High contrast homogenisation in nonlinear elasticity under small loads

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Abstract. We study the homogenisation of *geometrically nonlinear elastic* composites with high contrast. The composites we analyse consist of a perforated matrix material, which we call the "stiff" material, and a "soft" material that fills the remaining pores. We assume that the pores are of size $0 < \varepsilon \ll 1$ and are periodically distributed with period ε. We also assume that the stiffness of the soft material degenerates with rate $\varepsilon^{2\gamma}$, $\gamma > 0$, so that the contrast between the two materials becomes infinite as $\varepsilon \downarrow 0$. We study the homogenisation limit $\varepsilon \downarrow 0$ in a low energy regime, where the displacement of the stiff component is infinitesimally small. We derive an effective two-scale model, which, depending on the scaling of the energy, is either a quadratic functional or a partially quadratic functional that still allows for large strains in the soft inclusions. In the latter case, averaging out the small scale-term justifies a single-scale model for high-contrast materials, which features a non-linear and non-monotone effect describing a coupling between microscopic and the effective macroscopic displacements.

Keywords: High-contrast homogenisation, nonlinear elasticity, two-scale Γ-convergence

1. Introduction

We consider a geometrically nonlinear elastic composite material that consists of a "stiff" matrix material and periodically distributed pores filled by a "soft" material: for $\varepsilon > 0$ and a fixed scaling parameter $\gamma > 0$ we consider the energy functional of non-linear elasticity

$$\mathcal{I}_\varepsilon(u) := \int_\Omega \left(\varepsilon^{2\gamma} W^0(\nabla u) \chi_\varepsilon + W^1(\nabla u)(1 - \chi_\varepsilon) \right) dx - \int_\Omega f_\varepsilon \cdot u \, dx, \quad u \in H^1(\Omega). \qquad (1)$$

Here Ω denotes a Lipschitz domain in $\mathbb{R}^d$ (the reference domain of the elastic body) and $u : \Omega \to \mathbb{R}^d$ is a deformation satisfying clamped boundary conditions: $u(x) = x$ on $\partial\Omega$. We denote by $f_\varepsilon : \Omega \to \mathbb{R}^d$ the density of the applied body forces, W^0 and W^1 are frame-indifferent, non-degenerate energy densities (see Section 2 below for the precise assumptions), and χ_ε denotes the indicator function of the pores, *i.e.* of the domain occupied by the "soft" material component. As will be made precise in Section 2, we assume that the pores are of size ε and are periodically distributed in the interior of Ω with period ε. As can be seen from (1), in the homogenisation limit $\varepsilon \downarrow 0$ the stiffness of the "soft" material degenerates

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with rate $\varepsilon^{2\gamma}$ ($\gamma > 0$), while the stiffness of the "stiff" material remains unchanged. Hence, the contrast between the soft material (occupying the pores) and the stiff material (occupying the perforated matrix) becomes infinite in the limit $\varepsilon \downarrow 0$. We therefore refer to the corresponding limit procedure as *high-contrast homogenisation.* Our goal is to identify the effective behaviour of the minimisation problem associated with $\mathcal{I}_\varepsilon$ by studying its limit under a proper rescaling.

Summary and discussion of our result.　　To illustrate our result, here in the introduction we restrict ourselves to the special case $\gamma = 1$. If we assume that the density of the body forces is small in magnitude, in the sense that $f_\varepsilon = \varepsilon^\alpha f$ for some $\alpha \geqslant 1$, and has vanishing first moment, *i.e.* $\int_\Omega f(x) \cdot x \, dx = 0$, then (1) can be expressed as

$$
\begin{aligned}
I_\varepsilon^\alpha(\varphi) &:= \frac{1}{\varepsilon^{2\alpha}} \mathcal{I}_\varepsilon(u) \\
&= \int_\Omega \left(\frac{1}{\varepsilon^{2(\alpha-1)}} W^0 \big(I + \varepsilon^\alpha \nabla \varphi \big) \chi_\varepsilon + \frac{1}{\varepsilon^{2\alpha}} W^1 \big(I + \varepsilon^\alpha \nabla \varphi \big)(1 - \chi_\varepsilon) \right) dx - \int_\Omega f \cdot \varphi \, dx,
\end{aligned}
\tag{2}
$$

where

$$
\varphi(x) = \frac{u(x) - x}{\varepsilon^\alpha}, \quad x \in \Omega, \varphi \in H_0^1\big(\Omega; \mathbb{R}^d\big),
\tag{3}
$$

denotes the scaled displacement, and I stands for the identity matrix in $\mathbb{R}^{d \times d}$. In this paper we analyse the asymptotics of the minimisation problem associated with I_ε^α in the limit $\varepsilon \downarrow 0$ by appealing to the concept of Γ-convergence. The latter goes back to De Giorgi (e.g. see [20] for a standard reference). In a metric setting it is defined as follows:

Definition 1 (Γ-convergence). Let (X, d) denote a metric space. A sequence of functionals $I_\varepsilon : X \to [-\infty, \infty]$ Γ-converges to a functional $I : X \to [-\infty, \infty]$, if

　(a) (lower bound). For every $x_0 \in X$ and every $x_\varepsilon \to x_0$ in X we have $\liminf_{\varepsilon \to 0} I_\varepsilon(x_\varepsilon) \geqslant I_0(x_0)$.
　(b) (recovery sequence). For every $x_0 \in X$ there exists a sequence $x_\varepsilon \to x_0$ in X such that $\lim_{\varepsilon \to 0} I_\varepsilon(x_\varepsilon) = I_0(x_0)$.

In that case we call I_0 the Γ-limit of the sequence I_ε.

　A fundamental property of Γ-convergence is the following fact: If a sequence of functionals Γ-converges and the functionals are *equicoercive*, then the associated sequence of minima (resp. minimisers) converge (up to a subsequence) to the minimum (resp. a minimiser) of the Γ-limit, and any minimiser of the Γ-limit can be obtained as a limit of a minimizing sequence of the original functionals. Thanks to this property, Γ-convergence is especially useful for the study of the asymptotics of parametrised minimisation problems. The Γ-limit, if it exists, is unique; yet, the question whether a sequence of functionals Γ-converges or not, and the form of the Γ-limit depend on the topology of X. In particular, a sequence of functionals is more likely to Γ-converge in a stronger topology, while it is more likely to be equicoercive in a weaker topology. Therefore, it is natural to consider the strongest notion of convergence on X for which the functionals remain equicoercive. In the situation we are interested in, namely the asymptotics of the functionals I_ε^α defined on $X = H_0^1(\Omega)$, it turns out that due to the presence of high-contrast, a natural and appropriate notion of convergence on $H_0^1(\Omega)$ is a variant of two-scale convergence that we discuss next. Let us first recall the standard notion of two-scale convergence from [32] and [1]:

Definition 2. We say that a sequence $f_\varepsilon \in L^2(\Omega)$ weakly two-scale converges to $f \in L^2(\Omega \times Y)$ if the sequence f_ε is bounded and

$$\lim_{\varepsilon \downarrow 0} \int_\Omega f_\varepsilon(x)\varphi(x, x/\varepsilon)\, dx = \iint_{\Omega \times Y} f(x, y)\varphi(x, y)\, dx\, dy$$

for all $\varphi \in L^2(\Omega, C_\#(Y))$, where $C_\#(Y)$ denotes the Banach space of continuous, $Y := (0, 1)^d$-periodic functions on $\mathbb{R}^d$. We say that a sequence $f_\varepsilon \in L^2(\Omega)$ strongly two-scale converges to $f \in L^2(\Omega \times Y)$ if the sequence f_ε weakly two-scale converges to f and one has $\|f_\varepsilon\|_{L^2(\Omega)} \to \|f\|_{L^2(\Omega \times Y)}$ as $\varepsilon \downarrow 0$. For vector-valued functions two-scale convergence is defined component-wise.

In the study of the asymptotics of I_ε^α we work with a variant of this definition that is tailor-made to capture the effects of high-contrast. For $\gamma = 1$ it can be summarised as follows (for details and the general case see Section 3.1): Given a sequence of displacements φ_ε in $H_0^1(\Omega)$ we consider the unique decomposition $\varphi_\varepsilon = g_\varepsilon^0 + g_\varepsilon^1$ into a function $g_\varepsilon^0 \in H_0^1(\Omega_\varepsilon^0)$ and a contribution $g_\varepsilon^1 \in H_0^1(\Omega)$ that is harmonic in Ω_ε^0. We then write $\varphi_\varepsilon \overset{2}{\rightharpoonup} (g^0, g^1)$ and say that φ_ε converges to a pair (g^0, g^1) with $g^0 \in L^2(\Omega; H_0^1(Y_0))$ and $g^1 \in H_0^1(\Omega)$, if $g_\varepsilon^1 \rightharpoonup g^1$ weakly in $H^1(\Omega)$, and $g_\varepsilon^0 \overset{2}{\rightharpoonup} g^0$, $\varepsilon\nabla g_\varepsilon^0 \overset{2}{\rightharpoonup} \nabla_y g^0$ weakly two-scale. The component g^1 of a limit pair (g^0, g^1) describes the (scaled) macroscopic displacement of the body, and g^0 is a two-scale function describing the (scaled) microscopic displacement on the pores relative to the deformed matrix material. As a main result (see Theorem 1 and Theorem 3) we prove (in fact in a slightly more general situation) that I_ε^α Γ-converges with respect to the type of convergence introduced above. It turns out that two different regimes emerge for $\alpha > 1$ and $\alpha = 1$.

In the **small strain regime** ($\alpha > 1$), the strain $\varepsilon^\alpha \nabla\varphi$ becomes infinitesimally small in the entire domain Ω, and the limit behaviour is expressed by a linearised, two-scale energy $I_{\text{small}} : L^2(\Omega; H_0^1(Y_0)) \times H_0^1(\Omega; \mathbb{R}^d) \to \mathbb{R}$,

$$I_{\text{small}}(g^0, g^1) := \int_{\Omega \times Y} Q^0(\nabla_y g^0(x, y))\, dy + Q_{\text{hom}}^1(\nabla g^1(x))\, dx \tag{4}$$

$$- \int_\Omega \left(\int_{Y_0} g^0(x, y)\, dy + g^1(x) \right) \cdot f(x)\, dx. \tag{5}$$

Here Q^0 and Q^1 are the quadratic forms of the quadratic expansions of W^0 and W^1 at the identity, and Q_{hom}^1 denotes the homogenised energy density obtained from Q^1, see (12) and (23) for details. The functional I_{small} is the *two-scale Γ-limit* of the sequence I_ε^α in the sense that (cf. Theorem 1):

(a) (lower bound). For every (g^0, g^1) and every $\varphi_\varepsilon \overset{2}{\rightharpoonup} (g^0, g^1)$ we have $\liminf_{\varepsilon \downarrow 0} I_\varepsilon^\alpha(\varphi_\varepsilon) \geqslant I_{\text{small}}(g^0, g^1)$.

(b) (recovery sequence). For every (g^0, g^1) there exists a sequence $\varphi_\varepsilon \overset{2}{\rightharpoonup} (g^0, g^1)$ such that $\lim_{\varepsilon \downarrow 0} I_\varepsilon^\alpha(\varphi_\varepsilon) \geqslant I_{\text{small}}(g^0, g^1)$.

In addition, we prove that the functionals $(I_\varepsilon^\alpha)_{\varepsilon > 0}$ are equicoercive and deduce convergence of the associated minimisation problems (see Theorem 1 and Proposition 1). In Theorem 2 we establish a two-scale expansion showing that if φ^ε is an (almost) minimiser of I_ε^α, then

$$\varphi_\varepsilon(x) \approx g_*^1(x) + \varepsilon\psi(x, x/\varepsilon) + g_*^0(x, x/\varepsilon), \tag{6}$$

where (g_*^0, g_*^1) is a minimiser of I_{small}, and ψ denotes a corrector function that only depends on g_*^1. Finally, we illustrate that by averaging out the fast variable y the limit I_{small} can be further simplified. In fact, Proposition 2 shows that I_ε^α Γ-converges (with respect to weak convergence in $L^2(\Omega)$) to the functional $\bar{I}_{\text{small}} : L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$ given by

$$\bar{I}_{\text{small}}(\varphi) := \min\left\{\int_\Omega Q_{\text{hom}}^1(\nabla g^1) + \overline{Q}(\overline{G}) : g^1 \in H_0^1(\Omega), \overline{G} \in L^2(\Omega) \text{ with } g^1 + \overline{G} = \varphi\right\}$$
$$- \int_\Omega f \cdot \varphi \, dx, \tag{7}$$

with a (positive definite) quadratic form $\overline{Q} : \mathbb{R}^d \to [0, \infty)$ defined by

$$\overline{Q}(\overline{G}) := \min\left\{\int_{Y^0} Q^0(\nabla g^0(y)) \, dy : g^0 \in H_0^1(Y^0), \int_{Y^0} g^0 \, dy = \overline{G}\right\}.$$

$\overline{Q}$ captures the influence of the pores (and their geometry) on the effective behavior. The minimiser $\varphi_* \in L^2(\Omega)$ to $\bar{I}_{\text{small}}$ takes the form $\varphi_* = g_*^1 + \overline{G}_*$ with $\overline{G}_* := \int_{Y_0} g_*^0(\cdot, y) \, dy$. In view of (6) the field $\overline{G}_*$ can be interpreted as the gap between the macroscopic displacement and the microscopic displacements in the pores.

In the **finite strain regime**, which corresponds to $\alpha = 1$, the displacement gradient $\varepsilon \nabla \varphi_\varepsilon$ becomes infinitesimally small only in the stiff component, while large strains still may occur in the soft pores. Therefore, the Γ-limit is a non-convex (partially linearised) functional of the form

$$I_{\text{finite}}(g^0, g^1) := \int_{\Omega \times Y} \mathcal{Q}W^0(I + \nabla_y g^0(x, y)) \, dy + Q_{\text{hom}}^1(\nabla g^1(x)) \, dx$$
$$- \int_\Omega \left(\int_{Y_0} g^0(x, y) \, dy + g^1(x)\right) \cdot f(x) \, dx,$$

where $\mathcal{Q}W^0$ denotes the quasiconvex envelope of W^0. Similarly to the small strain regime, one can average out the fast scale y and obtain Γ-convergence of I_ε^0 to the functional $\bar{I}_{\text{finite}} : L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$ given by

$$\bar{I}_{\text{finite}}(\varphi) := \min\left\{\int_\Omega Q_{\text{hom}}^1(\nabla g^1) + \overline{V}(\overline{G}) : g^1 \in H_0^1(\Omega), \overline{G} \in L^2(\Omega) \text{ with } g^1 + \overline{G} = \varphi\right\} \tag{8}$$
$$- \int_\Omega f \cdot \varphi \, dx,$$

with non-convex potential $\overline{V} : \mathbb{R}^d \to [0, \infty)$ defined by

$$\overline{V}(\overline{G}) := \min\left\{\int_{Y^0} \mathcal{Q}W^0(I + \nabla g^0(y)) \, dy : g^0 \in H_0^1(Y^0), \int_{Y^0} g^0 \, dy = \overline{G}\right\},$$

see Remark 2. In contrast to the small strain regime, where $\overline{Q}$ is quadratic, the potential $\overline{V}$ is non-convex and expresses a *nonlinear (and non-monotone) coupling* between the macroscopic and microscopic displacement.

Connection to acoustic wave propagation in high-contrast materials. The sequence of functionals $\varepsilon^{-2\alpha}\mathcal{I}_\varepsilon$, in either of the two regimes described above, occupies an intermediate position between a fully nonlinearly elastic composite and fully linearised models, as $\varepsilon \to 0$. Notably, linear models with high contrast, which are suitable for the description of small displacement fields (that often occur, say, in acoustic wave propagation) already exhibit a coupling between the macroscopic part g^0 and microscopic part g^1 of the minimiser of I_{small}, which in our case is obtained as a limit in the small-strain regime $\alpha > 1$. This can be seen by considering the time-harmonic solitons to the equations of elastodynamics with the elastic part of the energy given by (4), away from the sources of the elastic motion. In this case the function f in (5) (which in our analysis we assume to be independent of the fast variable x/ε for simplicity, an assumption that can be relaxed with no changes in the proofs needed) has to be replaced by the sum $g^0 + g^1$, with the integration in (5) carried over Y_0 and Q at the same time, *i.e.* the work of the external forces (5) is replaced by the expression for the work of "self-forces"

$$-\omega^2 \int_\Omega \int_{Y_0} \big(g^0(x, y) + g^1(x)\big) \cdot \big(g^0(x, y) + g^1(x)\big) \, dy \, dx,$$

where ω is the frequency. The solution to the Euler–Lagrange equation for the resulting functional is a coupled system of equations for g^0, g^1, so that when the equation for g^0 is solved in terms of g^1 and substituted into the second equation, it takes the form (away from the sources):

$$\mathcal{A}^{\text{hom}}g^1 = \omega^2\beta\big(\omega^2\big)g^1, \tag{9}$$

for some non-negative self-adjoint differential operator $\mathcal{A}^{\text{hom}}$ and a special nonlinear function β, which takes positive and negative values on alternating intervals of the real axis (leading to "lacunae", or "band gaps" in the spectrum of the corresponding operator) and is obtained from the spectral decomposition of g^0 and the subsequent averaging over Y_0, see [38]. From this point of view, the non-quadratic finite-strain functional I_{finite} is a "matching", "partially quadratic", homogenised model corresponding, *e.g.*, to finite-amplitude, rather than small-amplitude, wave motions that can no longer be treated using a quadratic model such as I_{small} but can still be used in place of models of nonlinear elasticity where the elastic energy terms on both components of the composite (stiff and soft) are non-quadratic.

