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A note on local energy decay results for wave equations with a potential

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Abstract. In this paper, we derive uniform local energy decay results for wave equations with a short-range potential in an exterior domain. In this study, we considered this problem within the framework of non-compactly supported initial data, unlike previously reported studies. The essential parts of analysis are both L^2-estimates of the solution itself and the weighted energy estimates. Only a multiplier method is used, and we do not rely on any resolvent estimates.

Keywords: Wave equation, short-range potential, exterior mixed problem, non-compact support, initial data, local energy, algebraic decay

1. Introduction and statement of results

In this paper, we are concerned with the following initial-boundary value problem:

$$u_{tt} - \Delta u + V(x)u = 0, \quad t > 0, x \in \Omega, \tag{1.1}$$

$$u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x) \quad x \in \Omega, \tag{1.2}$$

$$u(t,x) = 0 \quad t > 0, x \in \partial\Omega, \tag{1.3}$$

where $\Omega \subset \mathbf{R}^n$ is an exterior domain with smooth compact boundary $\partial\Omega$ such that $0 \notin \bar{\Omega}$. Furthermore, let $\rho_0 > 0$ be a real number such that $\partial\Omega \subset B_{\rho_0}$ and assume that

(A-1) the obstacle $\mathcal{O} := \mathbf{R}^n \setminus \bar{\Omega}$ is star-shaped relative to the origin, that is, $x \cdot \nu(x) \leqslant 0$, $x \in \partial\Omega$, where $\nu(x)$ is the unit exterior normal at the point $x \in \partial\Omega$, and $B_r := \{x \in \mathbf{R}^n : |x| < r\}$ for $r > 0$.

Regarding the potential function $V(x)$, one assumes that $V \in C^1(\bar{\Omega})$, $V(x) \geqslant 0$ $(x \in \Omega)$, both $V(x)$ and $|\nabla V(x)|$ are bounded in $\bar{\Omega}$, and

(A-2) $\frac{1}{2}(x \cdot \nabla V(x)) + V(x) \leqslant 0$ for all $x \in \bar{\Omega}$.

Note that functions and solutions treated in this paper are all real-valued.

Example 1. One can present a typical example for $V(x)$ satisfying the assumption **(A-2)** as follows:

$$V(x) = V_0|x|^{-\alpha}, \qquad V_0 > 0,$$

where $\alpha \geqslant 2$. In general, the potential $V(x)$ is called short-range if $V(x) = O(|x|^{-\delta})$ ($|x| \to \infty$) and $\delta > 1$. This example shows that $V(x)$ is certainly a short-range potential.

Remark 1.1. It seems quite important for this type of problem whether the case $\alpha = 2$ can be included as an example of $V(x) = V_0|x|^{-\alpha}$ (cf., [11] and [29]). $\alpha = 2$ corresponds to the so-called scale-invariant case (see also [4]).

Remark 1.2. In the case of a radial function $V(x) = V(r)$ for $r := |x|$, assumption **(A-2)** can be replaced by

$$rV'(r) \leqslant -2V(r), \quad x \in \bar{\Omega}.$$

Example 2. If $\mathcal{O} := B_2$, then we can choose $V(x) := V_0 e^{-|x|}$ with $V_0 > 0$.

Remark 1.3. Let us compare our assumption on the potential with [7], which dealt with elastic waves in $\mathbf{R}^3$. Assumption **(A-2)** is weakened in [7] to the following condition:

$$\frac{1}{2}\big(x \cdot \nabla V(x)\big) + V(x) \leqslant \frac{\gamma}{2} V(x), \quad x \in \bar{\Omega}, \tag{1.4}$$

where $\gamma \in [0, 1)$; note, if we choose specifically $\gamma = 0$, condition (1.4) in [7] is the same as assumption (A-2). In that sense, condition (1.4) in [7] is weaker than assumption (A-2) here. However, the local energy decay in [7] was obtained under the strong assumption that $V(x)$ has compact support and a finite speed of propagation. Reference [7] considered a system of elastic waves in $\mathbf{R}^3$, but, of course, the results hold for a wave equation with the same type of potential.

We define the total energy for equation (1.1) by

$$E(t) := \int_\Omega e(t,x)\,dx := \frac{1}{2}\int_\Omega \big(\big|u_t(t,x)\big|^2 + \big|\nabla u(t,x)\big|^2 + V(x)\big|u(t,x)\big|^2\big)\,dx. \tag{1.5}$$

Then, under assumptions **(A-1)** and **(A-2)**, it is known that for each initial data $[u_0, u_1] \in H_0^1(\Omega) \times L^2(\Omega)$, the problem (1.1)–(1.3) has a unique weak solution $u \in C([0,\infty); H_0^1(\Omega)) \cap C^1([0,\infty); L^2(\Omega))$ satisfying the energy identity

$$E(t) = E(0).$$

Regarding this, the reader can refer to [6] and [12]. Note that in our case, the operator $L := -\Delta + V(\cdot)$ is nonnegative and self-adjoint in $L^2(\Omega)$ with its domain $D(L) = H^2(\Omega) \cap H_0^1(\Omega)$ because of the Kato–Rellich theorem.

