Asymptotic Analysis 100 (2016) 63–86
DOI 10.3233/ASY-161386
IOS Press

On the local existence of the free-surface Euler equation with surface tension

Mihaela Ignatova [a] and Igor Kukavica [b,*]
[a] *Department of Mathematics, Princeton University, Princeton, NJ 08544, USA*
E-mail: ignatova@math.princeton.edu
[b] *Department of Mathematics, University of Southern California, Los Angeles, CA 90089, USA*
E-mail: kukavica@usc.edu

Abstract. We address the existence of solutions for the free-surface Euler equation with surface tension in a bounded domain. Considering the problem in Lagrangian variables we provide a priori estimates leading to existence of local solutions with the initial velocity in $H^{3.5}$ for which the trace on the free boundary belongs to $H^{3.5}$.

Keywords: Euler equations, surface tension, free boundary

1. Introduction

In this paper, we address the local existence of solutions to the 3D free-surface incompressible Euler equations

$$\partial_t u + u \cdot \nabla u + \nabla p = 0 \quad \text{in } \Omega(t) \tag{1.1}$$

$$\nabla \cdot u = 0 \quad \text{in } \Omega(t) \tag{1.2}$$

where the free boundary $\partial\Omega(t)$ evolves according to the fluid velocity field $u(x,t)$, and the pressure obeys

$$p(x,t) = \sigma H \quad \text{on } \partial\Omega(t). \tag{1.3}$$

Here $\sigma > 0$ is the surface tension, while H represents twice the mean curvature of the boundary $\partial\Omega(t)$.

Problems related to local or global existence of solutions of free surface evolution under the Euler flow, with or without surface tension, have attracted considerable attention in the last decades. For both cases different approaches have been developed; however, the search is still in progress for the lowest regularity spaces where the existence or uniqueness of solutions hold. For the history of both problems, cf. [2,17,32] and references therein.

While in the zero surface tension case the problem is known to be unstable, and thus the Rayleigh-Taylor stability condition has to be imposed, this is not necessary when the surface tension is nonzero since the surface tension provides a stabilizing effect close to the boundary.

*Corresponding author. E-mail: kukavica@usc.edu.

We may divide the existing results of the rotational case, i.e., when the vorticity is nontrivial, into the Eulerian approach and the Lagrangian one. In the Eulerian approach, Schweitzer has obtained in [42] a local existence result with the initial velocity in $H^{4.5}$ and with a smallness assumption on the height of the interface. The primary tools in [42] are tangential and time differentiation, up to order three. In [17], Coutand and Shkoller used the Lagrangian formulation to obtain the local existence with initial data in $H^{4.5}$. The method used by Coutand and Shkoller is, as in [42], differentiation in space and time up to three times; however, the Lagrangian approach allowed to bypass the smallness assumption on the initial surface. We would like to stress that simple integrations by parts are not by themselves sufficient to close the estimates; additional care, including a careful treatment of the vorticity and the pressure equations, is necessary to close the estimates. In addition, in [42], a harmonic change of variables was used to overcome a lack of 1/2 derivative in the estimates resulting from tangential and time differentiation.

In [43] the authors employ ideas inspired by the geometrical description of Euler flows as geodesics on the infinite dimensional group of volume preserving diffeomorphisms to obtain conditional a priori energy estimates for the solutions when the initial velocity belongs to H^3. They also provide estimates which are uniform in surface tension, if additionally a Raleigh-Taylor condition is satisfied. Recently, in [20,21], using a different method, the authors established the local existence when the initial velocity belongs to $H^{3.5+\epsilon}$ for every $\epsilon > 0$.

Our goal is to revisit the Lagrangian approach to the free-surface rotational Euler equations and provide a priori estimates leading to local existence for the velocity in $H^{3.5}$ such that the trace on the free boundary belongs to H^3.5, lowering the regularity requirements from [17] and [42]. While the basic framework still involves time and tangential differentiation used in [17,42], we introduce two improvements which allow us to lower the required regularity. The first improvement is the use of Cauchy invariance [11,14,15,23,35,44] recently used in the zero surface tension case in [32,33]. The second improvement is a simple and direct treatment of the pressure, employing the Laplace problem with Neumann boundary conditions.