Methods and previous results. In this paper we appeal to analytic methods that have been developed in the last two decades in the areas of nonlinear elasticity and homogenisation. Among these are the notion of two-scale convergence introduced in [1,32] and periodic unfolding (see [18] and references therein). The convergence statements of our main results are expressed in the language of Γ-convergence (see [20] and references therein). In order to treat the geometric nonlinearity of the considered functional, we make use of the geometric rigidity estimate (see [23]). Since we consider a low energy regime, linearisation and homogenisation take place at the same time. The simultaneous treatment of both effects is inspired by recent works [24,28–31] of the third author, where various problems involving *simultaneous homogenisation, linearisation and dimension reduction* are studied. The homogenisation of the kind of high-contrast composites that we study is related to the homogenisation for periodically perforated domains (e.g. see [8,33]). For instance, we make use of extensions across the pores. As a side result we prove a version of the geometric rigidity estimate for perforated domains (see Lemma 4 below). We would like to remark that while the present work is one of the few papers, along with [9,16], that treat the fully nonlinear high-contrast case, during the last decade there has been a significant amount of literature devoted to the mathematical analysis of phenomena associated with, or modelled by, a high degree

of contrast between the properties of the materials constituting a composite, in the linearised setting. The first contributions in this direction are due to Zhikov [38], and Bouchitté and Felbacq [6], following an earlier paper by Allaire [1] and the collection of papers by Hornung [25] (see also the references therein), where the special role of high-contrast elliptic PDE was pointed out albeit not studied in detail. These works demonstrated that the behaviour of the field variable in such models is of a two-scale type in the homogenisation limit, *i.e.* the limit model cannot be reduced to a one-scale formulation and fields that depend on the fast variable remain in the effective model. They also noticed that the spectrum of such materials has a band-gap structure, as in (9), and indicated how this fact could be exploited for high-resolution imaging and cloaking. It has since been an adopted approach to the theoretical construction of "negative refraction" media, or more generally "metamaterials", which is now a hugely popular area of research in physics (see *e.g.* [34] and references therein). On the analytical side, a number of further works followed, in particular [3,4,7,10,11,13–15,17,26,35], where various consequences of high contrast (or, mathematically speaking, the property of non-uniform ellipticity) in the underlying equations have been explored. Among these are the "non-locality" and "micro-torsion" effects in materials with high-contrast inclusions in the shape of fibres extending in one or more directions, the "partial band-gap" wave propagation due to the high degree of anisotropy of one of the constituent media, and the localisation of energy in high-contrast media with a defect ("photonic crystal fibres"), all of which can be thought of as examples of "non-standard", or "non-classical", behaviour in composites, which is not available in the usual moderate-contrast materials. In the present paper we aim to develop further a rigorous high-contrast theory in the context of finite elasticity, where the underlying model is nonlinear.

With this paper we continue the multiscale theme initiated in [16], where the regime of large deformation gradients in the soft component of the composite was considered. Let us emphasise two points that contrast our contribution to some earlier work within the related field. First, we note that, apart from [9,16], a number of other articles (*e.g.* [5,7,12]) have treated high-contrast periodic composites in the nonlinear context. However, the related results are of limited relevance to nonlinear elasticity, due to the convexity or monotonicity assumptions made in these works. In the present paper we study a class of functionals subject to the requirement of material fame indifference (see assumption (W1) in Section 2), which makes our analysis fit the fully nonlinear elasticity framework, as opposed to the works mentioned. Second, as was discussed above, the analysis of composites with "soft" inclusions within a "stiff" matrix cannot be reduced to a "decoupled" model where the perforated medium obtained by removing the inclusions is considered first and the displacement within the inclusions is found independently, which from the physics perspective can be viewed as a kind of resonance phenomenon; *cf.* (9) in the linearisation regime, for which an inherent energy coupling, in the limit as $\varepsilon \to 0$, between the soft and stiff components of the composite is essential. On a related note, the proof of the key compactness statement (Lemma 1) involves the simultaneous analysis of the displacements on the two components. We would also like to highlight the fact that in [16] the order of the relative scaling of the displacements on the soft and stiff components of the composite are assumed from the outset, while in the present work it is the result of the above compactness argument itself.

Organisation of the paper.   In Section 2 we state the assumptions on the geometry of the composite and the material law. In Section 3 we present the main results, starting with results regarding two-scale compactness, convergence results in the small strain regime and finally the convergence result in the finite strain regime. All proofs are presented in Section 4.

1.1. Notation

Here we list some notation that we use throughout the text. Additional items will be introduced whenever they are first used in the text.

- $d \geqslant 2$ is the (integer) dimension of the space occupied by the material.
- $p \geqslant 1$ is the exponent in the notation L^p for a Lebesgue space.
- $Y := (0, 1)^d$ the reference period cell; Y^0 is an open Lipschitz set whose closure is contained in Y, and $Y^1 := Y \setminus \overline{Y^0}$.
- Ω, Ω_ε^0 and Ω_ε^1 denote the reference domains of the composite, the set occupied by the pore material, and the domain occupied by the matrix material, respectively, see Section 2 for precise definition.
- Unless stated otherwise, all function spaces $L^2(\Omega)$, $H^1(\Omega)$, $H_0^1(\Omega)$, *etc.* consist of functions taking values in $\mathbb{R}^d$.
- Function spaces whose notation contains subscript "c" consist of functions that vanish outside a compact set.
- The function spaces $H_\#^1$, $H_0^1(Y^0)$, and $\mathcal{A}(Y^0)$ are introduced in Section 3.1.
- We write $\cdot$ and $:$ for the canonical inner products in $\mathbb{R}^d$ and $\mathbb{R}^{d \times d}$, respectively.
- SO(d) denotes the set of rotations in $\mathbb{R}^{d \times d}$.
- $\lesssim$ stands for $\leqslant$ up to a multiplicative constant that only depends on d, Y^1, Ω, and on p if applicable.

2. Geometric and constitutive setup

The pore geometry. The set Y^0 defined above describes the "pores" contained within the cell Y. Note that Y^1 is an open, bounded, connected set with Lipschitz boundary. Therefore, to each $\varphi \in H^1(Y^1)$ we can associate (see *e.g.* [33]) a unique *harmonic* extension $g^1 \in H^1(Y)$ characterised by

$$g^1 = \varphi \quad \text{in } Y^1, \qquad \int_{Y^0} \nabla g^1 : \nabla \zeta \, dy = 0 \quad \forall \zeta \in H_0^1(Y^0). \tag{10}$$

For this extension the inequality

$$\left\| \nabla g^1 \right\|_{L^2(Y^0)} \leqslant C \| \nabla \varphi \|_{L^2(Y^1)} \tag{11}$$

holds with a constant C that only depends on Y^1.

For a given domain $\Omega \subset \mathbb{R}^d$ and $\varepsilon > 0$, we define the sets Ω_ε^0 and Ω_ε^1 as follows:

$$\Omega_\varepsilon^0 := \bigcup \left\{ \varepsilon \left(\xi + Y^0 \right) \mid \xi \in \mathbb{Z}^d, \varepsilon(\xi + Y) \subset \Omega \right\}, \qquad \Omega_\varepsilon^1 := \Omega \setminus \overline{\Omega_\varepsilon^0}.$$

Note that by construction Ω_ε^1 is a Lipschitz domain. In particular, it is connected and $\partial \Omega \subset \partial \Omega_\varepsilon^1$. We denote by χ_ε the indicator function of the set of pores:

$$\chi_\varepsilon(x) := \begin{cases} 1, & x \in \Omega_\varepsilon^0, \\ 0, & x \in \mathbb{R}^d \setminus \Omega_\varepsilon^0. \end{cases}$$

The composite.   The two materials are described by energy densities $W^i : \mathbb{R}^{d \times d} \to [0, +\infty]$, $i = 0, 1$. Unless stated otherwise, we assume that for $i = 0, 1$:

(W1) W^i is frame-indifferent, *i.e.* $W^i(RF) = W^i(F)$ for all $R \in SO(d)$ and all $F \in \mathbb{R}^{d \times d}$;

(W2) The identity matrix $I \in \mathbb{R}^{d \times d}$ is a "natural state", *i.e.* $W^i(I) = 0$, and W^i is non-degenerate:

$$W^i(F) \geqslant c_0 \operatorname{dist}^2\big(F, SO(d)\big), \quad \forall F \in \mathbb{R}^{d \times d}, c_0 > 0.$$

(W3) W^i has a quadratic expansion at I, *i.e.* there exists a non-negative quadratic form Q^i on $\mathbb{R}^{d \times d}$ and an increasing function $r^i : [0, \infty) \to [0, \infty]$ with $\lim_{s \downarrow 0} r^i(s) = 0$, such that

$$\big|W^i(I + G) - Q^i(G)\big| \leqslant |G|^2 r^i\big(|G|\big) \quad \forall G \in \mathbb{R}^{d \times d}. \tag{12}$$

As shown in [30, Lemma 2.7] the quadratic form Q^i associated with W^i via (W3) satisfies

$$c_1 |\operatorname{sym} G|^2 \leqslant Q^i(G) = Q^i(\operatorname{sym} G) \leqslant c_1^{-1} |\operatorname{sym} G|^2 \quad \forall G \in \mathbb{R}^{d \times d}, c_1 > 0. \tag{13}$$

In the finite strain regime we consider a different set of assumptions for W^0, which are listed in Section 3.3.

The scaling parameter γ.   Throughout the paper $\gamma > 0$ denotes a fixed scaling parameter. It is a quantitative measure of the relative contrast between the two components of the composite.

Energy functional.   We define the elastic energy as a functional of the displacement, as follows:

$$\mathcal{E}_\varepsilon(u) = \int_\Omega \big(\varepsilon^{2\gamma} W^0(I + \nabla u)\chi_\varepsilon + W^1(I + \nabla u)(1 - \chi_\varepsilon)\big)\, dx, \quad u \in H_0^1\big(\Omega, \mathbb{R}^d\big). \tag{14}$$

3. Main results

3.1. Compactness and two-scale convergence

We first present an *a priori* estimate and a two-scale compactness statement for sequences $\varphi_\varepsilon \in H_0^1(\Omega)$ whose energy is equi-bounded in the sense that

$$\limsup_{\varepsilon \downarrow 0} \Phi_\varepsilon^\gamma(v_\varepsilon) < \infty, \tag{15}$$

where

$$\Phi_\varepsilon^\gamma(v) := \int_\Omega \operatorname{dist}^2\big(I + \nabla v(x), SO(d)\big)\big(\varepsilon^{2\gamma} \chi_\varepsilon + (1 - \chi_\varepsilon)\big)\, dx.$$

Note that, by virtue of the non-degeneracy assumption (W2) the functional $\Phi_\varepsilon^\gamma(\cdot)$ bounds below $\mathcal{E}_\varepsilon(\mathrm{id} + \cdot)$, where $\mathrm{id}(x) = x$, $x \in \Omega$. As we shall see in the upcoming Lemma 1, the inequality (15) implies that the sequence φ_ε is bounded in $H^1(\Omega)$, and thus weakly converges (up to extracting a subsequence) to a limit displacement $\varphi \in H_0^1(\Omega)$. For our purpose we require a precise understanding of the oscillations that emerge along that limit. We achieve this by combining two concepts:

- We write a representation for φ_ε in the spirit of an asymptotic decomposition as $\varepsilon \downarrow 0$.
- We study the convergence properties of the terms in this decomposition by appealing to two-scale convergence.

In the following lemma we address the first item above.

Lemma 1. *Let $\varphi_\varepsilon \in H_0^1(\Omega)$ and $0 < \varepsilon \leqslant 1$.*

(a) *There exists a unique pair of functions $g_\varepsilon^0 \in H_0^1(\Omega_\varepsilon^0)$ and $g_\varepsilon^1 \in H_0^1(\Omega)$ such that*

$$\text{(i)} \quad \varphi_\varepsilon = g_\varepsilon^1 + \varepsilon^{1-\gamma} g_\varepsilon^0,$$

$$\text{(ii)} \quad \int_{\Omega_\varepsilon^0} \nabla g_\varepsilon^1 : \nabla \zeta = 0 \quad \forall \zeta \in H_0^1(\Omega_\varepsilon^0). \tag{16}$$

(b) *There exists a positive constant C that only depends on Ω, Y^0 such that*

$$\left\| g_\varepsilon^0 \right\|_{L^2(\Omega)}^2 + \left\| \varepsilon \nabla g_\varepsilon^0 \right\|_{L^2(\Omega)}^2 + \left\| g_\varepsilon^1 \right\|_{H^1(\Omega)}^2 \leqslant C \Phi_\varepsilon^\gamma(\varphi_\varepsilon), \tag{17}$$

where g_ε^0, g_ε^1 and φ_ε are related to each other as in (a).

As already explained in the introduction, for our purpose it is convenient to appeal to two-scale convergence, see Definition 2. We use the following shorthand notation:

$$f_\varepsilon \to f_0 \quad :\Leftrightarrow \quad f_\varepsilon \text{ strongly converges to } f_0 \text{ in } L^2(\Omega),$$

$$f_\varepsilon \rightharpoonup f_0 \quad :\Leftrightarrow \quad f_\varepsilon \text{ weakly converges to } f_0 \text{ in } L^2(\Omega),$$

$$f_\varepsilon \overset{2}{\rightharpoonup} f \quad :\Leftrightarrow \quad f_\varepsilon \text{ weakly two-scale converges to } f \text{ in } L^2(\Omega \times Y),$$

$$f_\varepsilon \overset{2}{\to} f \quad :\Leftrightarrow \quad f_\varepsilon \text{ strongly two-scale converges to } f \text{ in } L^2(\Omega \times Y).$$

The upcoming lemma states a two-scale compactness result for the displacements g_ε^0 and g_ε^1 that appear in the representation (16). Due to the differential constraint satisfied by g_ε^0, the corresponding two-scale limits automatically satisfy certain structural properties, which can be captured with the help of the following function spaces:

- $H_\#^1$ is the space of $[0, 1)^d$-periodic functions in $H_{\text{loc}}^1(\mathbb{R}^d)$.
- $H_0^1(Y^0)$ is the closed subspace of $H_\#^1$ consisting of functions $\psi \in H_\#^1$ with $\psi = 0$ on Y^1.
- $\mathcal{A}(Y^0)$ is the closed subspace of $H_\#^1$ consisting of functions $\psi \in H_\#^1$ that satisfy the identity

$$\int_{Y^0} \nabla_y \psi : \nabla_y \zeta \, dy = 0 \quad \forall \zeta \in H_0^1(Y^0).$$

Lemma 2. *Consider a sequence $\varphi_\varepsilon \in H_0^1(\Omega)$ and let $(g_\varepsilon^0, g_\varepsilon^1)$ be associated with φ_ε via (16). Suppose that there exists a sequence of positive numbers m_ε such that*

$$\limsup_{\varepsilon \downarrow 0} m_\varepsilon^{-2} \Phi_\varepsilon^\gamma(m_\varepsilon \varphi_\varepsilon) < \infty \quad \text{and} \quad m_\varepsilon = O(\varepsilon^\gamma).$$

Then there exist

$$g^0 \in L^2\big(\Omega, H_0^1\big(Y^0\big)\big), \qquad g^1 \in H_0^1(\Omega), \qquad \psi \in L^2\big(\Omega, \mathcal{A}\big(Y^0\big)\big) \tag{18}$$

such that, up to selecting a subsequence, one has

$$
\begin{aligned}
&g_\varepsilon^0 \overset{2}{\rightharpoonup} g^0, \qquad \varepsilon \nabla g_\varepsilon^0 \overset{2}{\rightharpoonup} \nabla_y g^0, \\
&g_\varepsilon^1 \rightharpoonup g^1 \quad \text{weakly in } H^1(\Omega) \quad \text{and} \quad \nabla g_\varepsilon^1 \overset{2}{\rightharpoonup} \nabla g^1 + \nabla_y \psi.
\end{aligned}
\tag{19}
$$

The identification obtained in the previous lemma is sharp, in the sense of the following statement.

Lemma 3. *Let $g^0 \in L^2(\Omega, H_0^1(Y^0))$, $g^1 \in H_0^1(\Omega)$ and $\psi \in L^2(\Omega, \mathcal{A}(Y^0))$. Let c_ε be an arbitrary sequence of positive numbers converging to zero. Then there exist function sequences $g_\varepsilon^0 \in H_0^1(\Omega_\varepsilon^0)$, $g_\varepsilon^1 \in H_0^1(\Omega)$ such that $(g_\varepsilon^0, g_\varepsilon^1)$ is related to $\varphi_\varepsilon := g_\varepsilon^1 + \varepsilon^{1-\gamma} g_\varepsilon^0$ as in (16), and*

$$
\begin{aligned}
&g_\varepsilon^0 \overset{2}{\to} g^0, \qquad \varepsilon \nabla g_\varepsilon^0 \overset{2}{\to} \nabla_y g^0, \\
&g_\varepsilon^1 \rightharpoonup g^1 \quad \text{weakly in } H^1(\Omega) \quad \text{and} \quad \nabla g_\varepsilon^1 \overset{2}{\to} \nabla g^1 + \nabla_y \psi, \\
&\limsup_{\varepsilon \downarrow 0} c_\varepsilon \|\nabla \varphi_\varepsilon\|_{L^\infty(\Omega)} = 0.
\end{aligned}
\tag{20}
$$

Our main result is formulated in terms of the notion of convergence described in the above lemmas. For convenience we use the following notation:

- Given $\varphi_\varepsilon \in H^1(\Omega)$ we write $g_\varepsilon^1 + \varepsilon^{1-\gamma} g_\varepsilon^0 \overset{(16)}{:=} \varphi_\varepsilon$, if $g_\varepsilon^1 \in H^1(\Omega)$, $g_\varepsilon^0 \in H_0^1(\Omega_\varepsilon^0)$, and both functions are related to φ_ε as in (16).
- We write $\varphi_\varepsilon \overset{2}{\rightharpoonup} (g^0, g^1)$, if $g_\varepsilon^1 + \varepsilon^{1-\gamma} g_\varepsilon^0 \overset{(16)}{:=} \varphi_\varepsilon$ and

$$g_\varepsilon^0 \overset{2}{\rightharpoonup} g^0, \qquad \varepsilon \nabla g_\varepsilon^0 \overset{2}{\rightharpoonup} \nabla_y g^0, \qquad g_\varepsilon^1 \rightharpoonup g^1 \quad \text{weakly in } H^1(\Omega). \tag{21}$$

- We write $\varphi_\varepsilon \overset{2}{\to} (g^0, g^1)$, if $g_\varepsilon^1 + \varepsilon^{1-\gamma} g_\varepsilon^0 \overset{(16)}{:=} \varphi_\varepsilon$ and

$$g_\varepsilon^0 \overset{2}{\to} g^0, \qquad \varepsilon \nabla g_\varepsilon^0 \overset{2}{\to} \nabla_y g^0, \qquad g_\varepsilon^1 \rightharpoonup g^1 \quad \text{weakly in } H^1(\Omega). \tag{22}$$

3.2. Convergence in the small strain regime $m_\varepsilon = o(\varepsilon^\gamma)$

Throughout this section we assume that the densities W^0 and W^1 satisfy the conditions (W1)–(W3). We show that in the small strain regime the limit functional

$$\mathcal{E}_{\text{small}} : L^2\big(\Omega, H_0^1\big(Y^0\big)\big) \times H_0^1(\Omega) \to [0, \infty),$$

is given by

$$\mathcal{E}_{\text{small}}\big(g^0, g^1\big) := \int_{\Omega \times Y} \big(Q^0\big(\nabla_y g^0(x, y)\big) + Q_{\text{hom}}^1\big(\nabla g^1(x)\big)\big) dx,$$

where

$$Q^1_{\text{hom}}(F) := \min_{\psi \in \mathcal{A}(Y^0)} \int_{Y^1} Q^1\big(F + \nabla_y \psi(y)\big)\, dy. \tag{23}$$

More precisely, the following theorem holds.