Our main purpose is to study the local energy decay problem of equation (1.1) with the short-range potential $V(x)$. Here, for each $R > 0$, the local energy $E_R(t)$ can be defined as follows:

$$E_R(t) := \frac{1}{2}\int_{B_R \cap \Omega} \big(\big|u_t(t,x)\big|^2 + \big|\nabla u(t,x)\big|^2 + V(x)\big|u(t,x)\big|^2\big)\,dx.$$

Before, proceeding, let us first discuss the related literature. For well-known local energy decay results, we note a study of C. Morawetz [19], where the uniform local energy decay result was derived by constructing the so-called Morawetz identity for equation (1.1) with $V(x) \equiv 0$. In fact, Morawetz derived $E_R(t) = O(t^{-1})$ $(t \to \infty)$ under a stronger geometrical constraint on the obstacle shape, specifically, for the star-shaped obstacle case. To obtain such results, Morawetz assumed that the initial data have compact support. One of the essential parts of the work in [19] was deriving the L^2-bound of the solution itself by using the compact support assumption on the initial data. Additionally, the estimate of the solution of the corresponding Poisson equation played a crucial role in [19]. In particular, in the three-dimensional case, it can be proved using Huygens principle that the local energy decays exponentially fast. In [20], the authors also treated the non-trapping obstacle case.

Following Morawetz, studies devoted to removing the compactness assumptions on the support of the initial data were conducted, as reported in [21,30], and [13,14,16]. These studies adopted the multiplier method, and in [30] and [21], the decay rate $O(t^{-2})$ and the integrability of the local energy were derived under a quite stronger weight condition on the initial data, while the decay rate $O(t^{-1})$ of the local energy was derived under a weaker weight condition on the initial data due to [13,14,16] (see also [8] for the variable coefficient case with Lipschitz wave speeds). It should be mentioned that the latter weight condition $(|x| \to \infty)$ imposed on the initial data seems to be the weakest assumption among the reported results. Other related deep investigations on the topic of local energy decay include [3,18,23–25], and [28], all under the condition of compactly supported initial data. In particular, in [1], one-dimensional wave equations with variable coefficients were adopted to capture the exponential decay of the local energy. To the best of the authors' knowledge, [1] was first to explore the one-dimension case deeply.

On the other hand, for equation (1.1) with potential $V(x)$, a few results are known. In particular, in [29], the sharp local energy decay rates in the short-range case such that $V(x) = V_0(1 + |x|^2)^{-\alpha/2}$ satisfying $\alpha > 2$ were investigated. In fact, the same author studied the Cauchy problem of (1.1) in $\mathbf{R}^n$ $(n \geqslant 3)$, and obtained the decay rate $O(t^{-2})$. In same study, the compactness of the support of the initial data was necessary. In connection with this, uniform weighted resolvent estimates were effectively adopted. Therefore, it seems that the local energy decay problem for equation (1.1) has yet to be considered without the compact support assumption on the initial data. We here develop our theory by using the multiplier method based on the expanded Morawetz identity. As a side note, in [7] and [26], local energy decay problems were investigated for elastic waves with time-independent potentials and wave equations with time-dependent potentials, respectively. However, in both cases, the problems were considered within the framework of compact support assumptions on both the potentials and initial data. It should be pointed out that assumption (A-2) in the present paper is stronger than the assumption in (8) of [7] (see Remark 1.2).

For later use, let us define a weight function $d_n(x)$ by

$$d_n(x) = \begin{cases} |x| & n \geqslant 3, \\ \log(B|x|) & n = 2, \end{cases}$$

with some constant $B > 0$ satisfying $B \inf\{|x| : x \in \Omega\} \geqslant 2$. Moreover, we denote the L^2-norm of $u \in L^2(\Omega)$ by $\|u\|$.

The main result of this paper is the following theorem.

Theorem 1.1. *Let $\rho_0 > 0$ be such that $\partial\Omega \subset B_{\rho_0}$. Further, for $n \geqslant 2$, require assumptions* **(A-1)** *and* **(A-2)** *above. Finally, let $R > \rho_0$ be an arbitrary fixed number. If $[u_0, u_1] \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$, then the* *unique smooth solution $u(t,x)$ to problem (1.1)–(1.3) satisfies*

$$E_R(t) \leqslant \frac{CK_0}{t-R}, \quad t > R,$$

for some constant $C > 0$, where

$$K_0 := \int_\Omega u_1(x)u_0(x)\,dx + \int_\Omega \big(u_1(x)\big(x\cdot\nabla u_0(x)\big)\big)\,dx + \sqrt{E(0)}\big(\|u_0\| + \big\|d_n(\cdot)u_1\big\|\big)$$
$$+ \int_\Omega (1+|x|)\big(\big|u_1(x)\big|^2 + \big|\nabla u_0(x)\big|^2 + V(x)\big|u_0(x)\big|^2\big)\,dx.$$

Remark 1.4. In some sense, assumption (A-2) on potential $V(x)$ is a technical condition; however, it includes an important example $\alpha = 2$, that is, $V(x) = V_0|x|^{-2}$, as a critical potential. An important fact is that such a singular potential is unique such that the perturbed wave equation still follows Huygen's principle in dimension $n = 3$. For the perturbed wave equation with a regular potential, the Huygens' principle never holds (see [9]).