Before discussing the organization of the paper, we briefly recall the history of free surface Euler equation problems. Early works on the free surface Euler equations involve results on small analytic data [19,38,51]. The important work [7] considered the viscous case, employing a Lagrangian set-up, subsequently used in many works on the inviscid problem. In [47,48] Wu obtained existence of solutions of the free surface Euler equations in 2D and 3D cases respectively, both addressing irrotational, no surface-tension cases. Positive surface tension was considered by Ambrose and Masmoudi in [4,5], who also studied the zero surface tension limit. The works [17,42,43] then constructed local solutions for the nonzero-surface tension Euler equations; cf. also works [29,40,41,46,50] for the positive surface-tension Navier-Stokes system. For other works on the zero-surface tension case, see [2,3,8–10,13,18,22,27,30, 31,34,36–38,45,52,53], for other works on non-zero surface tension, cf. [1,39] while for global existence of solutions, see [24–26,28,49].

The paper is organized as follows. In Section 2, we introduce the Lagrangian setting of the problem and state the main result, Theorem 2.1. Section 3 contains a preliminary lemma containing a priori estimates on the Lagrangian map η and the cofactor matrix a. Section 4 contains the proof of the main statement. It is subdivided into four subsections containing the v_{ttt}, v_{tt}, v_t estimates, and the div-curl estimate. In the final section, we collect all the available inequalities and apply the Gronwall lemma.

2. The main result

We consider the 3D Euler equation in the Lagrangian framework over a fixed domain Ω. Let $\eta(\cdot, t)\colon \Omega \to \Omega(t)$ be the flow map under which the initial domain configuration Ω evolves with time, such that $\Omega(t) = \eta(\Omega, t)$. For simplicity, we assume that the initial domain Ω is flat, i.e.,

$$\Omega = \left\{x = (x_1, x_2, x_3) : (x_1, x_2) \in \mathbb{R}^2, 0 < x_3 < 1\right\} \tag{2.1}$$

with periodic boundary conditions with period 1 in the lateral directions. We denote the top of Ω (corresponding to the free-surface) by

$$\Gamma_1 = \mathbb{T}^2 \times \{x_3 = 1\} \tag{2.2}$$

and the stationary bottom by

$$\Gamma_0 = \mathbb{T}^2 \times \{x_3 = 0\}. \tag{2.3}$$

Then the incompressible Euler equation has the form

$$v_t^i + \partial_k\left(a_i^k q\right) = 0 \quad \text{in } \Omega \times (0, T), i = 1, 2, 3 \tag{2.4}$$

$$a_i^k \partial_k v^i = 0 \quad \text{in } \Omega \times (0, T), \tag{2.5}$$

where $v(x, t) = \eta_t(x, t) = u(\eta(x, t), t)$ and $q(x, t) = p(\eta(x, t), t)$ denote the Lagrangian velocity and the pressure of the fluid over the initial domain Ω. The dynamics of the Lagrangian matrix $a(x, t) = [\nabla\eta(x, t)]^{-1}$ and the flow map $\eta(x, t)$ are described by the ODEs

$$a_t = -a : \nabla v : a \quad \text{in } \Omega \times (0, T) \tag{2.6}$$

$$\eta_t = v \quad \text{in } \Omega \times (0, T) \tag{2.7}$$

where the symbol : denotes the matrix multiplication, with the initial conditions

$$a(x, 0) = I \tag{2.8}$$

$$\eta(x, 0) = x \tag{2.9}$$

in Ω. The condition (2.6) can be written in coordinates as

$$\partial_t a_k^i = -a_l^k \partial_j v^l a_i^j, \quad i, k = 1, 2, 3. \tag{2.10}$$

We assume $v \cdot N = 0$ on $\Gamma_0 \times (0, T)$ and

$$a_i^k q N_k = -\Delta_2 \eta^i \quad \text{on } \Gamma_1 \times (0, T) \tag{2.11}$$

for $i = 1, 2, 3$, where $N = (N_1, N_2, N_3)$ is the unit outward normal with respect to Ω and $\Delta_2 = \partial_1^2 + \partial_2^2$. Note that we have set the surface tension to be 1, for simplicity.