Theorem 1. *Let m_ε be a sequence of positive numbers and assume that $m_\varepsilon = o(\varepsilon^\gamma)$ as $\varepsilon \downarrow 0$.*

(a) *(Compactness). Suppose that $\varphi_\varepsilon \in H^1_0(\Omega)$ satisfy*

$$\limsup_{\varepsilon \downarrow 0} m_\varepsilon^{-2} \mathcal{E}_\varepsilon(m_\varepsilon \varphi_\varepsilon) < \infty.$$

Then, up to a subsequence, one has $\varphi_\varepsilon \overset{2}{\rightharpoonup} (g^0, g^1)$ for some $g^0 \in L^2(\Omega, H^1_0(Y^0))$ and $g^1 \in H^1_0(\Omega)$.

(b) *(Lower bound). Consider $\varphi_\varepsilon \in H^1_0(\Omega)$ and suppose that $\varphi_\varepsilon \overset{2}{\rightharpoonup} (g^0, g^1)$ for some $g^0 \in L^2(\Omega, H^1_0(Y^0))$ and $g^1 \in H^1_0(\Omega)$. Then the estimate*

$$\liminf_{\varepsilon \downarrow 0} m_\varepsilon^{-2} \mathcal{E}_\varepsilon(m_\varepsilon \varphi_\varepsilon) \geqslant \mathcal{E}_{\text{small}}\big(g^0, g^1\big)$$

holds.

(c) *(Recovery sequence). For all $g^0 \in L^2(\Omega, H^1_0(Y^0))$ and $g^1 \in H^1_0(\Omega)$ there exists a sequence $\varphi_\varepsilon \in H^1_0(\Omega)$ such that $\varphi_\varepsilon \overset{2}{\rightarrow} (g^0, g^1)$ and*

$$\lim_{\varepsilon \downarrow 0} m_\varepsilon^{-2} \mathcal{E}_\varepsilon(m_\varepsilon \varphi_\varepsilon) = \mathcal{E}_{\text{small}}\big(g^0, g^1\big).$$

In the next result we consider a minimisation problem that involves the density of the "body forces" $\ell_\varepsilon \in L^2(\Omega)$. We study the variational limit of the (scaled) total energy

$$I_\varepsilon(\varphi) := \frac{1}{m_\varepsilon^2}\left(\mathcal{E}_\varepsilon(m_\varepsilon \varphi) - \int_\Omega \ell_\varepsilon \cdot (m_\varepsilon \varphi)\, dx \right), \tag{24}$$

where the scaling factor m_ε is determined by the body forces via

$$m_\varepsilon := \varepsilon^{1-\gamma} \|\ell_\varepsilon\|_{L^2(\Omega^0_\varepsilon)} + \|\ell_\varepsilon\|_{L^2(\Omega)}. \tag{25a}$$

In the small strain regime we assume that the body forces are small in the sense that

$$m_\varepsilon = o\big(\varepsilon^\gamma\big), \quad \varepsilon \downarrow 0. \tag{25b}$$

Moreover, we assume that the (scaled) body-force densities converge, as $\varepsilon \downarrow 0$, in the following way:

$$m_\varepsilon^{-1} \varepsilon^{1-\gamma} \chi_\varepsilon \ell_\varepsilon \overset{2}{\rightarrow} \ell^0, \qquad m_\varepsilon^{-1} \ell_\varepsilon \rightharpoonup \ell^1. \tag{25c}$$

It follows from Theorem 1 that the variational limit of the total energy (24) is given by the functional

$$I_{\mathrm{small}}\big(g^0, g^1\big) := \mathcal{E}_{\mathrm{small}}\big(g^0, g^1\big) - \int_{\Omega}\left(\int_{Y^0} \ell^0 \cdot g^0\, dy + \ell^1 \cdot g^1\right) dx. \tag{26}$$

Proposition 1. *Assume that (25a)–(25c) hold.*

(a) *(Convergence of infima). One has*

$$\lim_{\varepsilon \downarrow 0}\ \inf_{\varphi \in H_0^1(\Omega)}\ I_\varepsilon(\varphi) = \min I_{\mathrm{small}}\big(g^0, g^1\big),$$

where the minimum on the right-hand side is taken over all $g^0 \in L^2(\Omega, H_0^1(Y^0))$ and $g^1 \in H_0^1(\Omega)$. Moreover, the minimum is attained for a unique pair (g_*^0, g_*^1).

(b) *(Convergence of minimisers). Let $\varphi_\varepsilon \in H_0^1(\Omega)$ be a sequence of almost minimisers, i.e.*

$$I_\varepsilon(\varphi_\varepsilon) \leqslant \inf_{\varphi \in H_0^1(\Omega)}\ I_\varepsilon(\varphi) + o(1), \quad \varepsilon \downarrow 0. \tag{27}$$

Then

$$\varphi_\varepsilon \overset{2}{\rightharpoonup} \big(g_*^0, g_*^1\big) \quad and \quad \nabla g_\varepsilon^1 \overset{2}{\rightharpoonup} \nabla g_*^1 + \nabla_y \psi_*,$$

where $\psi_* \in L^2(\Omega, \mathcal{A}(Y^0))$ denotes the unique "corrector" characterised by

$$Q_{\mathrm{hom}}^1\big(\nabla g_*^1(x)\big) = \int_{Y^1} Q^1\big(\nabla g_*^1(x) + \nabla_y \psi_*(x, y)\big)\, dy, \quad \int_Y \psi_*(x, y)\, dy = 0, \tag{28}$$

for almost every $x \in \Omega$.

Next, we prove that almost minimisers φ_ε satisfy the asymptotic relation

$$\varphi_\varepsilon = g_{*,\varepsilon}^1(x) + \varepsilon^{1-\gamma} g_{*,\varepsilon}^0(x) + o(1), \quad \varepsilon \downarrow 0,\ \text{in}\ W^{1,p}(\Omega)\ (p < 2), \tag{29}$$

where $g_{*,\varepsilon}^1$ and $g_{*,\varepsilon}^0$ formally obey the "ansatz"

$$g_{*,\varepsilon}^0(x) \overset{\mathrm{formally}}{=} g_*^0(x, x/\varepsilon), \qquad g_{*,\varepsilon}^1(x) \overset{\mathrm{formally}}{=} g_*^1(x) + \varepsilon\psi_*(x, x/\varepsilon). \tag{30}$$

Here (g_*^0, g_*^1) and ψ_* denote the minimising pair and corrector from Proposition 1. Since the functions on the right-hand sides in (30) are in general not smooth enough to define $g_{*,\varepsilon}^0$ and $g_{*,\varepsilon}^1$ by (30) directly, we use instead the approximation associated with (g_*^0, g_*^1, ψ_*) via Lemma 3.

In addition to the properties of Y^0 assumed in Section 2, we require the following assumption on the regularity of Y^0:

Assumption 1. There exist an exponent $p < 2$ and a constant C such that for all $\varphi \in H^1(Y^1)$ and $g^1 \in H^1(Y)$ related via (10) we have $\|\nabla g^1\|_{L^p(Y^0)} \leqslant C\|\nabla \varphi\|_{L^p(Y^1)}$.

Note that Assumption 1 is satisfied if Y^0 can be written as the disjoint union of a finite number of Lipschitz domains $Y_1^0, \ldots, Y_N^0$ with $\partial Y_i^0 \cap \partial Y_j^0 = \emptyset$ for $i \neq j$.

Theorem 2. *Assume that (25a)–(25c) hold, and let Assumption 1 be satisfied. Let $\varphi_\varepsilon \in H_0^1(\Omega)$ be a sequence of almost minimisers, i.e.*

$$I_\varepsilon(\varphi_\varepsilon) \leqslant \inf_{\varphi \in H_0^1(\Omega)} I_\varepsilon(\varphi) + o(1), \quad \varepsilon \downarrow 0.$$

Let (g_*^0, g_*^1) be the minimiser of I_{small} and let ψ_* be defined through (28). Let $(g_{*,\varepsilon}^0, g_{*,\varepsilon}^1)$ and $\varphi_{*,\varepsilon} = g_{*,\varepsilon}^1 + \varepsilon^{1-\gamma} g_{*,\varepsilon}^0$ be associated with (g_*^0, g_*^1, ψ_*) as in Lemma 3, i.e. $\varphi_{*,\varepsilon} \xrightarrow{2} (g_*^0, g_*^1)$ and $\nabla g_{*,\varepsilon}^1 \xrightarrow{2} \nabla g_*^1 + \nabla_y \psi_*$. Then for $g_\varepsilon^1 + \varepsilon^{1-\gamma} g_\varepsilon^0 \overset{(16)}{:=} \varphi_\varepsilon$ one has

$$\left\| g_\varepsilon^0 - g_{*,\varepsilon}^0 \right\|_{L^p(\Omega_\varepsilon^0)} + \left\| \varepsilon \nabla g_\varepsilon^0 - \varepsilon \nabla g_{*,\varepsilon}^0 \right\|_{L^p(\Omega_\varepsilon^0)} + \left\| g_\varepsilon^1 - g_{*,\varepsilon}^1 \right\|_{W^{1,p}(\Omega)} \to 0 \quad \text{as } \varepsilon \downarrow 0. \tag{31}$$

Remark 1. To illustrate the result of Theorem 2, consider the case $\gamma = 1$ with $\ell_\varepsilon := m_\varepsilon \ell(x, \frac{x}{\varepsilon})$, where $\ell(x, y)$ is smooth both in x and y and is periodic in y. If the domain Ω and the pore set Y^0 are sufficiently regular, the minimisers (g_*^0, g_*^1) and ψ_* are smooth, by the classical elliptic regularity theory, see *e.g.* [21]. In that case we may set

$$g_{*,\varepsilon}^0(x) := g_*^0(x, x/\varepsilon) \quad \text{and} \quad g_{*,\varepsilon}^1 := g_*^1(x) + \varepsilon \psi_*(x, x/\varepsilon),$$

and the asymptotic formula for φ_ε reads

$$\varphi_\varepsilon = g_*^0(x, x/\varepsilon) + g_*^1(x) + \varepsilon \psi_*(x, x/\varepsilon) + R_\varepsilon(x),$$

where $\|R_\varepsilon\|_{W^{1,p}(\Omega)} \to 0$ as $\varepsilon \downarrow 0$.

In the remainder of this section, we restrict to the special case of the introduction (in the small strain regime), i.e. we assume that $\alpha > 1$, $\gamma = 1$, $m_\varepsilon = \varepsilon^\alpha$ and $\ell_\varepsilon = \varepsilon^\alpha f$ for some $f \in L^2(\Omega)$, so that the functionals I_ε^α and I_ε defined in (2) and (24) are identical. Hence, Theorem 1 and Proposition 1 prove two-scale Γ-convergence of I_ε^α and the convergence of the associated minimisation problems (as claimed in the Introduction), and Theorem 2 yields the two-scale expansion (6). We argue that the functionals I_ε^α Γ-converge to the (single scale) limit $\bar{I}_{\mathrm{small}}$:

Proposition 2. *For $\alpha > 1$ and $f \in L^2(\Omega)$ consider I_ε^α and $\bar{I}_{\mathrm{small}}$ defined in (2) and (7). We extend I_ε^α to a functional on $L^2(\Omega)$ by setting $I_\varepsilon^\alpha := +\infty$ on $L^2(\Omega) \setminus H_0^1(\Omega)$. Then:*

(a) *(Compactness). Suppose that $\varphi_\varepsilon \in L^2(\Omega)$ satisfy*

$$\limsup_{\varepsilon \downarrow 0} I_\varepsilon^\alpha(\varphi_\varepsilon) < \infty.$$

Then, up to a subsequence, we have $\varphi_\varepsilon \rightharpoonup \varphi$ weakly in $L^2(\Omega)$.

(b) *(Lower bound). For every $\varphi_\varepsilon \in L^2(\Omega)$ with $\varphi_\varepsilon \rightharpoonup \varphi$ weakly in $L^2(\Omega)$ we have*

$$\liminf_{\varepsilon \downarrow 0} I_\varepsilon^\alpha(\varphi_\varepsilon) \geqslant \bar{I}_{\text{small}}(\varphi)$$

(c) *(Upper bound). For every $\varphi \in L^2(\Omega)$ we can find $\varphi_\varepsilon \rightharpoonup \varphi$ weakly in $L^2(\Omega)$ such that*

$$\lim_{\varepsilon \downarrow 0} I_\varepsilon^\alpha(\varphi_\varepsilon) = \bar{I}_{\text{small}}(\varphi)$$

(d) *(Convergence of the minimisation problem). Let $\varphi_\varepsilon \in H_0^1(\Omega)$ denote an infimizing sequence of I_ε^α, i.e.*

$$I_\varepsilon^\alpha(\varphi_\varepsilon) \leqslant \inf_{\varphi \in L^2(\Omega)} I_\varepsilon^\alpha(\varphi) + o(1), \quad \varepsilon \downarrow 0.$$

Let (g_*^0, g_*^1) denote the unique minimiser of I_{small} and φ_* denote the unique minimiser of $\bar{I}_{\text{small}}$. Then $\inf_{L^2(\Omega)} I_\varepsilon^\alpha \to \min_{L^2(\Omega)} \bar{I}_{\text{small}}$, $\varphi_\varepsilon \rightharpoonup \varphi_*$ weakly in $L^2(\Omega)$, and

$$\varphi_* = g_*^1 + \int_{Y_0} g^0 \, dy, \qquad \bar{I}_{\text{small}}(\varphi_*) = I_{\text{small}}(g_*^0, g_*^1).$$

3.3. Convergence in the finite strain regime $m_\varepsilon = \varepsilon^\gamma$

Throughout this section we assume that

- W^1 satisfies the conditions (W1)–(W3).
- $W^0 : \mathbb{R}^{d \times d} \to [0, \infty)$ is continuous and satisfies the growth condition

$$c_0 \operatorname{dist}^2(F, \text{SO}(d)) \leqslant W^0(F) \leqslant c_0^{-1}(1 + |F|^2) \quad \forall F \in \mathbb{R}^{d \times d}, \tag{32a}$$

and the local Lipschitz condition

$$\left| W^0(F + G) - W^0(F) \right| \leqslant c_0^{-1}(1 + |F| + |G|)|G| \quad \forall F, G \in \mathbb{R}^{d \times d}. \tag{32b}$$

We prove that in the finite strain regime the limit functional

$$\mathcal{E}_{\text{finite}} : L^2(\Omega, H_0^1(Y^0)) \times H_0^1(\Omega) \to [0, \infty)$$

is given by

$$\mathcal{E}_{\text{finite}}(g^0, g^1) := \iint_{\Omega \times Y^0} \mathcal{Q}W^0(I + \nabla_y g^0(x, y)) \, dy \, dx + \int_\Omega Q_{\text{hom}}^1(\nabla g^1(x)) \, dx,$$

where $\mathcal{Q}W^0$ denotes the quasiconvex envelope of W^0 (see *e.g.* [19]). The associated limit of the total energy I^ε, see (24), is given by (*cf.* (26))

$$I_{\text{finite}}(g^0, g^1) := \mathcal{E}_{\text{finite}}(g^0, g^1) - \int_\Omega \left(\int_{Y^0} \ell^0 \cdot g^0 \, dy + \ell^1 \cdot g^1 \right) dx,$$

where ℓ^0, ℓ^1 are defined in the same way as in (25c).

Theorem 3.