It should be emphasized that the constant $C > 0$ determined in Theorem 1.1 does not depend on $R > \rho_0$, and that $R > \rho_0$ is independent of the size of support of the initial data. These imply that one never relies on the finite speed of the propagation property as is usually discussed (cf., [19]). This is our essential contribution, and the condition $(u_0, u_1) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$ imposed on the initial data is not essential. Using density, one can discuss the same local energy decay in the framework of $H_0^1(\Omega) \times L^2(\Omega)$. For this purpose, we introduce the weighted Sobolev space (see [5] and [17]).

Set $w(x) := (1+|x|)$. We first define the weighted L^2-space by

$$L^2(\Omega, w) := \left\{u \in L^2(\Omega) : \int_\Omega \big|u(x)\big|^2 w(x)\,dx < +\infty\right\}.$$

Next, we denote by $W^{1,2}(\Omega, w)$ the set of all functions $u \in L^2(\Omega, w)$ for which the weak derivatives $\partial_j u$ $(j = 1, 2, \ldots, n)$ belong to $L^2(\Omega, w)$. The norm of $u \in W^{1,2}(\Omega, w)$ can be defined by

$$\|u\|_{W^{1,2}(\Omega,w)} := \left(\int_\Omega \big(\big|u(x)\big|^2 + \big|\nabla u(x)\big|^2\big) w(x)\,dx\right)^{1/2}.$$

Note that $w^{-1}, w \in L^1_{\mathrm{loc}}(\Omega)$ and that $C_0^\infty(\Omega)$ is a subset of $W^{1,2}(\Omega, w)$. Thus, one can introduce the space $W_0^{1,2}(\Omega, w)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{1,2}(\Omega,w)}$. From the definition of the weight function $w(x)$, we see that $W_0^{1,2}(\Omega, w) \hookrightarrow H_0^1(\Omega)$.

Now we are ready to state a refinement of Theorem 1.1.

Theorem 1.2. *Let $n \geqslant 2$ and require assumptions **(A-1)** and **(A-2)**. Further, let $R > \rho_0$ be an arbitrary fixed number. If $[u_0, u_1] \in W_0^{1,2}(\Omega, w) \times L^2(\Omega)$, then the unique weak solution $u \in C([0,\infty); H_0^1(\Omega)) \cap C^1([0,\infty); L^2(\Omega))$ to problem (1.1)–(1.3) satisfies*

$$E_R(t) \leqslant \frac{CK_0}{t-R}, \quad t > R,$$

for some constant $C > 0$, where K_0 is as defined in Theorem 1.1, provided that

$$\int_\Omega |x|\left|u_1(x)\right|^2 dx < +\infty \quad (n=2), \qquad \int_\Omega |x|^2\left|u_1(x)\right|^2 dx < +\infty \quad (n \geqslant 3).$$

Remark 1.5. Unfortunately, the constant coefficient case $V(x) = m^2$ ($m > 0$) cannot be included as an example. Actually, $V(x) = m^2$ does not satisfy assumption (A-2), since $V(x) = m^2$ does not decrease radially. This is the so-called Klein–Gordon equation case, which seems to be a difficult case to address with our method. Assumption (A-2) may express a small perturbation from the pure wave equation case with $V(x) \equiv 0$. For the sharp local energy decay of the Klein–Gordon equation by using compactness assumptions on the initial data, see the recent paper [22]. Note that if one can derive the estimate $\int_0^\infty \|u(s,\cdot)\|^2 ds < +\infty$ for the Klein–Gordon equation, then one may obtain the local energy decay as stated in Theorem 1.2. This can be observed from Lemma 2.1 below with $V(x) = m^2$.

Remark 1.6. In the assumptions on the initial velocity $u_1(x)$ of Theorem 1.2, it is easy to see that in the case when $n = 2$, the condition $\|d_2(\cdot)u_1\| < +\infty$ can be absorbed into $\int_\Omega |x||u_1(x)|^2 dx < \infty$, while in the case of $n \geqslant 3$, $\|d_n(\cdot)u_1\| < +\infty$ implies $\int_\Omega |x||u_1(x)|^2 dx < \infty$. Incidentally, the condition $\int_\Omega |x|V(x)|u_0(x)|^2 dx < +\infty$ can be controlled by the quantity $\int_\Omega |x||u_0(x)|^2 dx$ because of the boundedness of the potential $V(x)$.

Note that the concrete case $V(x) := V_0|x|^{-\alpha}$ with $\alpha \geqslant 2$ can be included as an example, and in this case, from Theorem 1.1, one has

$$\frac{V_0}{2R^\alpha}\int_{B_R\cap\Omega}\left|u(t,x)\right|^2 dx \leqslant \frac{1}{2}\int_{B_R\cap\Omega} V(x)\left|u(t,x)\right|^2 dx \leqslant E_R(t) \leqslant \frac{CK_0}{t-R}, \quad t > R,$$

so that one also has a local L^2-decay result:

$$\int_{B_R\cap\Omega}\left|u(t,x)\right|^2 dx \leqslant \frac{2R^\alpha}{V_0}\frac{CK_0}{t-R}, \quad t > R. \tag{1.6}$$

The decay result (1.6) is closely related to that of [29, Theorem 1.2]. In [29], the critical case $\alpha = 2$ cannot be included as an example.

The rest of the present paper is organized into three sections. Section 2 is dedicated to sharing some preliminary results, which are used in the proof of Theorem 1.1. In Section 3, we prove our main result, Theorem 1.1. In Section 4, we observe the energy concentration area as a direct consequence of Theorem 1.2. An outline of the proof of Theorem 1.2 is given in the appendix.