We now state the main result of this paper.

Theorem 2.1. *Assume that $v(\cdot,t) = v_0 \in H^{3.5}(\Omega)$ is divergence-free and is such that $v_0|_{\Gamma_1} \in H^{3.5}(\Gamma_1)$. Then there exists a local-in-time solution (v, q, a, η) to (2.4)–(2.11) which satisfies*

$$v \in L^\infty\big([0,T]; H^{3.5}(\Omega)\big)$$
$$v_t \in L^\infty\big([0,T]; H^{2.5}(\Omega)\big)$$
$$v_{tt} \in L^\infty\big([0,T]; H^{1.5}(\Omega)\big)$$
$$v_{ttt} \in L^\infty\big([0,T]; L^2(\Omega)\big)$$

with $q \in L^\infty([0,T]; H^{3.5}(\Omega))$, $q_t \in L^\infty([0,T]; H^{2.5}(\Omega))$, $q_{tt} \in L^\infty([0,T]; H^1(\Omega))$, $a \in L^\infty([0,T]; H^{2.5}(\Omega))$, and $\eta \in C([0,T]; H^{3.5}(\Omega))$.

3. Preliminary results

In this section, we give formal a priori estimates on time derivatives of the unknown functions needed in the proof of Theorem 2.1. We begin with an auxiliary result providing bounds on the flow map η and the matrix a.

Lemma 3.1. *Assume that $\|v\|_{L^\infty([0,T];H^{3.5})} \leqslant M$. Let $p \in [1,\infty]$ and $i, j = 1,2,3$. With $T \in [0, 1/CM]$, where C is a sufficiently large constant, the following statements hold:*

(i) $\|\eta\|_{H^{3.5}} \leqslant C$ *for* $t \in [0,T]$*;*
(ii) $\|a\|_{H^{2.5}} \leqslant C$ *for* $t \in [0,T]$*;*
(iii) $\|a_t\|_{L^p} \leqslant C\|\nabla v\|_{L^p}$ *for* $t \in [0,T]$*;*
(iv) $\|\partial_i a_t\|_{L^p} \leqslant C\|\nabla v\|_{L^{p_1}}\|\partial_i a\|_{L^{p_2}} + C\|\nabla \partial_i v\|_{L^p}$ *for* $i = 1,2,3$ *and* $t \in [0,T]$ *where* $1 \leqslant p, p_1, p_2 \leqslant \infty$ *are such that* $1/p = 1/p_1 + 1/p_2$*;*
(v) $\|a_t\|_{H^r} \leqslant \|\nabla v\|_{H^r}$*, for* $r \in [0, 2.5]$ *and* $t \in [0,T]$*;*
(vi) $\|a_{tt}\|_{H^\sigma} \leqslant C\|\nabla v\|_{H^\sigma}\|\nabla v\|_{L^\infty} + C\|\nabla v_t\|_{H^\sigma}$ *for* $\sigma \in [0, 1.5]$ *and* $t \in [0,T]$ *and* $\|a_{tt}\|_{H^1(\Omega)} \leqslant C\|\nabla v\|^2_{H^{5/4}} + C\|\nabla v_t\|_{H^1}$*;*
(vii) $\|a_{ttt}\|_{L^p} \leqslant C\|\nabla v\|_{L^p}\|\nabla v\|^2_{L^\infty} + C\|\nabla v_t\|_{L^p}\|\nabla v\|_{L^\infty} + C\|\nabla v_{tt}\|_{L^p}$ *for* $t \in [0,T]$*;*
(viii) *for every* $\epsilon \in (0, 1/2]$ *and all* $t \leqslant T^* = \min\{\epsilon/CM^2, T\}$*, we have*

$$\big\|\delta_{jk} - a_l^j a_l^k\big\|^2_{H^{2.5}} \leqslant \epsilon, \quad j,k = 1,2,3 \tag{3.1}$$

and

$$\big\|\delta_{jk} - a_k^j\big\|^2_{H^{2.5}} \leqslant \epsilon, \quad j,k = 1,2,3. \tag{3.2}$$

In particular, the form $a_l^j a_l^k \xi_j^i \xi_k^i$ satisfies the ellipticity estimate

$$a_l^j a_l^k \xi_j^i \xi_k^i \geqslant \frac{1}{C}|\xi|^2, \quad \xi \in \mathbb{R}^{n^2} \tag{3.3}$$

for all $t \in [0,T^*]$ and $x \in \Omega$, provided $\epsilon \leqslant 1/C$ with C sufficiently large.