(a) *(Lower bound). Consider a sequence $\varphi_\varepsilon \in H_0^1(\Omega)$ and the associated decomposition $\varepsilon^{1-\gamma} g_\varepsilon^0 + g_\varepsilon^1 \overset{(16)}{:=} \varphi_\varepsilon$. If $\varphi_\varepsilon \overset{2}{\rightharpoonup} (g^0, g^1)$ and $g_\varepsilon^0 \overset{2}{\to} g^0$, then*

$$\liminf_{\varepsilon \downarrow 0} \varepsilon^{-2\gamma} \mathcal{E}_\varepsilon(\varepsilon^\gamma \varphi_\varepsilon) \geqslant \mathcal{E}_{\text{finite}}(g^0, g^1).$$

(b) *(Recovery sequence). For any $g^0 \in L^2(\Omega, H_0^1(Y^0))$ and $g^1 \in H_0^1(\Omega)$ there exists a sequence $\varphi_\varepsilon \in H_0^1(\Omega)$ such that $\varphi_\varepsilon \overset{2}{\rightharpoonup} (g^0, g^1)$ and*

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-2\gamma} \mathcal{E}_\varepsilon(\varepsilon^\gamma \varphi_\varepsilon) = \mathcal{E}_{\text{finite}}(g^0, g^1).$$

(c) *Suppose that the force densities $\ell_\varepsilon \in L^2(\Omega)$ satisfy (25a) and (25c) with $m_\varepsilon = \varepsilon^\gamma$. Then the infima converge, i.e.*

$$\lim_{\varepsilon \downarrow 0} \inf_{\varphi \in H_0^1(\Omega)} I_\varepsilon(\varphi) = \inf I_{\text{finite}}(g^0, g^1), \tag{33}$$

where the infimum on the right-hand side is taken over all functions $g^0 \in L^2(\Omega, H_0^1(Y^0))$ and $g^1 \in H_0^1(\Omega)$. Moreover, there exist a minimising pair (g_*^0, g_*^1) and a recovery sequence $\varphi_\varepsilon \in H_0^1(\Omega)$ with $\varphi_\varepsilon \overset{2}{\to} (g_*^0, g_*^1)$ such that

$$I_\varepsilon(\varphi_\varepsilon) \to I_{\text{finite}}(g_*^0, g_*^1) = \min I_{\text{finite}}(g^0, g^1) \quad \text{as } \varepsilon \downarrow 0. \tag{34}$$

Remark 2 (Example in Section 1). If we consider Theorem 3 and Proposition 1 in the case $\gamma = 1$, $m_\varepsilon = \varepsilon$ and $\ell_\varepsilon = \varepsilon f$ for some $f \in L^2(\Omega)$, then we recover the special case (in the finite strain regime) presented in the introduction. In particular, we deduce that the functionals I_ε^α two-scale Γ-converge (in the sense of Theorem 3) to I_{finite}. Arguing as in the small-strain regime, cf. Proposition 2, we deduce that I_ε^α Γ-converges (with respect to the weak topology in $L^2(\Omega)$) to $\tilde{I}_{\text{finite}}$, cf. (8). We leave the details to the readers.

4. Proofs

We start by proving the auxiliary results discussed in Section 3.1. Sections 4.2 and 4.3 contain the proofs of the main statements in the small strain and finite strain cases, respectively.

4.1. Proofs of Lemma 1, Lemma 2, and Lemma 3: A priori estimate, compactness and approximation

A key ingredient in the proof of Lemma 1 is the geometric rigidity estimate by Friesecke *et al.* [23]:

Theorem 4 (Geometric rigidity estimate, see [23]). *Let U be an open, bounded Lipschitz domain in $\mathbb{R}^d$, $d \geqslant 2$. There exists a constant $C(U)$ with the following property: for each $v \in H^1(U)$ there is a rotation $R \in \text{SO}(d)$ such that*

$$\int_U |\nabla v(x) - R|^2 \, dx \leqslant C(U) \int_U \text{dist}^2(\nabla v(x), \text{SO}(d)) \, dx.$$

Moreover, the constant $C(U)$ is invariant under uniform scaling of U.

In fact, we need the following modified version, which is adapted to perforated domains.

Lemma 4. *There exists a constant $C > 0$ that only depends on Ω and Y^1 such that for all $\varepsilon > 0$ and $v \in H^1(\Omega)$ satisfying*

$$\int_{\Omega_\varepsilon^0} \nabla v : \nabla \zeta \, dx = 0 \quad \forall \zeta \in H_0^1(\Omega_\varepsilon^0), \tag{35}$$

the estimates

$$\left\| \operatorname{dist}\!\big(\nabla v, \operatorname{SO}(d)\big) \right\|_{L^2(\Omega)} \leqslant C \left\| \operatorname{dist}\!\big(\nabla v, \operatorname{SO}(d)\big) \right\|_{L^2(\Omega_\varepsilon^1)}, \tag{36}$$

$$\| \nabla v - R \|_{L^2(\Omega_\varepsilon^1)} \leqslant C \left\| \operatorname{dist}\!\big(\nabla v, \operatorname{SO}(d)\big) \right\|_{L^2(\Omega_\varepsilon^1)}, \tag{37}$$

hold for some $R \in \operatorname{SO}(d)$, which may depend on v. In addition, if $v(x) = x + c$ on $\partial\Omega$ for some constant c, then we may set $R = I$.

Proof of Lemma 4. *Step* 1. The proof of the inequality (36).

Let $\widehat{\Omega}_\varepsilon := \bigcup \{\varepsilon(\xi + Y) | \xi \in \mathbb{Z}^d, \varepsilon(\xi + Y) \subset \Omega\}$ denote the union of ε-cells that are completely contained in Ω. Since $\Omega \setminus \widehat{\Omega}_\varepsilon \subset \Omega_\varepsilon^1$, it suffices to prove (36) for Ω replaced by $\widehat{\Omega}_\varepsilon$, respectively. In fact we shall prove the following stronger estimate: for all $\xi \in \mathbb{Z}^d$ with $\varepsilon(\xi + Y) \subset \Omega$ we have

$$\int_{\varepsilon(\xi+Y)} \operatorname{dist}^2\!\big(\nabla v, \operatorname{SO}(d)\big) \, dx \lesssim \int_{\varepsilon(\xi+Y^1)} \operatorname{dist}^2\!\big(\nabla v, \operatorname{SO}(d)\big) \, dx. \tag{38}$$

For the argument fix an admissible $\xi \in \mathbb{Z}^d$. Application of Theorem 4 with $U = \varepsilon(\xi + Y^1)$ yields a rotation $R \in \operatorname{SO}(d)$ such that

$$\int_{\varepsilon(\xi+Y^1)} |\nabla v - R|^2 \, dx \lesssim \int_{\varepsilon(\xi+Y^1)} \operatorname{dist}^2\!\big(\nabla v, \operatorname{SO}(d)\big) \, dx. \tag{39}$$

Note that the multiplicative constant in the estimate above only depends on Y^1, since $\varepsilon(\xi + Y^1)$ is a dilation and translation of Y^1. On the other hand, since $\varepsilon(\xi + Y^0) \ni x \mapsto (v(x) - Rx)$ is harmonic, we have (cf. (11)):

$$\int_{\varepsilon(\xi+Y)} \operatorname{dist}^2\!\big(\nabla v, \operatorname{SO}(d)\big) \, dx \leqslant \int_{\varepsilon(\xi+Y)} |\nabla v - R|^2 \lesssim \int_{\varepsilon(\xi+Y^1)} |\nabla v - R|^2.$$

Combined with (39), inequality (38) follows.

Step 2. The proof of the rigidity estimate (37).

From (36) and Theorem 4 (applied with $U = \Omega$) we deduce that for some $R \in \operatorname{SO}(d)$:

$$\| \nabla v - R \|_{L^2(\Omega)} \lesssim \left\| \operatorname{dist}\!\big(\nabla v, \operatorname{SO}(d)\big) \right\|_{L^2(\Omega_\varepsilon^1)}, \tag{40}$$

which in particular implies (37). Finally we argue that one can set $R = I$, if $v = x + c$ on $\partial\Omega$. In view of (40), it suffices to show that $\int_\Omega |\nabla v - I|^2 \, dx \leqslant \int_\Omega |\nabla v - R|^2 \, dx$ for all $R \in \mathrm{SO}(d)$. This inequality can be seen as follows: Consider $\varphi(x) := v(x) - x - c$ and note that φ vanishes on $\partial\Omega$, so that

$$\int_\Omega |\nabla v - I|^2 \, dx = \int_\Omega |\nabla\varphi|^2 \, dx \leqslant \int_\Omega |\nabla\varphi|^2 + |I - R|^2 \, dx = \int_\Omega \big|\nabla\varphi + (I - R)\big|^2 \, dx$$

$$= \int_\Omega |\nabla v - R|^2 \, dx,$$

which in fact holds for an arbitrary matrix R. □

We are now in position to present the proofs of Lemmas 1 and 2.

Proof of Lemma 1. In the following, the symbol $\lesssim$ stands for $\leqslant$ up to a multiplicative constant that only depends on Y^1 and Ω.

Step 1. Existence of the decomposition (16) and derivation of the estimate for g_ε^1.

Let g_ε^1 denote the unique function in $H^1(\Omega)$ characterised by $g_\varepsilon^1 = \varphi_\varepsilon$ in Ω_ε^1 and (16)(ii). Since $\partial\Omega_\varepsilon^0$ is Lipschitz, we deduce that $g_\varepsilon^0 := \varepsilon^{\gamma-1}(\varphi_\varepsilon - g_\varepsilon^1) \in H_0^1(\Omega_\varepsilon^0)$. This proves the existence of the decomposition. We claim that

$$\int_{\Omega_\varepsilon^0} \big|\nabla g_\varepsilon^1\big|^2 \, dx \lesssim \int_{\Omega_\varepsilon^1} |\nabla\varphi_\varepsilon|^2 \, dx = \int_{\Omega_\varepsilon^1} \big|\nabla g_\varepsilon^1\big|^2 \, dx. \tag{41}$$

Since Ω_ε^0 is defined as the union of the sets $\varepsilon(\xi + Y^0)$ with $\xi \in Z_\varepsilon := \{\xi \in \mathbb{Z}^d : \varepsilon(\xi + Y) \subset \Omega\}$, it suffices to prove $\int_{\varepsilon(\xi+Y^0)} |\nabla g_\varepsilon^1|^2 \, dx \lesssim \int_{\varepsilon(\xi+Y^1)} |\nabla\varphi_\varepsilon|^2 \, dx$. The latter follows from (11) by a scaling argument, since the rescaled functions $y \mapsto \varphi_\varepsilon(\varepsilon(\xi + y))$ and $y \mapsto g_\varepsilon^1(\varepsilon(\xi + y))$ satisfy (10).

Next, we prove (17). Consider $v_\varepsilon(x) := x + g_\varepsilon^1(x)$ and note that $v_\varepsilon(x)$ satisfies (35). Hence, (41) and Lemma 4 yield

$$\int_\Omega \big|\nabla g_\varepsilon^1\big|^2 \, dx \lesssim \int_{\Omega_\varepsilon^1} \big|\nabla g_\varepsilon^1\big|^2 \, dx \leqslant \int_\Omega |\nabla v_\varepsilon - I|^2 \overset{(37)}{\leqslant} C \int_{\Omega_\varepsilon^1} \mathrm{dist}^2\big(I + \nabla\varphi_\varepsilon(x), \mathrm{SO}(d)\big) \, dx$$

$$\leqslant \Phi_\varepsilon^\gamma(\varphi_\varepsilon). \tag{42}$$

Since g_ε^1 vanishes on the boundary of Ω, the estimate upgrades (by Poincaré's inequality) to $\|g_\varepsilon^1\|_{H^1(\Omega)} \leqslant C\Phi_\varepsilon^\gamma(\varphi_\varepsilon)$.

Step 2. Derivation of the estimate for g_ε^0.

Since we have an improved Poincaré inequality (see *e.g.* [25, Lemma 1.6]):

$$\forall g \in H_0^1\big(\Omega_\varepsilon^0\big): \quad \|g\|_{L^2(\Omega_\varepsilon^0)} \lesssim \varepsilon\|\nabla g\|_{L^2(\Omega_\varepsilon^0)}, \tag{43}$$

it suffices to prove

$$\big\|\varepsilon\nabla g_\varepsilon^0\big\|_{L^2(\Omega_\varepsilon^0)}^2 \lesssim \Phi_\varepsilon^\gamma(\varphi_\varepsilon).$$

To this end, notice that since φ_ε vanishes on the boundary of Ω, we have

$$\|\nabla\varphi_\varepsilon\|^2_{L^2(\Omega)} = \min_{R\in\mathrm{SO}(d)}\|I + \nabla\varphi_\varepsilon - R\|^2_{L^2(\Omega)} \overset{\text{Theorem 4}}{\lesssim} \int_\Omega \mathrm{dist}^2\big(I + \nabla\varphi_\varepsilon(x), \mathrm{SO}(d)\big)\,dx$$

$$\leqslant \varepsilon^{-2\gamma}\Phi^\gamma_\varepsilon(\varphi_\varepsilon). \tag{44}$$

Thanks to the first identity in (16), we get by triangle inequality:

$$\big\|\varepsilon^{1-\gamma}\nabla g^0_\varepsilon\big\|_{L^2(\Omega^0_\varepsilon)} \leqslant \|\nabla\varphi_\varepsilon\|_{L^2(\Omega)} + \big\|\nabla g^1_\varepsilon\big\|_{L^2(\Omega)}.$$

Combined with (42) and (44) we finally get

$$\big\|\varepsilon\nabla g^0_\varepsilon\big\|^2_{L^2(\Omega^0_\varepsilon)} = \varepsilon^{2\gamma}\big\|\varepsilon^{1-\gamma}\nabla g^0_\varepsilon\big\|^2_{L^2(\Omega^0_\varepsilon)} \lesssim \Phi^\gamma_\varepsilon(\varphi_\varepsilon). \qquad\square$$

Proof of Lemma 2. *Step* 1. *A priori* estimate and basic compactness.
From Lemma 1 we deduce that

$$\limsup_{\varepsilon\downarrow 0}\big(\big\|g^0_\varepsilon\big\|^2_{L^2(\Omega)} + \big\|\varepsilon\nabla g^0_\varepsilon\big\|^2_{L^2(\Omega)} + \big\|g^1_\varepsilon\big\|^2_{H^1(\Omega)}\big) \leqslant C\limsup_{\varepsilon\downarrow 0} m_\varepsilon^{-2}\Phi^\gamma_\varepsilon(m_\varepsilon\varphi_\varepsilon) < \infty. \tag{45}$$

Hence, by standard results concerning two-scale convergence (*cf.* [1, Proposition 1.14] and [36, Proposition 4.2]), there exist $g^1 \in H^1_0(\Omega)$, $\psi \in L^2(\Omega, H^1_\#)$ and $g^0 \in L^2(\Omega, H^1_\#)$ such that, up to a subsequence, one has

$$g^1_\varepsilon \rightharpoonup g^1 \quad\text{weakly in } H^1(\Omega), \qquad \nabla g^1_\varepsilon \overset{2}{\rightharpoonup} \nabla g^1 + \nabla_y\psi,$$

$$g^0_\varepsilon \overset{2}{\rightharpoonup} g^0, \qquad \varepsilon\nabla g^0_\varepsilon \overset{2}{\rightharpoonup} \nabla_y g^0.$$

Step 2. The proof of the inclusion $\psi \in L^2(\Omega, \mathcal{A}(Y^0))$.
By a density argument, it suffices to show that

$$\iint_{\Omega\times Y^0} \nabla_y\psi(x, y) : \nabla_y\big(\zeta_1(x)\zeta_2(y)\big)\,dx\,dy = 0 \tag{46}$$

for all scalar functions $\zeta_1 \in C^\infty_c(\Omega)$, and all $\zeta_2 \in C^\infty_c(Y^0)$. To this end, we identify ζ_2 with its unique Y-periodic extension to $\mathbb{R}^d$ that vanishes on Y^1, and set

$$\zeta_\varepsilon(x) := \varepsilon\zeta_1(x)\zeta_2(x/\varepsilon), \quad x \in \Omega.$$

Thanks to (16) we have

$$\int_\Omega \nabla g^1_\varepsilon : \nabla\zeta_\varepsilon\,dx = 0.$$

As can be easily checked, we have $\nabla \zeta_\varepsilon \xrightarrow{2} \zeta_1(x) \nabla_y \zeta_2(y)$, so that

$$
\begin{aligned}
0 &= \lim_{\varepsilon \downarrow 0} \int_\Omega \nabla g_\varepsilon^1 : \nabla \zeta_\varepsilon \, dx \\
&= \iint_{\Omega \times Y^0} \left(\nabla g^1(x) + \nabla_y \psi(x, y) \right) : \left(\zeta_1(x) \nabla_y \zeta_2(y) \right) dx \, dy \\
&= \iint_{\Omega \times Y^0} \nabla_y \psi(x, y) : \left(\zeta_1(x) \nabla_y \zeta_2(y) \right) dx \, dy,
\end{aligned}
$$

where the last identity holds thanks to the periodicity of ζ_2. This proves (46).

Step 3. The proof of the inclusion $g^0 \in L^2(\Omega, H_0^1(Y^0))$.

By a density argument, it suffices to show that

$$
\iint_{\Omega \times Y} g^0(x, y) \cdot \left(\zeta_1(x) \zeta_2(y) \right) dx \, dy = 0 \tag{47}
$$

for all scalar functions $\zeta_1 \in C_c^\infty(\Omega)$ and all $\zeta_2 \in H_\#^1$ with $\zeta_2 = 0$ on Y^0. We argue by considering the function $\zeta_\varepsilon(x) := \zeta_1(x) \zeta_2(\frac{x}{\varepsilon})$, $x \in \Omega$, the support of which is contained in Ω_ε^1 for $\varepsilon \ll 1$. Since $\zeta_\varepsilon \xrightarrow{2} \zeta_1(x) \zeta(y)$, and since g_ε^0 is supported in Ω_ε^0, we deduce that

$$
0 = \lim_{\varepsilon \downarrow 0} \int_\Omega g_\varepsilon^0(x) \cdot \zeta_\varepsilon(x) \, dx = \iint_{\Omega \times Y} g^0(x, y) \cdot \left(\zeta_1(x) \zeta_2(y) \right) dx \, dy.
$$

This completes the argument. $\quad \square$

In the proof of Lemma 3 we appeal to the construction of a diagonal sequence that is due to Attouch, see [2]:

Lemma 5. *For any $h : [0, \infty)^2 \to [0, +\infty]$, there exists a mapping $(0, 1) \ni \varepsilon \mapsto \delta(\varepsilon) \in (0, 1)$ such that*

$$
\lim_{\varepsilon \downarrow 0} \delta(\varepsilon) = 0 \quad and \quad \limsup_{\varepsilon \downarrow 0} h\left(\varepsilon, \delta(\varepsilon) \right) \leqslant \limsup_{\delta \downarrow 0} \limsup_{\varepsilon \downarrow 0} h(\varepsilon, \delta).
$$

Proof of Lemma 3. *Step* 1. Characterisation of strong two-scale convergence via unfolding.