2. Preliminaries

The following lemma is a kind of Morawetz identity of equations (1.1) and (1.2) obtained using the multiplier $m(u) = tu_t + x\cdot\nabla u + \frac{n-1}{2}u$. The Morawetz identity is useful when one needs to obtain some estimates on solutions, at least for hyperbolic equations. In [2] (Lemma 3.3), identities with generalized multipliers of Morawetz type were obtained to study the stabilization of solutions to a system of elastic waves with localized nonlinear dissipation.

Lemma 2.1. *Let $n \geqslant 2$, and $[u_0, u_1] \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$. Then, the corresponding smooth solution $u(t,x)$ to problem (1.1)–(1.3) satisfies the following identity: for $t \geqslant 0$, it holds that*

$$tE(t) + \frac{n-1}{2}\int_\Omega u_t(t,x)u(t,x)\,dx + \int_\Omega u_t(t,x)\big(x\cdot\nabla u(t,x)\big)\,dx$$
$$-\int_0^t\int_\Omega\left(\frac{1}{2}\big(x\cdot\nabla V(x)\big)+V(x)\right)\big|u(s,x)\big|^2\,dx\,ds = J_0 + \frac{1}{2}\int_0^t\int_{\partial\Omega}\left(\frac{\partial u}{\partial\nu}\right)^2\sigma\cdot\nu(\sigma)\,dS_\sigma\,ds,$$

where

$$J_0 := \frac{n-1}{2}\int_\Omega u_1(x)u_0(x)\,dx + \int_\Omega u_1(x)\big(x\cdot\nabla u_0(x)\big)\,dx,$$

and $\nu(\sigma)$ is the unit outward normal vector at each $\sigma \in \partial\Omega$.

Proof. Outline of proof Since we multiply both sides of (1.1) by $m(u) = tu_t + x\cdot\nabla u + \frac{n-1}{2}u$ in order to get the desired identity, it suffices to notice the following five identities.

$$\int_\Omega(-\Delta u)(x\cdot\nabla u)\,dx$$
$$= -\frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial u}{\partial\nu}\right)^2\sigma\cdot\nu(\sigma)\,dS_\sigma - \frac{n-2}{2}\int_\Omega|\nabla u|^2\,dx, \tag{2.1}$$

where we used the boundary condition (1.3) to derive (2.1) (cf. [12]). Furthermore, one has

$$\int_\Omega u_{tt}(x\cdot\nabla u)\,dx = \frac{d}{dt}\int_\Omega u_t(x\cdot\nabla u)\,dx + \frac{n}{2}\int_\Omega|u_t|^2\,dx, \tag{2.2}$$

and

$$\int_\Omega V(x)u(x\cdot\nabla u)\,dx = \frac{1}{2}\int_\Omega V(x)\big(x\cdot\nabla|u|^2\big)\,dx$$
$$= \frac{1}{2}\int_\Omega\nabla\cdot\big(V(x)|u|^2x\big)\,dx - \frac{n}{2}\int_\Omega V(x)|u|^2\,dx - \frac{1}{2}\int_\Omega|u|^2\big(x\cdot\nabla V(x)\big)\,dx$$
$$= -\frac{n}{2}\int_\Omega V(x)|u|^2\,dx - \frac{1}{2}\int_\Omega|u|^2\big(x\cdot\nabla V(x)\big)\,dx, \tag{2.3}$$

where the divergence formula (see Remark 3.1) and boundary condition (1.3) were used. Finally, the following identities hold:

$$\frac{d}{dt}\big(tE(t)\big) - E(t) = 0, \tag{2.4}$$

and

$$\frac{n-1}{2}\frac{d}{dt}\int_\Omega u_t u\,dx - \frac{n-1}{2}\int_\Omega|u_t|^2\,dx + \frac{n-1}{2}\int_\Omega|\nabla u|^2\,dx + \frac{n-1}{2}\int_\Omega V(x)|u|^2\,dx = 0. \tag{2.5}$$

By summing the five identities in (2.1)–(2.5) integrated over $[0, t]$, the desired identity can be derived. □

We also need the weighted energy estimate below, which is a modified version of an estimate introduced originally by Todorova-Yordanov [27](see also the Appendix in [16]). For this, we will use the following notation for the pointwise total energy and the weight function, respectively:

$$e(t,x) := \frac{1}{2}\left(\left|u_t(t,x)\right|^2 + \left|\nabla u(t,x)\right|^2 + V(x)\left|u(t,x)\right|^2\right), \quad t > 0, x \in \Omega,$$

and $\psi \in C^1([0,\infty) \times \bar{\Omega})$ satisfying $\psi_t(t,x) \neq 0$ for all $(t,x) \in [0,\infty) \times \bar{\Omega}$.