Above and in the sequel, if the domain of the norm is not specified, it is understood to be Ω.

Proof of Lemma 3.1. The assertion (i) follows immediately from (2.7), while for (ii), we have by (2.6)

$$\|a(t)\|_{H^{2.5}} \leqslant C + \int_0^t \|a(s)\|_{H^{2.5}}^2 \|\nabla v(s)\|_{H^{2.5}} \, ds \tag{3.4}$$

and (ii) is obtained by using the Gronwall lemma, provided $T \leqslant 1/CM$. Next, (2.6) implies

$$\|a_t\|_{L^p} \leqslant C\|a\|_{L^\infty}\|\nabla v\|_{L^p}\|a\|_{L^\infty} \leqslant C\|\nabla v\|_{L^p} \tag{3.5}$$

using (ii) in the last inequality. The inequality (v) is proved analogously, using the Sobolev multiplicative inequality instead of (3.5). The estimates (iv) and (vi) are proven similarly. For (viii), we write

$$\delta_{jk} - a_l^j a_l^k = -\int_0^t \partial_t\left(a_l^j a_l^k\right) ds \tag{3.6}$$

where $j, k \in \{1, 2, 3\}$. Therefore,

$$\begin{aligned} \|\delta_{jk} - a_l^j a_l^k\|_{H^{2.5}} &\leqslant \int_0^t \|\partial_t\left(a_l^j a_l^k\right)\|_{H^{2.5}} \, ds \\ &\leqslant C \int_0^t \|a_l^j\|_{H^{2.5}} \|\partial_t a_l^k\|_{H^{2.5}} \, ds \leqslant C \int_0^t \|a_t\|_{H^{2.5}} \, ds \leqslant CMt. \end{aligned} \tag{3.7}$$

The estimate (3.1) then follows if $CM^2T^2 \leqslant \epsilon$. The other assertions in (viii) are obtained analogously. □

In order to estimate the second derivative of the pressure, we need the following regularity lemma for an elliptic equation with Neumann boundary condition in a smooth (bounded) domain Ω. Assume that b_{ij} satisfies $\|b\|_{L^\infty} \leqslant M$ and that $b_{ij}(x)\xi_i\xi_j \geqslant M^{-1}|\xi|^2$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$, where $n \in \{2, 3, \ldots\}$.

Lemma 3.2 ([16]). *Let q be an H^1 solution of the*

$$\partial_i(b_{ij}\partial_j q) = \operatorname{div}\pi \quad \text{in } \Omega \tag{3.8}$$

$$b_{mk}\partial_k q N^m = g \quad \text{on } \partial\Omega \tag{3.9}$$

where $\pi, \operatorname{div}\pi \in L^2(\Omega)$ and $g \in H^{-1/2}(\partial\Omega)$ with the compatibility condition

$$\int_{\partial\Omega} (\pi \cdot N - g) = 0. \tag{3.10}$$

If

$$\|b - I\|_{L^\infty} \leqslant \epsilon_0 \tag{3.11}$$

where $\epsilon_0 > 0$ *is a sufficiently small constant depending on* M*, then we have*

$$\|q - \bar{q}\|_{H^1} \leqslant C\|\pi\|_{L^2} + C\|g - \pi \cdot N\|_{H^{-1/2}(\partial\Omega)} \tag{3.12}$$

where $\bar{q} = (1/|\Omega|)\int q\,dx$.

The existence of solutions of this problem under the given conditions has been established in [6]. However, we believe that the inequality (3.12), which does not contain the L^2-norm of $\operatorname{div}\pi$, is new.