For $f_\varepsilon : \Omega \to \mathbb{R}$ and $f : \Omega \times Y \to \mathbb{R}$ define

$$
d_\varepsilon(f_\varepsilon, f) := \int_{\mathbb{R}^d} \int_Y \left| \tilde{f}_\varepsilon \left(\varepsilon \lfloor x/\varepsilon \rfloor + \varepsilon y \right) - \tilde{f}(x, y) \right|^2 dy \, dx,
$$

where $\tilde{f}_\varepsilon$ denotes the extension by zero of f_ε to $\mathbb{R}^d$, $\tilde{f}$ denotes the extension by zero of f to $\mathbb{R}^d \times Y$, and $\lfloor z \rfloor$ denotes the unique element in $\mathbb{Z}^d$ with $z - \lfloor z \rfloor \in [0, 1)^d$. We recall from [36] that

$$
f_\varepsilon \xrightarrow{2} f \quad \Leftrightarrow \quad d_\varepsilon(f_\varepsilon, f) \to 0. \tag{48}
$$

The characterisation extends in the obvious way to vector-valued functions.

Step 2. Construction of g_ε^1.

We claim that there exists a sequence g_ε^1 in $H_0^1(\Omega)$ whose elements satisfy (16)(ii) and

$$g_\varepsilon^1 \rightharpoonup g^1 \quad \text{weakly in } H^1(\Omega), \qquad \nabla g_\varepsilon^1 \overset{2}{\to} g^1 + \nabla_y \psi, \qquad \limsup_{\varepsilon \downarrow 0} c_\varepsilon \big\| \nabla g_\varepsilon^1 \big\|_{L^\infty(\Omega)} = 0. \tag{49}$$

Indeed, by a density argument there exist $g^{1,\delta} \in C_c^\infty(\Omega)$, $\delta \in (0,1)$, and $\psi \in C_c^\infty(\Omega, C_\#^\infty(Y))$ such that

$$\big\| g^{1,\delta} - g^1 \big\|_{H^1(\Omega)} + \big\| \nabla_y \psi^\delta - \nabla_y \psi \big\|_{L^2(\Omega \times Y)} \leqslant \delta \quad \forall \delta.$$

For $\varepsilon > 0$, $\delta \in (0,1)$, define

$$g_\varepsilon^{1,\delta}(x) := g^{1,\delta}(x) + \varepsilon \psi^\delta(x, x/\varepsilon),$$

and set

$$d_\varepsilon^\delta := d_\varepsilon\big(\nabla g_\varepsilon^{1,\delta}, \nabla g^1 + \nabla_y \psi\big) + \big\| g_\varepsilon^{1,\delta} - g^1 \big\|_{L^2(\Omega)} + c_\varepsilon \big\| \nabla g_\varepsilon^{1,\delta} \big\|_{L^\infty(\Omega)}.$$

By construction, we have $\lim_{\delta \downarrow 0} \limsup_{\varepsilon \downarrow 0} d_\varepsilon^\delta = 0$, and Lemma 5 yields a function $\varepsilon \mapsto \delta(\varepsilon)$ with $\lim_{\varepsilon \downarrow 0} d_\varepsilon^{\delta(\varepsilon)} = 0$. In view of Step 1, this implies that the diagonal sequence $\tilde{g}_\varepsilon^1 := g_\varepsilon^{1,\delta(\varepsilon)}$ satisfies (49). Now, for each $\varepsilon > 0$, let g_ε^1 denote the function satisfying (16)(ii) and such that $g_\varepsilon^1 = \tilde{g}_\varepsilon^1$ on Ω_ε^1. To conclude the argument, we only need to show that g_ε^1 satisfies (49). Consider the difference $\eta_\varepsilon := \tilde{g}_\varepsilon^1 - g_\varepsilon^1$. Since η_ε is bounded in $H^1(\Omega)$ and $\eta_\varepsilon = 0$ in Ω_ε^1, we have $\eta_\varepsilon \rightharpoonup 0$ in $H^1(\Omega)$, and, up to a subsequence, $\nabla \eta_\varepsilon \overset{2}{\rightharpoonup} \nabla_y \varphi$ for some $\varphi \in L^2(\Omega, H_0^1(Y^0))$. On the other hand, since g_ε^1 satisfies (16)(ii) and $\tilde{g}_\varepsilon^1$ satisfies (49), we deduce that

$$\| \nabla \eta_\varepsilon \|_{L^2(\Omega)}^2 = \int_{\Omega_\varepsilon^0} \nabla \eta_\varepsilon : \nabla \eta_\varepsilon \, dx = \int_{\Omega_\varepsilon^0} \nabla \tilde{g}_\varepsilon^1 : \nabla \eta_\varepsilon \, dx \to \iint_{\Omega \times Y^0} \big(\nabla g^1 + \nabla_y \psi \big) : \nabla_y \varphi \, dx \, dy.$$

Since ∇g^1 is independent of y, and because $\psi \in L^2(\Omega, \mathcal{A}(Y^0))$, the integral on the right-hand side vanishes. Hence, $\| \nabla \eta_\varepsilon \|_{L^2(\Omega)}^2 \to 0$, and thus g_ε^1 satisfies (49).

Step 3. Conclusion.

As can be shown by appealing to a combination of a density argument and a diagonal-sequence argument, similar to Step 1, there exists a sequence $g_\varepsilon^0 \in H_0^1(\Omega_\varepsilon^0)$ such that

$$g_\varepsilon^0 \overset{2}{\to} g^0, \qquad \varepsilon \nabla g_\varepsilon^0 \overset{2}{\to} \nabla_y g^0, \qquad \limsup_{\varepsilon \downarrow 0} c_\varepsilon \big\| \varepsilon^{1-\gamma} \nabla g_\varepsilon^0 \big\|_{L^\infty(\Omega)} = 0.$$

Now define $\varphi_\varepsilon(x) := \varepsilon^{1-\gamma} g_\varepsilon^0 + g_\varepsilon^1$, and note that $(g_\varepsilon^0, g_\varepsilon^1)$ satisfy (16). In view of the convergence of g_ε^0 and g_ε^1, the sequence φ_ε has the required properties. $\quad\square$

4.2. Proof of Theorems 1, 2 and Propositions 1, 2: Small strain regime

As a preliminary remark, we note that two different effects play a role when passing to the limit $\varepsilon \downarrow 0$ in the small strain regime:

- The non-convex energy functional is linearised at identity map (which is a stress-free state for $\mathcal{E}_\varepsilon$) – this corresponds to the passage from nonlinear to linearised elasticity.
- The obtained linearised, still oscillating, convex-quadratic energy is homogenised.

The following lemma is used to treat both effects simultaneously. Its proof combines convex homogenisation methods (e.g. [37, Proposition 1.3]) with a "careful Taylor expansion" in the spirit of [23, Proof of Theorem 6.2]. For notational convenience, we introduce two "linearised" functionals:

- For $G_\varepsilon = (G_\varepsilon^0, G_\varepsilon^1) \in L^2(\Omega, \mathbb{R}^{d \times d}) \times L^2(\Omega, \mathbb{R}^{d \times d})$ set

$$\mathcal{Q}_\varepsilon(G_\varepsilon) = \mathcal{Q}_\varepsilon\big(G_\varepsilon^0, G_\varepsilon^1\big) := \int_{\Omega_\varepsilon^0} Q^0\big(G_\varepsilon^0(x)\big)\, dx + \int_{\Omega_\varepsilon^1} Q^1\big(G_\varepsilon^1(x)\big)\, dx.$$

- For $G = (G^0, G^1) \in L^2(\Omega \times Y, \mathbb{R}^{d \times d}) \times L^2(\Omega \times Y, \mathbb{R}^{d \times d})$ set

$$\mathcal{Q}(G) = \mathcal{Q}\big(G^0, G^1\big) := \int_{\Omega \times Y^0} Q^0\big(G^0(x, y)\big)\, dx\, dy + \int_{\Omega \times Y^1} Q^1\big(G^1(x, y)\big)\, dx\, dy.$$

Lemma 6. *Consider sequences $g_\varepsilon^0, g_\varepsilon^1 \in H^1(\Omega)$ that satisfy*

$$\limsup_{\varepsilon \downarrow 0}\big(\big\|\varepsilon \nabla g_\varepsilon^0\big\|_{L^2(\Omega)} + \big\|\nabla g_\varepsilon^1\big\|_{L^2(\Omega)}\big) < \infty.$$

Set $\varphi_\varepsilon := \varepsilon^{1-\gamma} g_\varepsilon^0 + g_\varepsilon^1$ and $G_\varepsilon = (G_\varepsilon^0, G_\varepsilon^1) := (\varepsilon \nabla g_\varepsilon^0, \nabla g_\varepsilon^1)$.

(a) *If $G_\varepsilon \xrightarrow{2} G$ and $m_\varepsilon = o(\varepsilon^\gamma)$ as $\varepsilon \downarrow 0$, then*

$$\liminf_{\varepsilon \downarrow 0} m_\varepsilon^{-2} \mathcal{E}_\varepsilon(m_\varepsilon \varphi_\varepsilon) \geqslant \liminf_{\varepsilon \downarrow 0} \mathcal{Q}_\varepsilon(\theta_\varepsilon G_\varepsilon) \geqslant \mathcal{Q}(G), \tag{50}$$

where $\theta_\varepsilon : \Omega \to \{0, 1\}$ is defined by

$$\theta_\varepsilon(x) := \begin{cases} 1, & \text{if } |\nabla \varphi_\varepsilon| \leqslant (m_\varepsilon \varepsilon^\gamma)^{-1/2}, \\ 0, & \text{otherwise.} \end{cases}$$

(b) *If $G_\varepsilon \xrightarrow{2} G$, $m_\varepsilon = o(\varepsilon^\gamma)$ as $\varepsilon \downarrow 0$, and*

$$\limsup_{\varepsilon \downarrow 0} \|m_\varepsilon \nabla \varphi_\varepsilon\|_{L^\infty(\Omega)} = 0, \tag{51}$$

then

$$\lim_{\varepsilon \downarrow 0} m_\varepsilon^{-2} \mathcal{E}_\varepsilon(m_\varepsilon \varphi_\varepsilon) = \mathcal{Q}(G).$$

Remark 3. In Lemma 6, following [23], the function θ_ε is introduced, in order to truncate the peaks of G_ε. This is needed for exploiting the quadratic expansion (W3). Since both $\varepsilon \nabla g_\varepsilon^0$ and ∇g_ε^1 are assumed to be bounded sequences in $L^2(\Omega)$, we deduce, from the definition of θ_ε, the fact that $m_\varepsilon \varepsilon^{-\gamma} = o(1)$ and the Chebyshev inequality, that

$$\forall r < \infty \quad \|\theta_\varepsilon - 1\|_{L^r(\Omega)} \to 0, \quad \text{and} \quad \|\theta_\varepsilon\|_{L^\infty(\Omega)} \leqslant 1. \tag{52}$$

In the proof of Lemma 6 we need to pass to the limit in products of the form $f_\varepsilon \theta_\varepsilon$, where θ_ε satisfies (52), or $f_\varepsilon \chi_\varepsilon$, where χ_ε denotes the indicator of Ω_ε^0. This is done by appealing to the next two lemmas, the proofs of which are elementary and left to the reader.

Lemma 7. *Let $f_\varepsilon, \theta_\varepsilon$ be sequences in $L^2(\Omega)$ and assume that θ_ε satisfies (52), then the following implications are valid:*

$$\limsup_{\varepsilon \downarrow 0} \|f_\varepsilon\|_{L^2(\Omega)} < \infty \quad \Rightarrow \quad \|\theta_\varepsilon f_\varepsilon - f_\varepsilon\|_{L^p(\Omega)} \to 0 \quad \forall p < 2,$$

$$f_\varepsilon \rightharpoonup f \quad \Rightarrow \quad \theta_\varepsilon f_\varepsilon \rightharpoonup f,$$

$$f_\varepsilon \overset{2}{\rightharpoonup} f \quad \Rightarrow \quad \theta_\varepsilon f_\varepsilon \overset{2}{\rightharpoonup} f,$$

$$\left\{ \begin{matrix} f_\varepsilon \rightharpoonup f \\ (|f_\varepsilon|^2) \ equi\text{-}integrable \end{matrix} \right\} \quad \Rightarrow \quad \|\theta_\varepsilon f_\varepsilon - f_\varepsilon\|_{L^2(\Omega)} \to 0.$$

Lemma 8. *Suppose that f_ε be a sequence in $L^2(\Omega)$ and, as above, let χ_ε denote the set indicator function of Ω_ε^0. Then the following implications hold:*

$$f_\varepsilon \overset{2}{\rightharpoonup} f \quad \Rightarrow \quad \chi_\varepsilon f_\varepsilon \overset{2}{\rightharpoonup} \chi(y) f(x, y),$$

$$f_\varepsilon \overset{2}{\to} f \quad \Rightarrow \quad \chi_\varepsilon f_\varepsilon \overset{2}{\to} \chi(y) f(x, y),$$

where χ denotes the indicator function of Y^0.

Proof of Lemma 6. *Step* 1. Linearisation.
   We claim that the following statement holds for $i = 1, 2$: Let F_ε denote a sequence in $L^2(\Omega, \mathbb{R}^{d \times d})$, and let c_ε be a sequence of positive numbers converging to zero, such that

$$\limsup_{\varepsilon \to 0} c_\varepsilon \|F_\varepsilon\|_{L^\infty(\Omega)} = 0. \tag{53}$$

Then the convergence

$$\lim_{\varepsilon \downarrow 0} \left| c_\varepsilon^{-2} \int_\Omega W^i(I + c_\varepsilon F_\varepsilon) \, dx - \int_\Omega Q^i\big(F_\varepsilon(x)\big) \, dx \right| = 0 \tag{54}$$

holds. Indeed, thanks to (W3) we have

$$\left| c_\varepsilon^{-2} W^i(I + c_\varepsilon F_\varepsilon) - Q^i(F_\varepsilon) \right| \leqslant |F_\varepsilon|^2 r^i \big(c_\varepsilon |F_\varepsilon|\big) \leqslant |F_\varepsilon|^2 r^i \big(c_\varepsilon \|F_\varepsilon\|_{L^\infty(\Omega)}\big) \quad \text{a.e.}$$

Thanks to (53), and since F_ε is bounded in $L^2(\Omega)$, the right-hand side converges to zero in $L^1(\Omega)$, and (54) follows.

Step 2. Proof of part (a).

Since the energy densities W^0, W^1 are minimised at the identity, *cf.* (W2), we have

$$m_\varepsilon^{-2}\mathcal{E}_\varepsilon(\varphi_\varepsilon) \geqslant \left(m_\varepsilon\varepsilon^{-\gamma}\right)^{-2}\int_\Omega W^0\left(I + m_\varepsilon\varepsilon^{-\gamma}\theta_\varepsilon F_\varepsilon^0(x)\right)dx + m_\varepsilon^{-2}\int_\Omega W^1\left(I + m_\varepsilon\theta_\varepsilon F_\varepsilon^1(x)\right)dx, \quad (55)$$

where

$$F_\varepsilon^0 := \chi_\varepsilon\left(G_\varepsilon^0 + \varepsilon^\gamma G_\varepsilon^1\right), \qquad F_\varepsilon^1(x) := (1 - \chi_\varepsilon)G_\varepsilon^1. \quad (56)$$

Thanks to the definition of θ_ε we have $\|m_\varepsilon\varepsilon^{-\gamma}\theta_\varepsilon F_\varepsilon^0\|_{L^\infty(\Omega)} + \|m_\varepsilon\theta_\varepsilon F_\varepsilon^1\|_{L^\infty(\Omega)} \to 0$, so that we may apply (54) to the right-hand side in (55). We get

$$\liminf_{\varepsilon\downarrow0} m_\varepsilon^{-2}\mathcal{E}_\varepsilon(\varphi_\varepsilon) \geqslant \liminf_{\varepsilon\downarrow0}\mathcal{Q}_\varepsilon\left(\theta_\varepsilon F_\varepsilon^0, \theta_\varepsilon F_\varepsilon^1\right) = \liminf_{\varepsilon\downarrow0}\mathcal{Q}_\varepsilon\left(\theta_\varepsilon G_\varepsilon^1, \theta_\varepsilon G_\varepsilon^0\right),$$

where for the last identity we used the facts that $F_\varepsilon^1 = G_\varepsilon^1$ on Ω_ε^1 and

$$\left\|F_\varepsilon^0 - G_\varepsilon^0\right\|_{L^2(\Omega_\varepsilon^0)} = \left\|\varepsilon^\gamma\nabla g_\varepsilon^1\right\|_{L^2(\Omega_\varepsilon^0)} \to 0.$$

It remains to argue that

$$\liminf_{\varepsilon\downarrow0}\mathcal{Q}_\varepsilon(\theta_\varepsilon G_\varepsilon) \geqslant \mathcal{Q}(G).$$

In order to show this, notice that

$$Q_\varepsilon(\theta_\varepsilon G_\varepsilon) = \int_\Omega Q^0\left(\theta_\varepsilon\chi_\varepsilon G_\varepsilon^0\right)dx + \int_\Omega Q^1\left(\theta_\varepsilon(1 - \chi_\varepsilon)G_\varepsilon^1\right)dx. \quad (57)$$

From $G_\varepsilon \overset{2}{\rightharpoonup} G$ we deduce, using Lemma 7, Remark 3 and Lemma 8, that

$$\theta_\varepsilon\chi_\varepsilon G_\varepsilon^0 \overset{2}{\rightharpoonup} \chi(y)G^0(x, y), \qquad \theta_\varepsilon(1 - \chi_\varepsilon)G_\varepsilon^1 \overset{2}{\rightharpoonup} \left(1 - \chi(y)\right)G^1(x, y).$$

By appealing to the lower semicontinuity of convex integral functionals with respect to weak two-scale convergence (*cf.* [37, Proposition 1.3]), we deduce that the $\liminf$ of the right-hand side in (57) is bounded below by $\mathcal{Q}(G)$. This completes the argument.

