuple $(z, w_1, \ldots, w_{n^\alpha})$ such that $z \in \Sigma^{n^\alpha}$ and each $w_i \in \Sigma^{<n^\beta}$.
- A string y is *accepted with advice* $(z, w_1, \ldots, w_{n^\alpha})$ if $M'(y)$ queries $w_1, \ldots, w_{n^\alpha}$ and accepts y when M' is given $z[1], \ldots, z[n^\alpha]$ as the query answers.
- An advice $(z, w_1, \ldots, w_{n^\alpha})$ is *safe* if for all $y \in \Sigma^n$, y is accepted with advice $(z, w_1, \ldots, w_{n^\alpha})$ implies $y \in A$.

We claim that for all $x \in A_{=n}$, there is a safe advice $(z, \vec{w})$ such that x is accepted with advice $(z, \vec{w})$.

Suppose the claim is false. Then we can find the least $x \in A_{=n}$ that does not have a safe advice. We can specify the correct answer sequence $z \in \Sigma^{n^\alpha}$ for $M(x)$ when querying oracle R_C. With this correct answer sequence z, M must query some string in R_C that is not in $\Sigma^{<n^\beta}$. Therefore we can describe a string r with $C(r) \geq n^\beta$ by describing n, z, and the index of r in $M(x)$'s query set on query answer sequence z. Thus $C(r) \leq n^\alpha + O(\log n)$, which is a contradiction since $\alpha < \beta$.

We define a family of sets $\mathcal{B}$ by putting into B_n all advices $(z, w_1, \ldots, w_{n^\alpha})$ that are safe. Let $1 > \gamma > \alpha + \beta$. The total number of possible advices is at most $2^{n^\alpha} \cdot (2^{n^\beta})^{n^\alpha} < 2^{n^\gamma}$, so $|B_n| < 2^{n^\gamma}$. We have that A NP-reduces to $\mathcal{B}$ and A disjunctively reduces in 2^n time to $\mathcal{B}$, so the theorem follows from Lemmas 2.3 and 2.4. □

Corollary 5.2. *If* $\mathrm{NP} \subseteq \mathrm{P}_{n^\alpha-\mathrm{T}}(R_C)$ *for some* $\alpha < \frac{1}{2}$, *then* $\mathrm{NP} \subseteq \mathrm{coNP/poly}$.

Corollary 5.3. *If the polynomial-time hierarchy does not collapse, then* $\mathrm{NP} \not\subseteq \mathrm{P}_{n^\alpha-\mathrm{T}}(R_C)$ *for any* $\alpha < \frac{1}{2}$.

Corollary 5.4. *For any* $\alpha < \frac{1}{2}$, $\mathrm{E} \not\subseteq \mathrm{P}_{n^\alpha-\mathrm{T}}(R_C)$.

6. Open Problems

We believe the following open problems should be tractable but appear to require techniques beyond those used in this paper.

Problem 6.1. Show that $\mathrm{E} \not\subseteq \mathrm{P}_{n^\alpha-\mathrm{T}}(R_C)$ for $\frac{1}{2} \leq \alpha < 1$.

Problem 6.2. Show that $\mathrm{NP} \not\subseteq \mathrm{P}_{n^\alpha-\mathrm{T}}(R_C)$ for $\frac{1}{2} \leq \alpha < 1$ under a reasonable hypothesis.

It is unknown whether even every decidable problem is polynomial-time Turing reducible to R_C. We conjecture that $\mathrm{ESPACE} \not\subseteq \mathrm{P_T}(R_C)$ and that this can be proved using resource-bounded dimension or measure:

Problem 6.3. Show that $\mathrm{P_T}(R_C) \cap \mathrm{DEC}$ has pspace-dimension 0.

Lastly, we know $\mathrm{SAT} \leq_{\mathrm{dtt}} R_C$ and that $\mathrm{SAT} \leq^{\mathrm{p}}_{\mathrm{dtt}} R_C$ iff $\mathrm{P} = \mathrm{NP}$. What more can be said about the amount of time it takes to disjunctively reduce SAT to R_C?

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