Proof of Lemma 3.2. First, using (3.8)–(3.9), we have

$$\int_\Omega b_{mk}\partial_k q\,\partial_m\phi = -\int_\Omega \phi\operatorname{div}\pi + \int_{\partial\Omega} g\phi, \quad \phi \in H^1(\Omega) \tag{3.13}$$

and thus

$$\int_\Omega b_{mk}\partial_k q\,\partial_m\phi = \int_\Omega \pi\cdot\nabla\phi + \int_{\partial\Omega}(g - \pi\cdot N)\phi, \quad \phi \in H^1(\Omega). \tag{3.14}$$

Using the Cauchy–Schwarz inequality, we obtain

$$\left|\int_\Omega b_{mk}\partial_k q\,\partial_m\phi\right| \leqslant C\big(\|\pi\|_{L^2} + \|g - \pi\cdot N\|_{H^{-1/2}(\partial\Omega)}\big)\|\phi\|_{H^1}, \quad \phi \in H^1(\Omega). \tag{3.15}$$

Since also $\int_\Omega |\phi||q| \leqslant \|\phi\|_{L^2}\|q\|_{L^2}$ for all $\phi \in L^2(\Omega)$, we get

$$\|q\|_{H^1} \leqslant C\big(\|\pi\|_{L^2} + \|g - \pi\cdot N\|_{H^{-1/2}(\partial\Omega)} + \|q\|_{L^2}\big). \tag{3.16}$$

Next, we aim to improve this inequality by estimating the L^2-norm of q. For this purpose, for every $f \in L^2(\Omega)$ such that $\int_\Omega f = 0$, solve

$$\begin{aligned} &\Delta\tilde{\phi}_f = f \quad \text{in } \Omega \\ &\frac{\partial\tilde{\phi}_f}{\partial N} = 0 \quad \text{on } \partial\Omega \\ &\int_\Omega \tilde{\phi}_f = 0. \end{aligned} \tag{3.17}$$

Note that, by the energy inequality,

$$\int_\Omega |\nabla\tilde{\phi}_f|^2 \leqslant C\|f\|_{L^2}^2. \tag{3.18}$$

Since $\partial\tilde{\phi}_f/\partial N = 0$ on $\partial\Omega$, we have $\int_\Omega q\Delta\tilde{\phi}_f + \int_\Omega \nabla q\cdot\nabla\tilde{\phi}_f = 0$ and thus

$$\left|\int_\Omega qf\right| = \left|\int_\Omega \nabla q\cdot\nabla\tilde{\phi}_f\right|. \tag{3.19}$$

In order to estimate $\int_\Omega \nabla q \cdot \nabla \tilde{\phi}_f$, we write

$$\int_\Omega \partial_i q \partial_i \tilde{\phi}_f = \int_\Omega b_{mk} \partial_k q \partial_m \tilde{\phi}_f + \int_\Omega (\delta_{mk} - b_{mk}) \partial_k q \partial_m \tilde{\phi}_f. \tag{3.20}$$

Using (3.15) on the first term, we get

$$\left| \int_\Omega q f \right| \leqslant C \big(\|\pi\|_{L^2} + \|g - \pi \cdot N\|_{H^{-1/2}(\partial\Omega)} \big) \|\nabla \tilde{\phi}_f\|_{L^2} + C\epsilon_0 \|\nabla q\|_{L^2} \|\nabla \tilde{\phi}_f\|_{L^2} \tag{3.21}$$

whence

$$\left| \int_\Omega q f \right| \leqslant C \big(\|\pi\|_{L^2} + \|g - \pi \cdot N\|_{H^{-1/2}(\partial\Omega)} + \epsilon_0 \|\nabla q\|_{L^2} \big) \|f\|_{L^2}. \tag{3.22}$$

Since this inequality holds for all $f \in L^2(\Omega)$ such that $\int_\Omega f = 0$, we obtain

$$\|q - \bar{q}\|_{L^2} \leqslant C \big(\|\pi\|_{L^2} + \|g - \pi \cdot N\|_{H^{-1/2}(\partial\Omega)} + \epsilon_0 \|\nabla q\|_{L^2} \big). \tag{3.23}$$

On the other hand, by (3.16), we also have

$$\big\| \nabla (q - \bar{q}) \big\|_{L^2} \leqslant C \big(\|\pi\|_{L^2} + \|g - \pi \cdot N\|_{H^{-1/2}(\partial\Omega)} + \|q - \bar{q}\|_{L^2} + \|\bar{q}\|_{L^2} \big). \tag{3.24}$$