Step 3. Proof of part (b).

We claim that

$$\lim_{\varepsilon\downarrow0}\left|\frac{1}{m_\varepsilon^2}\mathcal{E}_\varepsilon(m_\varepsilon\varphi_\varepsilon) - \mathcal{Q}_\varepsilon\left(\varepsilon\nabla g_\varepsilon^0, \nabla g_\varepsilon^1\right)\right| = 0. \quad (58)$$

Note that

$$m_\varepsilon^{-2} \mathcal{E}_\varepsilon(\varphi_\varepsilon) = \left(m_\varepsilon \varepsilon^{-\gamma}\right)^{-2} \int_\Omega W^0\left(I + m_\varepsilon \varepsilon^{-\gamma} F_\varepsilon^0(x)\right) dx + m_\varepsilon^{-2} \int_\Omega W^1\left(I + m_\varepsilon F_\varepsilon^1(x)\right) dx,$$

where F_ε^0 and F_ε^1 are defined in (56). By (51) we have $\|m_\varepsilon \varepsilon^{-\gamma} F_\varepsilon^0\|_{L^\infty(\Omega)} + \|m_\varepsilon F_\varepsilon^1\|_{L^\infty(\Omega)} \to 0$ and (54) yields

$$\left| m_\varepsilon^{-2} \mathcal{E}_\varepsilon(m_\varepsilon \varphi_\varepsilon) - \mathcal{Q}_\varepsilon\left(F_\varepsilon^0, F_\varepsilon^1\right) \right| \to 0 \quad \text{as } \varepsilon \downarrow 0.$$

Since $G_\varepsilon \xrightarrow{2} G$ we have, thanks to Lemma 8:

$$\chi_\varepsilon F_\varepsilon^0 \xrightarrow{2} \chi(y) G^0(x, y), \qquad (1 - \chi_\varepsilon) F_\varepsilon^1 \xrightarrow{2} \left(1 - \chi(y)\right) G^1(x, y).$$

Hence, the continuity of convex integral functionals with respect to strong two-scale convergence (*cf.* [36]) yields

$$\mathcal{Q}_\varepsilon\left(F_\varepsilon^0, F_\varepsilon^1\right) \to \mathcal{Q}(G),$$

which completes the argument. □

We are now in a position to prove the Γ-convergence statement for the energies $\mathcal{E}_\varepsilon$.

Proof of Theorem 1. *Step* 1. Part (a) (Compactness).
Thanks to (W2) we have

$$m_\varepsilon^{-2} \mathcal{E}_\varepsilon(m_\varepsilon \varphi_\varepsilon) \geqslant c_0 m_\varepsilon^{-2} \Phi_\varepsilon^\gamma(m_\varepsilon \varphi_\varepsilon).$$

Hence, the claim of Theorem 1(a) directly follows from Lemma 2.
Step 2. Part (b) (Lower bound).
Without loss of generality we assume that

$$\liminf_{\varepsilon \downarrow 0} m_\varepsilon^{-2} \mathcal{E}_\varepsilon(m_\varepsilon \varphi_\varepsilon) = \limsup_{\varepsilon \downarrow 0} m_\varepsilon^{-2} \mathcal{E}_\varepsilon(m_\varepsilon \varphi_\varepsilon) < \infty.$$

Furthermore, thanks to Lemma 2, we can assume in addition that $\nabla g_\varepsilon^1 \xrightarrow{2} \nabla g^1 + \nabla_y \psi$ for some $\psi \in L^2(\Omega, \mathcal{A}(Y^0))$, so that

$$G_\varepsilon := \left(\varepsilon \nabla g_\varepsilon^0, \nabla g_\varepsilon^1\right) \xrightarrow{2} \left(\nabla_y g^0, \nabla g^1 + \nabla_y \psi\right) =: G.$$

Applying Lemma 6(a) yields

$$\liminf_{\varepsilon \downarrow 0} m_\varepsilon^{-2} \mathcal{E}_\varepsilon(m_\varepsilon \varphi_\varepsilon) \geqslant \int_{\Omega \times Y^0} Q^0\left(\nabla_y g^0(x, y)\right) dx \, dy + \int_{\Omega \times Y^1} Q^1\left(\nabla g^1(x) + \nabla_y \psi(x, y)\right) dx \, dy.$$

This completes the argument, since the right-hand side is bounded from below by $\mathcal{E}_{\text{small}}(g^0, g^1)$.

Step 3. Part (c) (Upper bound).

Choose $\psi \in L^2(\Omega, \mathcal{A}(Y^0))$ such that

$$\int_{\Omega \times Y^1} Q(\nabla g^1(x) + \nabla_y \psi(x, y)) \, dx \, dy = \int_{\Omega} Q_{\text{hom}}^1(\nabla g^1(x)) \, dx. \tag{59}$$

Let φ_ε denote the sequence associated with g^0, g^1 and ψ via Lemma 3 with $c_\varepsilon := m_\varepsilon$. In view of (20), applying Lemma 6(b) yields

$$\lim_{\varepsilon \downarrow 0} m_\varepsilon^{-2} \mathcal{E}_\varepsilon(m_\varepsilon \varphi_\varepsilon) = \int_{\Omega \times Y} Q^0(\nabla_y g^0(x, y)) \, dx \, dy + \int_{\Omega \times Y^1} Q^1(\nabla g^1(x) + \nabla_y \psi(x, y)) \, dx \, dy.$$

It follows from (59) that the right-hand side equals $\mathcal{E}_{\text{small}}(g^0, g^1)$. $\quad\square$

Proof of Proposition 1. *Step* 1. *A priori* estimate.

We claim that for every sequence φ_ε in $H_0^1(\Omega)$ the following implication holds:

$$\limsup_{\varepsilon \downarrow 0} I_\varepsilon(\varphi_\varepsilon) < \infty \quad \Rightarrow \quad \limsup_{\varepsilon \downarrow 0} m_\varepsilon^{-2} \mathcal{E}_\varepsilon(m_\varepsilon \varphi_\varepsilon) < \infty. \tag{60}$$

Indeed, we have

$$\left| \int_{\Omega} \ell_\varepsilon \cdot m_\varepsilon \varphi_\varepsilon \, dx \right| \leq m_\varepsilon \big(\|\ell_\varepsilon\|_{L^2(\Omega)} \|g_\varepsilon^1\|_{L^2(\Omega)} + \varepsilon^{1-\gamma} \|\ell_\varepsilon\|_{L^2(\Omega_\varepsilon^0)} \|g_\varepsilon^0\|_{L^2(\Omega_\varepsilon^0)} \big)$$

$$\overset{(25a)}{\lesssim} m_\varepsilon \big(\|m_\varepsilon g_\varepsilon^1\|_{L^2(\Omega)} + \|m_\varepsilon g_\varepsilon^0\|_{L^2(\Omega)} \big) \overset{(17)}{\lesssim} m_\varepsilon^2 \sqrt{m_\varepsilon^{-2} \phi_\varepsilon^\gamma(m_\varepsilon \varphi_\varepsilon)}$$

$$\overset{(W2)}{\lesssim} m_\varepsilon^2 \sqrt{m_\varepsilon^{-2} \mathcal{E}_\varepsilon(m_\varepsilon \varphi_\varepsilon)}.$$

Combining this with the definition of I_ε we get

$$m_\varepsilon^{-2} \mathcal{E}_\varepsilon(m_\varepsilon \varphi_\varepsilon) \lesssim I_\varepsilon(\varphi_\varepsilon) + \sqrt{m_\varepsilon^{-2} \mathcal{E}_\varepsilon(\varphi_\varepsilon)},$$

which implies (60).

Step 2. The proof of parts (a) and (b).

The existence of a minimiser to I_{small} follows by the direct method. The minimiser (g_*^0, g_*^1) is unique, since the implication

$$\int_{\Omega \times Y} Q^0(\nabla_y \tilde{g}^0(x, y)) \, dx \, dy + \int_{\Omega} Q_{\text{hom}}^1(\nabla \tilde{g}^1(x)) \, dx = 0 \quad \Rightarrow \quad \tilde{g}^0 = \tilde{g}^1 = 0$$

holds for all $\tilde{g}^0 \in L^2(\Omega, H_0^1(Y^0))$ and $\tilde{g}^1 \in H_0^1(\Omega)$.

The remaining claims of Proposition 1 follow from the standard Γ-convergence arguments (*cf.* [20, Corollary 7.20]), provided the functionals I_ε, $\varepsilon > 0$, are equi-coercive and Γ-converge to I_{small}. Indeed, thanks to (25c), it is easy to check that $\varphi_\varepsilon \overset{2}{\rightharpoonup} (g^0, g^1)$ implies

$$\lim_{\varepsilon \downarrow 0} \frac{1}{m_\varepsilon^2} \int_\Omega \ell_\varepsilon \cdot m_\varepsilon \varphi_\varepsilon \, dx = \int_\Omega \left(\int_{Y^0} \ell^0 \cdot g^0 \, dy + \ell^1 \cdot g^1 \right) dx, \tag{61}$$

since the integral on the left-hand side only involves products of weakly and strongly two-scale convergent factors, *cf.* [36, Proposition 2.8]). In combination with Theorem 1, this implies that I_ε Γ-converges to I_{small}. In addition, the trivial inequality

$$\inf_{\varphi \in H_0^1(\Omega)} I_\varepsilon(\varphi) \leqslant I_\varepsilon(0) = 0,$$

combined with (60) and Lemma 2, proves that the functionals I_ε are equi-coercive. $\square$

For the proof of Theorem 2 we make use of the following lemma:

Lemma 9 (Decomposition Lemma, see [22,27]). *Let $v \in H_0^1(\Omega)$ and let $v_\varepsilon \in H^1(\Omega)$ be a sequence with $v_\varepsilon \rightharpoonup v$ in $H^1(\Omega)$. Then there exists a sequence $\overline{V}_\varepsilon \in H^1(\Omega)$ such that the following properties hold for a subsequence of v_ε (not relabelled):*

(a) *$\overline{V}_\varepsilon \rightharpoonup v$ in $H^1(\Omega)$;*
(b) *$\overline{V}_\varepsilon = v_\varepsilon$ in a neighbourhood of $\partial\Omega$;*
(c) *$(|\nabla \overline{V}_\varepsilon|^2)$ is equi-integrable;*
(d) *$|\{x \in \Omega : v_\varepsilon(x) \neq \overline{V}_\varepsilon(x)\}| \to 0$ as $\varepsilon \downarrow 0$.*

Proof of Theorem 2. It suffices to prove the theorem for a subsequence. Throughout the proof we write

$$G_\varepsilon := \left(\varepsilon \nabla g_\varepsilon^0, \nabla g_\varepsilon^1 \right), \qquad G := \left(\nabla_y g_*^0, \nabla g_*^1 + \nabla_y \psi_* \right).$$

Furthermore, we make use of the functionals $\mathcal{Q}_\varepsilon$ and $\mathcal{Q}$ introduced at the beginning of Section 4.2. Recall that

$$\mathcal{E}_{\text{small}} \left(g_*^0, g_*^1 \right) = Q(G).$$

Step 1. Convergence of φ_ε and of the corresponding energy values.
We claim that, as $\varepsilon \downarrow 0$, one has

$$G_\varepsilon \overset{2}{\rightharpoonup} G, \tag{62}$$

$$I_\varepsilon(\varphi_\varepsilon) \to I_{\text{small}} \left(g_*^0, g_*^1 \right), \tag{63}$$

$$m_\varepsilon^{-2} \mathcal{E}_\varepsilon(m_\varepsilon \varphi_\varepsilon) \to \mathcal{E}_{\text{small}} \left(g_*^0, g_*^1 \right). \tag{64}$$

Indeed, from Proposition 1 we immediately deduce that $\varphi_\varepsilon \overset{2}{\rightharpoonup} (g_*^0, g_*^1)$ and (63). Furthermore, in view of the continuity of the loading term, *cf.* (61), this implies (64). For (62), it remains to argue that $\nabla g_\varepsilon^1 \overset{2}{\rightharpoonup}$

$\nabla g_*^1 + \nabla_y \psi_*$. Thanks to $\varphi_\varepsilon \xrightarrow{2} (g_*^0, g_*^1)$ and Lemma 2 we have, up to a subsequence, $\nabla g_\varepsilon^1 \xrightarrow{2} \nabla g_*^1 + \nabla_y \psi$ for some $\psi \in L^2(\Omega, \mathcal{A}(Y^0))$. Furthermore, from (64) and Lemma 6(a) we infer that

$$E_{\text{small}}\big(g_0^*, g_1^*\big) = \liminf_{\varepsilon \downarrow 0} m_\varepsilon^{-2} \mathcal{E}_\varepsilon(m_\varepsilon \varphi_\varepsilon) \geqslant \int_{\Omega \times Y^0} Q^0\big(\nabla_y g_*^0\big) \, dx \, dy + \int_{\Omega \times Y^1} Q^1\big(\nabla g_*^1 + \nabla_y \psi\big) \, dx \, dy.$$

This, in particular, implies

$$\int_{\Omega \times Y^1} Q^1\big(\nabla g_*^1 + \nabla_y \psi\big) \, dx \, dy = \int_\Omega Q_{\text{hom}}^1\big(\nabla g_*^1\big) \, dx.$$

In view of (28) we conclude that $\psi = \psi_*$ and (62) follows.

Step 2. Equi-integrable decomposition.

We claim that for a subsequence (not relabelled) there exist sequences $\bar{g}_\varepsilon^0, \overline{G}_\varepsilon^1 \in H_0^1(\Omega)$ such that $\overline{G}_\varepsilon := (\varepsilon \nabla \overline{G}_\varepsilon^0, \nabla \overline{G}_\varepsilon^1)$ satisfies

$$\overline{G}_\varepsilon - G_\varepsilon \xrightarrow{2} 0, \qquad \|\overline{G}_\varepsilon - G_\varepsilon\|_{L^p(\Omega)} \to 0, \tag{65}$$

and

$$\mathcal{Q}_\varepsilon(\overline{G}_\varepsilon) \to \mathcal{E}_{\text{small}}\big(g_*^0, g_*^1\big). \tag{66}$$

To show the above, notice that thanks to Lemma 9 there exist sequences $\overline{G}_\varepsilon^0, \overline{G}_\varepsilon^1 \in H_0^1(\Omega)$ such that

- $(\varepsilon^2 |\nabla \overline{G}_\varepsilon^0|^2)$ and $(|\nabla \bar{g}_\varepsilon^1|^2)$ are equi-integrable,
- the indicator function $\bar{\theta}_\varepsilon$ defined by

$$\bar{\theta}_\varepsilon(x) := \begin{cases} 1, & \text{if } \overline{G}_\varepsilon^0(x) = g_\varepsilon^0(x) \text{ and } \overline{G}_\varepsilon^1(x) = g_\varepsilon^1(x), \\ 0, & \text{otherwise}, \end{cases} \tag{67}$$

satisfies (52).

Since $\overline{G}_\varepsilon - G_\varepsilon = (1 - \bar{\theta}_\varepsilon)(\overline{G}_\varepsilon - G_\varepsilon)$ (and $p < 2$), the convergence (65) follows from the boundedness of the sequence $(\overline{G}_\varepsilon - G_\varepsilon)$ in $L^2(\Omega)$, Lemma 7, and Hölder's inequality.

We prove (66). Thanks to (62) we have $\overline{G}_\varepsilon \xrightarrow{2} G$, so that (due to the lower semicontinuity of convex integral functionals with respect to weak two-scale convergence, *cf.* [37, Proposition 1.3]):

$$\liminf_{\varepsilon \downarrow 0} \mathcal{Q}_\varepsilon(\overline{G}_\varepsilon) \geqslant \mathcal{Q}(G) = \mathcal{E}_{\text{small}}\big(g_*^0, g_*^1\big).$$

Hence, for (66) it suffices to prove the opposite estimate, i.e. $\limsup_{\varepsilon \downarrow 0} \mathcal{Q}_\varepsilon(\overline{G}_\varepsilon) \leqslant \mathcal{E}_{\text{small}}(G)$, which, thanks to (50) and (64), follows from

$$\limsup_{\varepsilon \downarrow 0}\big(\mathcal{Q}_\varepsilon(\overline{G}_\varepsilon) - \mathcal{Q}_\varepsilon(\theta_\varepsilon G_\varepsilon)\big) \leqslant 0. \tag{68}$$

In order to show (68) notice that since the supports of $\bar{\theta}_\varepsilon$ and $(1 - \bar{\theta}_\varepsilon)$ are disjoint, and because $\bar{\theta}_\varepsilon \theta_\varepsilon G_\varepsilon = \bar{\theta}_\varepsilon \theta_\varepsilon \overline{G}_\varepsilon$ (cf. (67)), an expansion of the squares yields

$$\mathcal{Q}_\varepsilon(\overline{G}_\varepsilon) - \mathcal{Q}_\varepsilon(\theta_\varepsilon G_\varepsilon) = \mathcal{Q}_\varepsilon(\bar{\theta}_\varepsilon \overline{G}_\varepsilon) + \mathcal{Q}_\varepsilon((1 - \bar{\theta}_\varepsilon)\overline{G}_\varepsilon) - \mathcal{Q}_\varepsilon(\bar{\theta}_\varepsilon \theta_\varepsilon G_\varepsilon) - \mathcal{Q}_\varepsilon((1 - \bar{\theta}_\varepsilon)\theta_\varepsilon G_\varepsilon)$$

$$= \mathcal{Q}_\varepsilon(\bar{\theta}_\varepsilon(1 - \theta_\varepsilon)\overline{G}_\varepsilon) + \mathcal{Q}_\varepsilon((1 - \bar{\theta}_\varepsilon)\overline{G}_\varepsilon) - \mathcal{Q}_\varepsilon((1 - \bar{\theta}_\varepsilon)\theta_\varepsilon G_\varepsilon)$$

$$\leqslant \mathcal{Q}_\varepsilon(\bar{\theta}_\varepsilon(1 - \theta_\varepsilon)\overline{G}_\varepsilon) + \mathcal{Q}_\varepsilon((1 - \bar{\theta}_\varepsilon)\overline{G}_\varepsilon).$$

It is easy to check that $\bar{\theta}_\varepsilon(1 - \theta_\varepsilon)$ and $1 - \bar{\theta}_\varepsilon$ converge to zero in $L^r(\Omega)$ for all $r < \infty$. Hence, since $|\overline{G}_\varepsilon|^2$ is equi-integrable, Lemma 7 implies that the right-hand side of the previous estimate converges to zero, and (68) follows.