Lemma 2.2. *Let $n \geqslant 2$ and $[u_0, u_1] \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$. Then, the corresponding smooth solution $u(t,x)$ to problem (1.1)–(1.3) satisfies the following identity:*

$$\begin{aligned}0 &= \frac{\partial}{\partial t}\left(\psi(t,x)e(t,x)\right) - \nabla\cdot\left(\psi(t,x)u_t(t,x)\nabla u(t,x)\right) - \frac{V(x)}{2}\left|u(t,x)\right|^2\psi_t(t,x)\\ &\quad - \frac{1}{2\psi_t(t,x)}\left|\psi_t(t,x)\nabla u(t,x) - u_t(t,x)\nabla\psi(t,x)\right|^2 + \frac{|u_t(t,x)|^2}{2\psi_t(t,x)}\left(\left|\nabla\psi(t,x)\right|^2 - \psi_t(t,x)^2\right),\\ &\quad t > 0, x \in \Omega.\end{aligned}$$

Proof. We first note that the solution $u(t,x)$ is sufficiently smooth, and the following identity holds:

$$\nabla\cdot(\psi u_t \nabla u) = u_t\nabla\psi\cdot\nabla u + \psi\nabla u_t\cdot\nabla u + \psi u_t\Delta u.$$

Since $\Delta u = u_{tt} + V(x)u$, it follows that

$$\nabla\cdot(\psi u_t\nabla u) = \frac{1}{2\psi_t}2\psi_t u_t\nabla\psi\cdot\nabla u + \frac{\partial}{\partial t}\left(\psi(t,x)e(t,x)\right) - \frac{\psi_t}{2}\left(|\nabla u|^2 + |u_t|^2 + V(x)|u|^2\right).$$

Here, the following identity is crucial:

$$2\psi_t u_t\nabla\psi\cdot\nabla u = -|\psi_t\nabla u - u_t\nabla\psi|^2 + \psi_t^2|\nabla u|^2 + |u_t|^2|\nabla\psi|^2.$$

By substitution and cancellation, it follows that

$$\nabla\cdot(\psi u_t\nabla u) = \frac{\partial}{\partial t}\left(\psi(t,x)e(t,x)\right) - \frac{1}{2\psi_t}|\psi_t\nabla u - u_t\nabla\psi|^2 + \frac{|u_t|^2}{2\psi_t}\left(|\nabla\psi|^2 - |\psi_t|^2\right) - \frac{\psi_t}{2}V(x)|u|^2.$$

This implies the desired identity. □

To prove the following L^2-estimate of the solution, one can use a method similar to one introduced in [15] (see also [16, Lemma 2.2]). Since the proof relies on the Hardy inequality in the exterior domains for $n \geqslant 2$, the weight function $d_n(x)$ appears in the statement (see [10]).

Lemma 2.3. *Let $n \geqslant 2$, and $[u_0, u_1] \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$. Then, the corresponding smooth solution $u(t,x)$ to problem (1.1)–(1.3) satisfies the following estimate:*

$$\|u(t,\cdot)\| \leqslant C\bigl(\|u_0\| + \|d_n(\cdot)u_1\|\bigr), \quad t \geqslant 0.$$

Proof. Note that the function $v(t,x) := \int_0^t u(s,x)\,ds$ is the solution of the problem

$$v_{tt} - \Delta v + V(x)v = u_1, \quad t > 0, x \in \Omega,$$
$$v(0) = 0, \qquad v_t(0) = u_0.$$

Using the multiplier v_t, for $\varepsilon > 0$, we obtain

$$\|v_t\|^2 + \|\nabla v\|^2 + \bigl\|\sqrt{V(\cdot)}v\bigr\|^2 = \|u_0\|^2 + 2\int_\Omega u_1(x)v(t,x)\,dx,$$
$$\leqslant \|u_0\|^2 + \frac{1}{2\varepsilon}\bigl\|d_n(\cdot)u_1\bigr\|^2 + \frac{\varepsilon}{2}\int_\Omega \frac{v^2(t,x)}{d_n^2(x)}\,dx, \quad t > 0.$$

Applying the Hardy inequality for dimension $n \geqslant 2$ and choosing a suitable $\varepsilon > 0$, the proof of the lemma follows from $u = v_t$. □

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by using Lemmas 2.1, 2.2, and 2.3.

We first use assumptions **(A-1)** and **(A-2)** and Lemma 2.1 to get the inequality

$$tE(t) + \frac{n-1}{2}\int_\Omega u_t(t,x)u(t,x)\,dx + \int_\Omega u_t(t,x)\bigl(x \cdot \nabla u(t,x)\bigr)\,dx \leqslant J_0, \tag{3.1}$$

where we made use of the fact that assumption **(A-1)** implies $\sigma \cdot \nu(\sigma) \leqslant 0$ for each $\sigma \in \partial\Omega$. Thus, it suffices to estimate two quantities included in (3.1):

$$I_1(t) := \left|\int_\Omega u_t(t,x)u(t,x)\,dx\right|, \tag{3.2}$$

$$I_2(t) := \left|\int_\Omega u_t(t,x)\bigl(x \cdot \nabla u(t,x)\bigr)\,dx\right|. \tag{3.3}$$

$I_1(t)$ can be estimated by applying Lemma 2.3, and $I_2(t)$ can be evaluated with Lemma 2.2.

(I) *Finding a bound for $I_1(t)$.*

From the Schwarz inequality, we can obtain

$$I_1(t) \leqslant \int_\Omega \bigl|u(t,x)\bigr|\bigl|u_t(t,x)\bigr|\,dx \leqslant \bigl\|u_t(t,\cdot)\bigr\|\bigl\|u(t,\cdot)\bigr\|.$$

Then, from (1.5), we see that

$$\frac{1}{2}\left\|u_t(t,\cdot)\right\|^2 \leqslant E(t) = E(0),$$

so that

$$\left\|u_t(t,\cdot)\right\| \leqslant \sqrt{2E(0)}.$$

Thus, combining the last relation with Lemma 2.3, we have

$$I_1(t) \leqslant C\sqrt{2E(0)}\big(\|u_0\| + \left\|d_n(\cdot)u_1\right\|\big) \quad t \geqslant 0. \tag{3.4}$$

(II) *Finding a bound for $I_2(t)$.*

For this purpose, we define a weight function $\psi(t,x)$ like one introduced in [16]:

$$\psi(t,x) = \begin{cases} (1+|x|-t) & |x| \geqslant t, x \in \mathbf{R}^n, \\ (1+t-|x|)^{-1} & |x| < t, x \in \mathbf{R}^n. \end{cases}$$

Then, it is easy to check that $\psi \in C^1([0,\infty)\times\mathbf{R}^n)$ satisfies

$$\psi_t(t,x) < 0, \quad t > 0, x \in \mathbf{R}^n, \tag{3.5}$$

$$\psi_t(t,x)^2 - \left|\nabla\psi(t,x)\right|^2 = 0, \quad t > 0, x \in \mathbf{R}^n. \tag{3.6}$$

Note that (3.6) is the so-called Eikonal equation for (1.1). Therefore, it follows from Lemma 2.2, $V(x) \geqslant 0$, (3.5), and (3.6) that

$$0 \geqslant \frac{\partial}{\partial t}\big(\psi(t,x)e(t,x)\big) - \nabla\cdot\big(\psi(t,x)u_t(t,x)\nabla u(t,x)\big), \quad t > 0, x \in \Omega.$$

By integrating both sides of the relation above over $[0,t]\times\Omega$ and using the divergence theorem and (1.3), we obtain a weighted energy estimate such that

$$\int_\Omega \psi(t,x)\big(\left|u_t(t,x)\right|^2 + \left|\nabla u(t,x)\right|^2 + V(x)\left|u(t,x)\right|^2\big)\,dx \\ \leqslant \int_\Omega \big(1+|x|\big)\big(\left|u_1(x)\right|^2 + \left|\nabla u_0(x)\right|^2 + V(x)\left|u_0(x)\right|^2\big)\,dx. \tag{3.7}$$

Now, let us estimate $I_2(t)$ based on (3.7). This is just a modification of [16, Lemma 2.4]. First, let $R > \rho_0$ be an arbitrary fixed number. Set $\Omega_R := \Omega \cap B_R$. Then, for $t > R$, it follows that

$$I_2(t) \leqslant \int_\Omega |x|\left|u_t(t,x)\right|\left|\nabla u(t,x)\right|\,dx$$

$$\leqslant R\int_{\Omega_R}|u_t(t,x)||\nabla u(t,x)|\,dx+\int_{|x|\geqslant R}|x||u_t(t,x)||\nabla u(t,x)|\,dx$$
$$\leqslant \frac{R}{2}\int_{\Omega_R}\left(|u_t(t,x)|^2+|\nabla u(t,x)|^2\right)dx+\int_{|x|\geqslant R}|x||u_t(t,x)||\nabla u(t,x)|\,dx$$
$$\leqslant \frac{R}{2}\int_{\Omega_R}\left(|u_t(t,x)|^2+|\nabla u(t,x)|^2+V(x)|u(t,x)|^2\right)dx$$
$$+\int_{|x|\geqslant R}|x||u_t(t,x)||\nabla u(t,x)|\,dx. \tag{3.8}$$

Let us estimate the last term of (3.8). We can write

$$\int_{|x|\geqslant R}|x||u_t(t,x)||\nabla u(t,x)|\,dx$$
$$\leqslant \int_{|x|\geqslant t}|x||u_t(t,x)||\nabla u(t,x)|\,dx+\int_{t\geqslant|x|\geqslant R}|x||u_t(t,x)||\nabla u(t,x)|\,dx$$
$$\leqslant \int_{|x|\geqslant t}\left(|x|-t\right)|u_t(t,x)||\nabla u(t,x)|\,dx+t\int_{|x|\geqslant t}|u_t(t,x)||\nabla u(t,x)|\,dx$$
$$+t\int_{t\geqslant|x|\geqslant R}|u_t(t,x)||\nabla u(t,x)|\,dx$$
$$\leqslant \frac{1}{2}\int_{|x|\geqslant t}\left(1+|x|-t\right)\left(|u_t(t,x)|^2+|\nabla u(t,x)|^2\right)dx$$
$$+\frac{t}{2}\int_{|x|\geqslant t}\left(|u_t(t,x)|^2+|\nabla u(t,x)|^2\right)dx+\frac{t}{2}\int_{t\geqslant|x|\geqslant R}\left(|u_t(t,x)|^2+|\nabla u(t,x)|^2\right)dx$$
$$\leqslant \frac{1}{2}\int_{|x|\geqslant t}\left(1+|x|-t\right)\left(|u_t(t,x)|^2+|\nabla u(t,x)|^2+V(x)|u(t,x)|^2\right)dx$$
$$+\frac{t}{2}\int_{|x|\geqslant R}\left(|u_t(t,x)|^2+|\nabla u(t,x)|^2+V(x)|u(t,x)|^2\right)dx. \tag{3.9}$$