Step 3. Error estimate.

We claim that

$$\int_{\Omega_\varepsilon^0} \varepsilon \big|\mathrm{sym}\nabla\big(\bar{g}_\varepsilon^0 - g_{*,\varepsilon}^0\big)\big|^2 \, dx + \int_{\Omega_\varepsilon^1} \big|\mathrm{sym}\nabla\big(\bar{g}_\varepsilon^1 - g_{*,\varepsilon}^1\big)\big|^2 \, dx \to 0 \quad \text{as } \varepsilon \downarrow 0. \tag{69}$$

For the argument set $G_{*,\varepsilon} := (\varepsilon\nabla g_{*,\varepsilon}^0, \nabla g_{*,\varepsilon}^1)$. In view of (13) it suffices to argue that

$$\mathcal{Q}_\varepsilon(\overline{G}_\varepsilon - G_{*,\varepsilon}) \to 0 \quad \text{as } \varepsilon \downarrow 0.$$

The latter can be seen as follows: We have

$$\mathcal{Q}_\varepsilon(\overline{G}_\varepsilon - G_{*,\varepsilon}) = \mathcal{Q}_\varepsilon(\overline{G}_\varepsilon) - \mathcal{Q}_\varepsilon(G_{*,\varepsilon}) + 2\mathcal{B}_\varepsilon(G_{*,\varepsilon}; G_{*,\varepsilon} - \overline{G}_\varepsilon),$$

where $\mathcal{B}_\varepsilon$ denotes the bilinear form associated with $\mathcal{Q}_\varepsilon$.

The difference of the two quadratic terms on the right-hand side converges to zero, since $G_{*,\varepsilon}$ is associated with a recovery sequence, and thanks to (66). On the other hand, since $G_{*,\varepsilon}$ strongly two-scale converges, and $G_{*,\varepsilon} - \bar{G}_\varepsilon \overset{2}{\rightharpoonup} 0$ by (65), we deduce that

$$\lim_{\varepsilon \downarrow 0} \mathcal{B}_\varepsilon(G_{*,\varepsilon}; G_{*,\varepsilon} - \overline{G}_\varepsilon) = 0,$$

as $\mathcal{B}_\varepsilon(G_{*,\varepsilon}; G_{*,\varepsilon} - \overline{G}_\varepsilon)$ only involves products between a weakly and a strongly two-scale convergent factor (*cf.* [36, Proposition 2.8]).

Step 4. Conclusion (Proof of (31)).

We split the estimate into

$$\int_\Omega \big(|g_\varepsilon^0 - g_{*,\varepsilon}^0|^p + |\varepsilon\nabla g_\varepsilon^0 - \varepsilon\nabla g_{*,\varepsilon}^0|^p\big) \, dx \to 0 \quad \text{as } \varepsilon \downarrow 0, \tag{70}$$

$$\int_\Omega \big(|g_\varepsilon^1 - g_{*,\varepsilon}^1|^p + |\nabla g_\varepsilon^1 - \nabla g_{*,\varepsilon}^1|^p\big) \, dx \to 0 \quad \text{as } \varepsilon \downarrow 0. \tag{71}$$

Thanks to (65) and Step 3 we have

$$\int_{\Omega_\varepsilon^0} \varepsilon \big|\mathrm{sym}\nabla\big(g_\varepsilon^0 - g_{*,\varepsilon}^0\big)\big|^p \, dx + \int_{\Omega_\varepsilon^1} \big|\mathrm{sym}\nabla\big(g_\varepsilon^1 - g_{*,\varepsilon}^1\big)\big|^p \, dx \to 0 \quad \text{as } \varepsilon \downarrow 0. \tag{72}$$

Argument for (70): Set $\eta_\varepsilon^0 := g_\varepsilon^0 - g_{*,\varepsilon}^0$. Since $\eta_\varepsilon^0 \in H_0^1(\Omega_\varepsilon^0) \subset H_0^1(\Omega)$, Korn's inequality yields

$$\int_\Omega |\nabla \eta_\varepsilon^0|^p \, dx \leqslant C \int_\Omega |\mathrm{sym} \nabla \eta_\varepsilon^0|^p \, dx = C \int_{\Omega_\varepsilon^0} |\mathrm{sym} \nabla \eta_\varepsilon^0|^p \, dx,$$

where $C > 0$ only depends on Ω, p, and d. Combined with the improved Poincaré inequality (43) and (72), (70) follows.

Argument for (71): We claim that (71) follows from

$$\left\| \mathrm{sym} \nabla \eta_\varepsilon^1 \right\|_{L^p(\Omega_\varepsilon^0)} \to 0, \tag{73}$$

where $\eta_\varepsilon^1 := g_\varepsilon^1 - g_{*,\varepsilon}^1$. Indeed, since η_ε^1 vanishes on $\partial \Omega$, (73), (72) and Korn's first inequality yield (71).

Thanks to the definition of Ω_ε^0, the argument for (73) can be reduced to the following statement: For all $\xi \in Z_\varepsilon := \{\xi \in \mathbb{Z}^d : \varepsilon(\xi + Y) \subset \Omega\}$ we have

$$\int_{\varepsilon(\xi+Y^0)} |\mathrm{sym} \nabla \eta_\varepsilon|^p \, dx \lesssim \int_{\varepsilon(\xi+Y^1)} |\mathrm{sym} \nabla \eta_\varepsilon|^p \, dx. \tag{74}$$

For the argument consider the rescaled function

$$\hat{\eta}_\varepsilon : Y \mapsto \mathbb{R}^d, \quad \hat{\eta}_\varepsilon(y) := \eta_\varepsilon\big(\varepsilon(\xi + y)\big) - Sy + c,$$

where $S \in \mathbb{R}_{\mathrm{skew}}^{d \times d}$ and $c \in \mathbb{R}^d$ are chosen such that the Poincaré and Korn inequalities yield

$$\int_{Y^1} \big(|\hat{\eta}_\varepsilon|^p + |\nabla \hat{\eta}_\varepsilon|^p\big) \, dy \lesssim \int_{Y^1} |\mathrm{sym} \nabla \hat{\eta}_\varepsilon|^p \, dy. \tag{75}$$

Since both g_ε^1 and $g_{*,\varepsilon}^1$ satisfy (16)(ii), we have $-\triangle \hat{\eta}_\varepsilon = 0$ in Y^0 in the distributional sense. Hence, thanks to Assumption 1 and (75), we have

$$\int_{Y^0} |\nabla \hat{\eta}_\varepsilon|^p \, dy \lesssim \int_{Y^1} |\mathrm{sym} \nabla \hat{\eta}_\varepsilon|^p \, dy,$$

and thus

$$\int_{\varepsilon(\xi+Y^0)} |\mathrm{sym} \nabla \eta_\varepsilon|^p \, dx = \varepsilon^{d-p} \int_{Y^0} |\mathrm{sym} \nabla \hat{\eta}_\varepsilon|^p \, dy \lesssim \varepsilon^{d-p} \int_{Y^1} |\mathrm{sym} \nabla \hat{\eta}_\varepsilon|^p \, dy$$

$$= \int_{\varepsilon(\xi+Y^1)} |\mathrm{sym} \nabla \eta_\varepsilon|^p \, dx. \qquad \square$$

Proof of Proposition 2. *Step* 1. Proof of (a).

Arguing as in Step 1 in the proof of Proposition 1 we find that $\limsup_{\varepsilon \downarrow 0} \varepsilon^{-2\alpha} \mathcal{E}_\varepsilon(\varepsilon^\alpha \varphi_\varepsilon) < \infty$, and thus the compactness part of Theorem 1 (and the fact that two-scale convergence implies weak convergence) yields (for a subsequence, which we do not relabel):

$$\varphi_\varepsilon \overset{2}{\rightharpoonup} \big(g^0, g^1\big) \quad \text{and} \quad \varphi_\varepsilon \rightharpoonup \varphi := g^1 + \int_{Y^0} g^0 \, dy. \tag{76}$$

Step 2. Proof of (b).

We may restrict to the case

$$\liminf_{\varepsilon \downarrow 0} I_\varepsilon^\alpha(\varphi_\varepsilon) = \limsup_{\varepsilon \downarrow 0} I_\varepsilon^\alpha(\varphi_\varepsilon) < \infty.$$

Thanks to Step 1 we may assume without loss of generality that (76) holds. From the lower bound part of Theorem 1 (and the fact that $\int_\Omega f \cdot \varphi_\varepsilon \, dx \to \int_\Omega f \cdot \varphi \, dx$) we thus deduce that

$$\liminf_{\varepsilon \downarrow 0} I_\varepsilon^\alpha(\varphi_\varepsilon) \geqslant I_{\text{small}}(g^0, g^1).$$

With the definition of $\overline{Q}$ from the introduction and $\overline{G} := \int_{Y_0} g^0 \, dy$ we get the inequality:

$$
\begin{aligned}
I_{\text{small}}(g^0, g^1) &\geqslant \int_\Omega Q_{\text{hom}}^1(\nabla g^1(x)) - (g^1(x) + \overline{G}(x)) \cdot f(x) \, dx \\
&\quad + \int_\Omega \min\left\{ \int_{Y^0} Q^0(\nabla_y \tilde{g}^0(y)) : \tilde{g}^0 \in H_0^1(Y^0), \int_{Y^0} \tilde{g}^0(y) = \overline{G}(x) \right\} \\
&= \int_\Omega Q_{\text{hom}}^1(\nabla g^1(x)) + \overline{Q}(\overline{G}(x)) - (g^1(x) + \overline{G}(x)) \cdot f(x) \, dx.
\end{aligned}
$$

Since $\varphi = g^1 + \bar{G}$ by (76), the right-hand side is bounded from below by $\bar{I}_{\text{small}}(\varphi)$, which completes the argument for (b).

Step 3. Proof of (c).

Let $\varphi \in L^2(\Omega)$. It suffices to argue that there exist $g^0 \in L^2(\Omega, H_0^1(Y^0))$ and $g^1 \in H_0^1(\Omega)$ with $\varphi = g^1 + \int_{Y^0} g^0 \, dy$ and

$$\bar{I}_{\text{small}}(\varphi) = I_{\text{small}}(g^0, g^1). \tag{77}$$

Indeed, in that case, we can find by part (c) of Theorem 1 a sequence $\varphi_\varepsilon \in H_0^1(\Omega)$ such that $\varphi_\varepsilon \overset{2}{\rightharpoonup} (g^0, g^1)$ and $\lim_{\varepsilon \downarrow 0} I_\varepsilon^\alpha(\varphi_\varepsilon) = I_{\text{small}}(g^0, g^1)$. Since $\varphi_\varepsilon \overset{2}{\rightharpoonup} (g^0, g^1)$ implies $\varphi_\varepsilon \rightharpoonup \varphi = g^1 + \int_{Y^0} g^0 \, dy$ weakly in $L^2(\Omega)$, we deduce from (77) that φ_ε is the sought for recovery sequence. In order to prove (77), let ρ_i ($i = 1, 2, 3$) denote the unique minimiser in $H_0^1(Y^0)$ to the functional

$$H_0^1(Y^0) \ni \rho \mapsto \int_{Y^0} Q^0(\nabla \rho(y)) \, dy \quad \text{subject to} \int_{Y^0} \rho \, dy = e_i,$$

and set

$$\bar{q}_i := \int_{Y^0} Q^0(\nabla \rho_i(y)) \, dy.$$

Then it is easy to check that

$$\overline{Q}(G) = \sum_{i=1}^3 G_i^2 \bar{q}_i \quad \text{for all } G \in \mathbb{R}^3,$$

and since $\bar{q}_1, \ldots, \bar{q}_3 > 0$, we deduce that $\overline{Q}$ is a positive definite quadratic form. Hence, since $Q^1_{\text{hom}}(F) \geqslant c|\text{sym}F|^2$ for some $c > 0$, we deduce that we can find a unique function $g^1 \in H^1_0(\Omega)$ that minimizes the functional

$$H^1_0(\Omega) \ni g \mapsto \int_\Omega Q^1_{\text{hom}}\big(\nabla g(x)\big) + \overline{Q}\big(\varphi(x) - g(x)\big) \, dx. \tag{78}$$

Setting $g^0(x, y) := \sum_{i=1}^3 ((\varphi(x) - g(x)) \cdot e_i)\rho_i(y)$ we deduce that $\int_\Omega \overline{Q}(\varphi(x) - g(x)) \, dx = \int_{\Omega \times Y^0} Q^0(\nabla_y g^0(x, y)) \, dy$, and thus (thanks to the definition of $\bar{I}_{\text{small}}$) (77) follows.

Step 4. Proof of (d).

By Proposition 1 we have $\varphi_\varepsilon \xrightarrow{2} (g^0_*, g^1_*)$ and thus $\varphi_\varepsilon \rightharpoonup \tilde{\varphi} := g^1_* + \int_{Y^0} g^0_* \, dy$. By definition of $\bar{I}_{\text{small}}$ we have

$$I_{\text{small}}\big(g^0_*, g^1_*\big) \geqslant \bar{I}_{\text{small}}(\tilde{\varphi}) \geqslant \inf_{L^2(\Omega)} \bar{I}_{\text{small}}. \tag{79}$$

On the other hand, the map $L^2(\Omega) \ni \varphi \mapsto g \in H^1_0(\Omega)$ with g minimizing the functional in (78) is linear and bounded; hence, we deduce that $\bar{I}_{\text{small}}$ is quadratic and strictly convex. It thus admits a unique minimiser $\varphi_* \in L^2(\Omega)$. By Step 3 (cf. (77)) we associated with φ_* a pair (g^0, g^1) such that $\bar{I}_{\text{small}}(\varphi_*) = I_{\text{small}}(g^0, g^1)$. Combined with (79) we get

$$\min_{L^2(\Omega)} \bar{I}_{\text{small}} = \bar{I}_{\text{small}}(\varphi_*) = I_{\text{small}}\big(g^0, g^1\big) \geqslant I_{\text{small}}\big(g^0_*, g^1_*\big) \geqslant \bar{I}_{\text{small}}(\tilde{\varphi}) \geqslant \min_{L^2(\Omega)} \bar{I}_{\text{small}}.$$

Hence, equality holds everywhere and the claimed identities follow from the strict convexity of $\bar{I}_{\text{small}}$. The convergence of $\inf_{L^2(\Omega)} I^\alpha_\varepsilon \to \min_{L^2(\Omega)} \bar{I}_{\text{small}}$ follows from part (b) and (c) by standard arguments from theory of Γ-convergence. $\square$

4.3. Proof of Theorem 3: Finite strain regime

We define, for $g^0_\varepsilon \in H^1_0(\Omega^0_\varepsilon)$, $g^0 \in L^2(\Omega, H^1_0(Y^0))$, and $g^1_\varepsilon, g^1 \in H^1_0(\Omega)$, the following functionals:

$$I^0_\varepsilon\big(g^0_\varepsilon\big) := \int_{\Omega^0_\varepsilon} \big(W^0\big(I + \varepsilon\nabla g^0_\varepsilon(x)\big) - \varepsilon^{1-2\gamma}\ell_\varepsilon \cdot g^0_\varepsilon\big) \, dx,$$

$$I^0_0\big(g^0\big) := \int_{\Omega \times Y^0} \big(\mathcal{Q}W^0\big(I + \nabla_y g^0(x, y)\big) - \ell^0 \cdot g^0\big) \, dx \, dy,$$

$$I^1_\varepsilon\big(g^1_\varepsilon\big) := \varepsilon^{-2\gamma} \int_{\Omega^1_\varepsilon} \left(W^1\big(I + \varepsilon^\gamma \nabla g^1_\varepsilon(x)\big) - \int_\Omega \varepsilon^{-\gamma}\ell_\varepsilon \cdot g^1_\varepsilon\right) dx,$$

$$I^1_0\big(g^1\big) := \int_\Omega \big(Q^1_{\text{hom}}\big(\nabla g^1(x)\big) - \ell^1 \cdot g^1\big) \, dx.$$

Thanks to the Lipschitz condition (32b), we can decompose I_ε into the sum $I^0_\varepsilon + I^1_\varepsilon$ at the expense of a small error. More precisely, the following lemma holds.