Thus, it follows from (3.8) and (3.9) that

$$I_2(t)\leqslant \frac{R}{2}\int_{\Omega_R}\left(|u_t(t,x)|^2+|\nabla u(t,x)|^2+V(x)|u(t,x)|^2\right)dx$$
$$+\frac{1}{2}\int_{|x|\geqslant t}\left(1+|x|-t\right)\left(|u_t(t,x)|^2+|\nabla u(t,x)|^2+V(x)|u(t,x)|^2\right)dx$$
$$+\frac{t}{2}\int_{|x|\geqslant R}\left(|u_t(t,x)|^2+|\nabla u(t,x)|^2+V(x)|u(t,x)|^2\right)dx$$
$$\leqslant \frac{R}{2}\int_{\Omega_R}\left(|u_t(t,x)|^2+|\nabla u(t,x)|^2+V(x)|u(t,x)|^2\right)dx$$

$$+\int_{|x|\geqslant t}\psi(t,x)e(t,x)\,dx+t\int_{|x|\geqslant R}e(t,x)\,dx,$$

which implies

$$\begin{aligned}I_2(t)&\leqslant RE_R(t)+\int_{\Omega}\psi(t,x)e(t,x)\,dx+t\int_{|x|\geqslant R}e(t,x)\,dx\\&\leqslant RE_R(t)+\int_{\Omega}\big(1+|x|\big)e(0,x)\,dx+t\int_{|x|\geqslant R}e(t,x)\,dx\end{aligned}\tag{3.10}$$

because of (3.7), where

$$\int_{\Omega}\big(1+|x|\big)e(0,x)\,dx=\int_{\Omega}\big(1+|x|\big)\big(\big|u_1(x)\big|^2+\big|\nabla u_0(x)\big|^2+V(x)\big|u_0(x)\big|^2\big)\,dx=:I_0.\tag{3.11}$$

Remark 3.1. In deriving (3.7), we only used the divergence formula in the unbounded domain Ω. This can be justified by noticing the fact that the solution $u(t,\cdot)$ is sufficiently smooth and supp $u(t,\cdot)$ is compact in $\bar{\Omega}$ for each fixed $t>0$. This is due to the finite propagation property of the wave equations since the initial data have compact support $u_j\in C_0^\infty(\Omega)$ $(j=0,1)$. Thus, there exists a large constant $L_t>0$ for each $t>0$ such that $u(t,x)=0$ for $|x|>L_t$. Consequently, one can apply the divergence formula in the bounded region $\bar{\Omega}\cap\bar{B}_{L_t}$ for each $t>0$ in order to derive (3.7). In this paper, we use this concept without specific mention.

Let us prove Theorem 1.1 at a stroke.

Proof of Theorem 1.1. Let $R>\rho_0$ be an arbitrary fixed number, and take $t>R$. From (3.1), we immediately obtain the following equality:

$$tE_R(t)+t\int_{|x|\geqslant R}e(t,x)\,dx\leqslant\frac{n-1}{2}I_1(t)+I_2(t)+J_0.\tag{3.12}$$

Because of (3.4), (3.10), and (3.12), we know

$$\begin{aligned}&tE_R(t)+t\int_{|x|\geqslant R}e(t,x)\,dx\\&\quad\leqslant J_0+C\frac{n-1}{2}\sqrt{E(0)}\big(\|u_0\|+\big\|d_n(\cdot)u_1\big\|\big)\\&\qquad+RE_R(t)+t\int_{|x|\geqslant R}e(t,x)\,dx+\frac{1}{2}\int_{\Omega}\big(1+|x|\big)\big(\big|u_1(x)\big|^2+\big|\nabla u_0(x)\big|^2+V(x)\big|u_0(x)\big|^2\big)\,dx,\end{aligned}$$

which implies the desired decay estimate for the local energy:

$$(t-R)E_R(t)\leqslant J_0+C\sqrt{E(0)}\big(\|u_0\|+\big\|d_n(\cdot)u_1\big\|\big)+\frac{I_0}{2}.$$

Note that the part related to the exterior energy $t\int_{|x|\geqslant R}e(t,x)\,dx$ cancels nicely in the computations above. This means one cannot have any information on decay in time to the exterior energy. □

Remark 3.2. If one uses the generalized assumption (1.4) in place of assumption (A-2), from the proof above, the following additional quantity must be estimated in our method:

$$\frac{\gamma}{2}\int_0^\infty \int_\Omega V(x)\big|u(s,x)\big|^2\,dx\,ds < +\infty.$$

Such an estimate may be extremely difficult, which is not the goal of this work.

4. Concluding remarks

In this section, we observe the energy concentration phenomenon as a consequence of the local energy decay. In this connection, it is important not to use the finite speed of the propagation property in the solution.

Let $t > R > \rho_0$. From (3.7) and (3.11), we have

$$\int_{|x|\geqslant t}\big(1+|x|-t\big)e(t,x)\,dx \leqslant \int_\Omega \big(1+|x|\big)e(0,x)\,dx = I_0.$$

Then, for any fixed small $\varepsilon > 0$, it follows that

$$\int_{|x|\geqslant(1+\varepsilon)t}\big(1+|x|-t\big)e(t,x)\,dx \leqslant I_0,$$

so that

$$\int_{|x|\geqslant(1+\varepsilon)t} e(t,x)\,dx \leqslant \frac{I_0}{1+\varepsilon t}.$$

This implies

$$\int_{|x|\geqslant(1+\varepsilon)t} e(t,x)\,dx = O\big(t^{-1}\big)\quad (t\to\infty)$$

for any $\varepsilon > 0$. While one knows the energy conservation identity such that $E(t) = E(0)$. Thus, the following decomposition of the total energy can be performed:

$$\int_{|x|\geqslant(1+\varepsilon)t} e(t,x)\,dx + \int_{R\leqslant|x|\leqslant(1+\varepsilon)t} e(t,x)\,dx + E_R(t) = E(0).$$

Therefore, one can observe the energy concentration phenomenon such that

$$\int_{R\leqslant|x|\leqslant(1+\varepsilon)t} e(t,x)\,dx = E(0) + O\big(t^{-1}\big)\quad (t\gg 1) \tag{4.1}$$

by using Theorem 1.2. We observe that (4.1) may express a typical wave property from the viewpoint of the energy propagation under the non-compact support condition on the initial data, that is, as time goes to infinity, almost all of the energy is concentrated in the region $\{x\in\Omega : R\leqslant|x|\leqslant(1+\varepsilon)t\}$ with a small ε-loss.

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Appendix

In this appendix, we give an outline of the proof of Theorem 1.2. For this purpose, we define a weight function as follows:

$$w_n(x) = \begin{cases} d_n(x) & n \geqslant 3, \\ \sqrt{|x|} & n = 2. \end{cases}$$

First of all, the initial data $[u_0, u_1] \in W_0^{1,2}(\Omega, w) \times L^2(\Omega)$ with $\int_\Omega w_n(x)^2 |u_1(x)|^2\,dx < +\infty$ can be approximated by smooth functions $[\phi_k, \psi_k] \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$ $(k = 1, 2, 3, \dots)$ such that

$$\|\phi_k - u_0\|_{W_0^{1,2}(\Omega,w)} \to 0 \quad (k \to \infty),$$
$$\int_\Omega \big(1 + w_n(x)^2\big)\big|\psi_k(x) - u_1(x)\big|^2\,dx \to 0 \quad (k \to \infty).$$

For each $k \in \mathbf{N}$, we consider the Cauchy problem

$$u_{tt}^{(k)}(t,x) - \Delta u^{(k)}(t,x) + V(x)u^{(k)}(t,x) = 0, \quad (t,x) \in (0,\infty) \times \Omega, \tag{A.1}$$
$$u^{(k)}(0,x) = \phi_k(x), \qquad u_t^{(k)}(0,x) = \psi_k(x), \quad x \in \Omega, \tag{A.2}$$
$$u^{(k)}(t,x) = 0, \quad x \in \partial\Omega, t > 0. \tag{A.3}$$

Then, it follows from Theorem 1.1 that for each $k \in \boldsymbol{N}$, the problem (A.1)–(A.3) admits a unique smooth solution $u^{(k)}(t,x)$ with compact support for each $t \geqslant 0$ satisfying

$$\int_{B_R \cap \Omega} \big(\big|u_t^{(k)}(t,x)\big|^2 + \big|\nabla u^{(k)}(t,x)\big|^2 + V(x)\big|u^{(k)}(t,x)\big|^2\big)\,dx \leqslant \frac{CK_{0,k}}{t-R}, \quad (t > R), \tag{A.4}$$

where $C > 0$ is independent of k, and

$$K_{0,k} := \int_\Omega \phi_k(x)\psi_k(x)\,dx + \int_\Omega \big(\psi_k(x)\big(x \cdot \nabla\phi_k(x)\big)\big)\,dx + \sqrt{E_k(0)}\big(\|\phi_k\| + \big\|d_n(\cdot)\psi_k\big\|\big)$$
$$+ \int_\Omega \big(1 + |x|\big)\big(\big|\psi_k(x)\big|^2 + \big|\nabla\phi_k(x)\big|^2 + V(x)\big|\phi_k(x)\big|^2\big)\,dx,$$
$$E_k(0) := \frac{1}{2}\int_\Omega \big(\big|\psi_k(x)\big|^2 + \big|\nabla\phi_k(x)\big|^2 + V(x)\big|\phi_k(x)\big|^2\big)\,dx.$$

Note that $K_{0,k} \to K_0$ as $k \to \infty$ (see also Remark 1.6). Furthermore, between the weak solution $u(t, x)$ and the approximate solution $u^{(k)}(t, x)$, it holds that

$$\sup_{t\in[0,\infty)} \left(\left\|u_t^{(k)}(t,\cdot) - u_t(t,\cdot)\right\| + \left\|\nabla u^{(k)}(t,\cdot) - \nabla u(t,\cdot)\right\| + \left\|\sqrt{V(\cdot)}\left(u^{(k)}(t,\cdot) - u(t,\cdot)\right)\right\|\right) \to 0$$

$$(k \to \infty),$$

$$\sup_{t\in[0,T]} \left\|u^{(k)}(t,\cdot) - u(t,\cdot)\right\| \to 0 \quad (k \to \infty),$$

for each $T > 0$. Thus, by letting $k \to \infty$ in (A.4), one obtains the desired estimate of Theorem 1.2.

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