Lemma 10. *Suppose that $m_\varepsilon = \varepsilon^\gamma$. There exists a constant $C > 0$ such that for all $\varepsilon > 0$ and $\varphi_\varepsilon \in H_0^1(\Omega)$, $\varepsilon^{1-\gamma} g_\varepsilon^0 + g_\varepsilon^1 \overset{(16)}{:=} \varphi_\varepsilon$ we have*

$$\left| I_\varepsilon(\varphi_\varepsilon) - \left(I_\varepsilon^0(g_\varepsilon^0) + I_\varepsilon^1(g_\varepsilon^1) \right) \right| \leqslant C\varepsilon^\gamma \left(1 + \Phi_\varepsilon^\gamma(\varphi_\varepsilon) \right).$$

Proof. Note that

$$\begin{aligned}
&\left| I_\varepsilon(\varphi_\varepsilon) - \left(I_\varepsilon^0(g_\varepsilon^0) + I_\varepsilon^1(g_\varepsilon^1) \right) \right| \\
&= \left| \int_{\Omega_\varepsilon^0} W^0\left(I + \varepsilon^\gamma \left(\varepsilon^{1-\gamma} \nabla g_\varepsilon^0 + \nabla g_\varepsilon^1 \right) \right) - W^0\left(I + \varepsilon \nabla g_\varepsilon^0 \right) dx \right|.
\end{aligned}$$

In view of (32b) and (17), the statement follows. □

The following lemma is a simple consequence of [16, Lemma 21, Lemma 22] and (25c):

Lemma 11. *Assume (25a), (25c) and $m_\varepsilon = e^\gamma$.*

(a) *Consider a sequence $g_\varepsilon^0 \in H_0^1(\Omega_\varepsilon^0)$. If $g_\varepsilon^0 \overset{2}{\to} g^0$ and $\varepsilon \nabla g_0^1 \overset{2}{\rightharpoonup} \nabla_y g^0$, then*

$$\liminf_{\varepsilon \downarrow 0} I_\varepsilon^0(g_\varepsilon^0) \geqslant I_0^0(g^0).$$

(b) *For all $g^0 \in L^2(\Omega, H_0^1(Y^0))$ there exists a sequence $g_\varepsilon^0 \in H_0^1(\Omega_\varepsilon^0)$ such that $g_\varepsilon^0 \overset{2}{\to} g^0$, $\varepsilon \nabla g_\varepsilon^0 \overset{2}{\rightharpoonup} \nabla_y g^0$, and*

$$\lim_{\varepsilon \downarrow 0} I_\varepsilon^0(g_\varepsilon^0) = I_0^0(g^0).$$

For the stiff part one can prove (similar to Lemma 6) the following lemma:

Lemma 12. *Assume that (25a)–(25c) hold.*

(a) *Consider $g_\varepsilon^1 \in H_0^1(\Omega)$. If $g_\varepsilon^1 \rightharpoonup g^1$ weakly in $H^1(\Omega)$, then*

$$\liminf_{\varepsilon \downarrow 0} I_\varepsilon^1(g_\varepsilon^1) \geqslant I_0^1(g^1).$$

(b) *For all $g^1 \in H_0^1(\Omega)$ there exists a sequence $g_\varepsilon^1 \in H_0^1(\Omega)$ such that*

$$g_\varepsilon^1 \rightharpoonup g^1 \quad \text{weakly in } H^1(\Omega), \quad \text{and} \quad \lim_{\varepsilon \downarrow 0} I_\varepsilon^1(g_\varepsilon^1) = I_0^1(g^1).$$

We proceed to the proof of Theorem 3.

Proof of Theorem 3. *Step* 1. The proof of parts (a) and (b).
Statement (a) and (b) directly follow from Lemma 10, Lemma 11 and Lemma 12.
Step 2. Proof of (33): convergence of the minima.

For brevity set

$$e_\varepsilon := \inf_{\varphi \in H_0^1(\Omega)} \varepsilon^{-2\gamma} I_\varepsilon(\varphi_\varepsilon), \qquad e_0 := \inf_{\substack{g^0 \in L^2(\Omega, H_0^1(Y^0)) \\ g^1 \in H_0^1(\Omega)}} I_{\text{finite}}(g^0, g^1).$$

We prove (33) in the form of the two inequalities

$$\limsup_{\varepsilon \downarrow 0} e_\varepsilon \leqslant e_0, \qquad \liminf_{\varepsilon \downarrow 0} e_\varepsilon \geqslant e_0. \tag{80}$$

The argument for the first inequality in (80) is standard: for $\delta > 0$ choose (g^0, g^1) with $I_{\text{finite}}(g^0, g^1) \leqslant e_0 + \delta$. By part (b) there exists a recovery sequence φ_ε, so that $I_\varepsilon(\varphi_\varepsilon) \to I_{\text{finite}}(g^0, g^1)$. Hence

$$\limsup_{\varepsilon \downarrow 0} e_\varepsilon \leqslant \limsup_{\varepsilon \downarrow 0} I_\varepsilon(\varphi_\varepsilon) = I_{\text{finite}}(g^0, g^1) \leqslant e_0 + \delta.$$

Since this is valid for all $\delta > 0$, the first inequality in (80) follows.

Next, we prove the second inequality in (80). Let φ_ε denote a sequence with the property $\liminf_{\varepsilon \downarrow 0} e_\varepsilon = \liminf_{\varepsilon \downarrow 0} I_\varepsilon(\varphi_\varepsilon)$; e.g. choose $\varphi_\varepsilon \in H_0^1(\Omega)$ such that $I_\varepsilon(\varphi_\varepsilon) \leqslant e_\varepsilon + \varepsilon$. Combining this with Lemma 10 we deduce that

$$\liminf_{\varepsilon \downarrow 0} e_\varepsilon = \liminf_{\varepsilon \downarrow 0} I_\varepsilon(\varphi_\varepsilon) = \liminf_{\varepsilon \downarrow 0}\bigl(I_\varepsilon^0(g_\varepsilon^0) + I_\varepsilon^1(g_\varepsilon^1)\bigr) \geqslant \liminf_{\varepsilon \downarrow 0} I_\varepsilon^0(g_\varepsilon^0) + \liminf_{\varepsilon \downarrow 0} I_\varepsilon^1(g_\varepsilon^1),$$

where $\varepsilon^{1-\gamma} g_\varepsilon^0 + g_\varepsilon^1 \overset{(16)}{:=} \varphi_\varepsilon$. By passing to a subsequence, we assume without loss of generality that $\varphi_\varepsilon \overset{2}{\rightharpoonup} (g^0, g^1)$. Since, thanks to Lemma 12, we have

$$\liminf_{\varepsilon \downarrow 0} I_\varepsilon^1(g_\varepsilon^1) \geqslant I_0^1(g^1) \geqslant \inf_{\tilde{g}^1 \in H_0^1(\Omega)} I_0^1(\tilde{g}^1),$$

it remains to argue that

$$\liminf_{\varepsilon \downarrow 0} I_\varepsilon^0(g_\varepsilon^0) \geqslant \inf_{\tilde{g}^0 \in L^2(\Omega, H_0^1(Y^0))} I_0^0(\tilde{g}^0). \tag{81}$$

We identify g_ε^0 with its extension by zero to $\mathbb{R}^d$, and consider the *periodic unfolding* of g_ε^0 defined as

$$G_\varepsilon^0 : \Omega \times Y \to \mathbb{R}^d, \qquad G_\varepsilon^0(x, y) := g_\varepsilon^0\bigl(\varepsilon \lfloor x/\varepsilon \rfloor + \varepsilon y\bigr),$$

where $\lfloor z \rceil$ stands the unique vector in $\mathbb{Z}^d$ such that $z - \lfloor z \rceil \in [0, 1)^d$. Further, note that $G_\varepsilon^0 \in L^2(\Omega, H_0^1(Y^0))$, and for $\xi \in Z_\varepsilon := \{\xi \in \mathbb{Z}^d : \varepsilon(\xi + Y) \subset \Omega\}$ and $y \in Y$ one has

$$g_\varepsilon^0\bigl(\varepsilon(\xi + y)\bigr) = G_\varepsilon^0\bigl(\varepsilon(\xi + y), y\bigr), \qquad \varepsilon \nabla g_\varepsilon^0\bigl(\varepsilon(\xi + y)\bigr) = \nabla_y G_\varepsilon^0\bigl(\varepsilon(\xi + y), y\bigr).$$

Now we consider $I_\varepsilon^0(g_\varepsilon^0)$, which involves an integral over the set Ω_ε^0. Since the latter can be written as a union of sets of the form $\varepsilon(\xi + Y^0)$ with $\xi \in Z_\varepsilon$, an elementary calculation shows that

$$
\begin{aligned}
I_\varepsilon(g_\varepsilon^0) &= \sum_{\xi \in Z_\varepsilon} \int_{\varepsilon(\xi+Y^0)} \left(W^0\big(I + \varepsilon \nabla g_\varepsilon^0(x)\big) - \varepsilon^{2\gamma-1}\ell_\varepsilon(x) \cdot g_\varepsilon^0(x)\right) dx \\
&= \sum_{\xi \in Z_\varepsilon} \varepsilon^d \int_{Y^0} \left(W^0\big(I + \nabla_y G_\varepsilon^0\big(\varepsilon(\xi + y), y\big)\big) - \varepsilon^{2\gamma-1}\ell_\varepsilon\big(\varepsilon(\xi + y)\big) \cdot G_\varepsilon^0\big(\varepsilon(\xi + y), y\big)\right) dy \\
&= \int_\Omega \int_{Y^0} \left(W^0\big(I + \nabla_y G_\varepsilon^0(x, y)\big) - \varepsilon^{2\gamma-1}\ell_\varepsilon\big(\varepsilon(\lfloor x/\varepsilon \rfloor + y)\big) \cdot G_\varepsilon^0(x, y)\right) dy\, dx.
\end{aligned}
$$

Thanks to (25c), the characterisation of strong two-scale convergence introduced in Step 1 of the proof of Lemma 3, and the fact that $\limsup_{\varepsilon \downarrow 0} \|g_\varepsilon^0\|_{L^2(\Omega)} < \infty$, we have

$$
\limsup_{\varepsilon \downarrow 0} \left| \int_{\Omega \times Y^0} \left(\varepsilon^{2\gamma-1}\ell_\varepsilon\big(\varepsilon(\lfloor x/\varepsilon \rfloor + y)\big) - \ell^0(x, y)\right) \cdot G_\varepsilon^0\big(\varepsilon(\lfloor x/\varepsilon \rfloor + y), y\big)\, dx\, dy \right| = 0.
$$

Hence, one has

$$
\begin{aligned}
\liminf_{\varepsilon \downarrow 0} I_\varepsilon(g_\varepsilon^0) &= \liminf_{\varepsilon \downarrow 0} \int_\Omega \int_{Y^0} \left(W^0\big(I + \nabla_y G_\varepsilon^0(x, y)\big) - \ell^0(x, y) \cdot G_\varepsilon^0(x, y)\right) dy\, dx \\
&\geqslant \inf_{g^0 \in L^2(\Omega, H_0^1(Y^0))} \int_\Omega \int_{Y^0} \left(W^0\big(I + \nabla_y g^0(x, y)\big) - \ell^0(x, y) \cdot g^0(x, y)\right) dy\, dx \\
&\geqslant \inf_{g^0 \in L^2(\Omega, H_0^1(Y^0))} \int_\Omega \int_{Y^0} \left(\mathcal{Q}W^0\big(I + \nabla_y g^0(x, y)\big) - \ell^0(x, y) \cdot g^0(x, y)\right) dy\, dx,
\end{aligned}
$$

which proves (81).

Step 3. The proof of the convergence (34).

Since $\mathcal{Q}W^0$ is quasiconvex and Q_{hom}^1 quadratic, there exists a pair (g_*^0, g_*^1) that minimises I_{finite}. Now the sequence associated with (g_*^0, g_*^1) via Theorem 3(b) satisfies (34). $\quad\square$

Acknowledgements

References

[1] G. Allaire, Homogenisation and two-scale convergence, *SIAM J. Math. Anal.* **23**(6) (1992), 1482–1518. doi:10.1137/0523084.

[2] H. Attouch, Variational convergence for functions and operators, *Pitman Advanced Pub. Program.* (1984).

[3] M. Bellieud, Torsion effects in elastic composites with high contrast, *SIAM J. Math. Anal.* **41**(6) (2010), 2514–2553. doi:10.1137/07069362X.

[4] M. Bellieud and I. Gruais, Homogenisation of an elastic material reinforced by very stiff or heavy fibres. Non-local effects. memory effects, *J. Math. Pures Appl.* **84**(9) (2005), 55–96. doi:10.1016/j.matpur.2004.02.003.

[5] G. Bouchitté and M. Bellieud, Homogenisation of a soft elastic material reinforced by fibers, *Asymptot. Anal.* **32**(2) (2002), 153–183.

[6] G. Bouchitté and D. Felbacq, Homogenisation near resonances and artificial magnetism from dielectrics, *C. R. Math. Acad. Sci. Paris* **339**(5) (2004), 377–382. doi:10.1016/j.crma.2004.06.018.

[7] A. Braides and M. Briane, Homogenisation of non-linear variational problems with thin low-conducting layers, *Appl. Math. Optim.* **55**(1) (2007), 1–29. doi:10.1007/s00245-006-0861-6.

[8] A. Braides and A. Defranceschi, *Homogenisation of Multiple Integrals*, Oxford Lecture Series in Mathematics and Its Applications, Vol. 12, Oxford University Press, New York, 1998.

[9] A. Braides, V. Chiadò Piat and A. Piatnitski, A variational approach to double-porosity problems, *Asymptot. Anal.* **39** (2004), 281–308.

[10] A. Braides, V. Chiadò Piat and A. Piatnitski, Homogenization of discrete high-contrast energies, *SIAM J. Math. Anal.* **47**(4) (2015), 3064–3091. doi:10.1137/140975668.

[11] M. Briane, Optimal conditions of convergence and effects of anisotropy in the homogenisation of non-uniformly elliptic problems, *Asymptot. Anal.* **25** (2001), 271–297.

[12] M. Briane, Homogenisation of a class of non-uniformly elliptic monotonic operators, *Nonlinear Anal. Ser. A: Theory Methods* **48**(1) (2002), 137–158. doi:10.1016/S0362-546X(00)00179-6.

[13] M. Briane, Homogenisation of non-uniformly bounded operators: Critical barrier for nonlocal effects, *Arch. Rational Mech. Anal.* **164** (2002), 73–101. doi:10.1007/s002050200196.

[14] M. Camar-Eddine and P. Seppecher, Determination of the closure of the set of elasticity functionals, *Arch. Ration. Mech. Anal.* **170**(3) (2003), 211–245. doi:10.1007/s00205-003-0272-7.

[15] M. Cherdantsev, Spectral convergence for high-contrast elliptic periodic problems with a defect via homogenisation, *Mathematika* **55**(1–2) (2009), 29–57. doi:10.1112/S0025579300000942.

[16] M. Cherdantsev and K.D. Cherednichenko, Two-scale Γ-convergence of integral functionals and its application to homogenisation of nonlinear high-contrast periodic composites, *Archive for Rational Mechanics and Analysis* **204** (2012), 445–478. doi:10.1007/s00205-011-0481-4.

[17] K.D. Cherednichenko, V.P. Smyshlyaev and V.V. Zhikov, Non-local homogenised limits for composite media with highly anisotropic periodic fibres, *Proc. Roy. Soc. Edinburgh Sect. A* **136**(1) (2006), 87–114. doi:10.1017/S0308210500004455.

[18] D. Cioranescu, A. Damlamian and R. De Arcangelis, Homogenisation of quasiconvex integrals via the periodic unfolding method, *SIAM J. Math. Anal.* **37**(5) (2006), 1435–1453. doi:10.1137/040620898.

[19] B. Dacorogna, *Direct Methods in Calculus of Variations*, Springer, 2000.

[20] G. Dal Maso, *An Introduction to Γ-Convergence*, Birkhäuser, 1993.

[21] L.C. Evans, *Partial Differential Equations*, American Mathematical Society, 1998.

[22] I. Fonseca, S. Müller and P. Pedregal, Analysis of concentration and oscillation effects generated by gradients, *SIAM J. Math. Anal.* **29**(3) (1998), 736–756. doi:10.1137/S0036141096306534.

[23] G. Friesecke, R.D. James and S. Müller, A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity, *Commun. Pure Appl. Math.* **55**(11) (2002), 1461–1506. doi:10.1002/cpa.10048.

[24] P. Hornung, S. Neukamm and I. Velčić, Derivation of a homogenised nonlinear plate theory from 3d elasticity, *Calc. Var. PDE* **51**(3–4) (2014), 677–699. doi:10.1007/s00526-013-0691-8.

[25] U. Hornung (ed.), *Homogenisation and Porous Media*, Springer, 1997.

[26] I.V. Kamotski and V.P. Smyshlyaev, Localised modes due to defects in high contrast periodic media via homogenisation, Bath Institute for Complex Systems, 2006, Preprint 3/06.

[27] J. Kristensen, Finite functionals and Young measures generated by gradients of Sobolev functions, MAT-Report 1994-34, Math. Inst., Technical University of Denmark, 1994.

[28] S. Müller and S. Neukamm, On the commutability of homogenisation and linearisation in finite elasticity, *Arch. Rat. Mech. Anal.* **201**(2) (2011), 465–500. doi:10.1007/s00205-011-0438-7.

[29] S. Neukamm, Homogenisation, linearisation and dimension reduction in elasticity with variational methods, PhD thesis, Technische Universität München, 2010.

[30] S. Neukamm, Rigorous derivation of a homogenised bending-torsion theory for inextensible rods from three-dimensional elasticity, *Arch. Ration. Mech. Anal.* **206**(2) (2012), 645–706. doi:10.1007/s00205-012-0539-y.

[31] S. Neukamm and I. Velčić, Derivation of a homogenised von-Kármán plate theory from 3*D* nonlinear elasticity, *Mathematical Models and Methods in Applied Sciences* **23**(14) (2013), 2701–2748. doi:10.1142/S0218202513500449.

[32] G. Nguetseng, A general convergence result for a functional related to the theory of homogenisation, *SIAM J. Math. Anal.* **20**(3) (1989), 608–623. doi:10.1137/0520043.

[33] O.A. Oleinik, A.S. Shamaev and G.A. Yosifian, *Mathematical Problems in Elasticity and Homogenisation*, North-Holland, 1992.

[34] J.B. Pendry, D. Schurig and D.R. Smith, Controlling electromagnetic fields, *Science* **312**(5781) (2006), 1780–1782. doi:10.1126/science.1125907.

[35] V.P. Smyshlyaev, Propagation and localization of elastic waves in highly anisotropic composites via homogenisation, *Mechanics of Materials* **41**(4) (2009), 434–447. doi:10.1016/j.mechmat.2009.01.009.

[36] A. Visintin, Towards a two-scale calculus, *ESAIM: Control, Optimisation and Calculus of Variations 12(3)* (2006), 371–397. doi:10.1051/cocv:2006012.

[37] A. Visintin, Two-scale convergence of some integral functionals, *Calculus of Variations and Partial Differential Equations* **29**(2) (2007), 239–265. doi:10.1007/s00526-006-0068-3.

[38] V.V. Zhikov, On an extension of the method of two-scale convergence and its applications, *Sb. Math.* **191**(7) (2000), 973–1014. doi:10.1070/SM2000v191n07ABEH000491.