Combining (3.23) and (3.24) and then choosing ϵ_0 sufficiently small then leads to (3.12). □

Now, let Ω be as in (2.1), and let q be as in Lemma 3.2. In order to bound $|\bar{q}|$, let H be a solution of the Dirichlet/Neumann problem

$$\Delta H = 1 \quad \text{in } \Omega \tag{3.25}$$

$$H = 0 \quad \text{on } \Gamma_1 \tag{3.26}$$

$$\frac{\partial H}{\partial N} = 0 \quad \text{on } \Gamma_0. \tag{3.27}$$

Using

$$\int_\Omega q \Delta H + \int_\Omega \nabla q \cdot \nabla H = \int_{\partial\Omega} \frac{\partial H}{\partial N} q \tag{3.28}$$

we obtain

$$\left| \int_\Omega q \, dx \right| \leqslant C \big\| \nabla (q - \bar{q}) \big\|_{L^2} + C \|q\|_{L^2(\Gamma_1)} \tag{3.29}$$

which combined with (3.12) leads to

$$\|q\|_{H^1} \leqslant C \|\pi\|_{L^2} + C \|g - \pi \cdot N\|_{H^{-1/2}(\partial\Omega)} + C \|q\|_{L^2(\Gamma_1)} \tag{3.30}$$

for solutions of the problem (3.8)–(3.9) under given boundedness and ellipticity conditions.

The bounds on the pressure and its derivatives are obtained by solving a linear elliptic equation with Neumann boundary conditions.

Lemma 3.3. *Assume that (v, q, a, η) solves the system (2.4)–(2.11) for a given coefficient matrix $a \in H^{2.5}(\Omega)$ satisfying (i)–(viii) from Lemma 3.1, with a sufficiently small constant $\epsilon = 1/C$. Then the estimate*

$$\|q\|_{H^{3.5}} \leqslant C\|\nabla v\|_{H^{2.5}}\|v\|_{H^{2.5}} + C\|v_t\|_{H^2(\Gamma_1)} + C \tag{3.31}$$

holds for all $t \in (0, T)$. Moreover, the time derivatives q_t and q_{tt} satisfy

$$\begin{aligned}\|q_t\|_{H^{2.5}} \leqslant{}& C\|\nabla v\|_{H^{1.5+\epsilon}}\big(\|q\|_{H^{2.5}} + \|v_t\|_{H^{1.5}}\big)\\ &+ C\big(\|\nabla v\|_{H^{1.5}}\|\nabla v\|_{L^\infty} + \|\nabla v_t\|_{H^{1.5}}\big)\|v\|_{H^{1.5+\epsilon}}\\ &+ C\|v_{tt}\|_{H^1(\Gamma_1)} + C\|v\|_{H^{2.5}} + C\|v_t\|_{H^{2.5}}\end{aligned} \tag{3.32}$$

and

$$\begin{aligned}\|q_{tt}\|_{H^1} \leqslant{}& C\big(\|v\|_{H^{1.5}}\|\nabla v\|_{L^\infty} + \|v_t\|_{H^{1.5}}\big)\big(\|q\|_{H^2} + \|v_t\|_{H^1}\big)\\ &+ C\|\nabla v\|_{L^\infty}\big(\|q_t\|_{H^1} + \|v_{tt}\|_{L^2}\big)\\ &+ C\big(\|v\|_{H^{1.5}}\|\nabla v\|_{L^\infty}^2 + \|v_t\|_{H^{1.5}}\|\nabla v\|_{L^\infty} + \|v_{tt}\|_{H^{1.5}}\big)\|v\|_{H^1} + C\|v_{ttt}\|_{L^2}\\ &+ C\|v\|_{H^2}^{3/2}\|v\|_{H^3}^{1/2} + C\|\nabla v\|_{L^\infty}\|v\|_{H^{2.5}} + C\|v_t\|_{H^{2.5}}\end{aligned} \tag{3.33}$$

for all $t \in (0, T)$, where $T \leqslant 1/CM$ for a sufficiently large constant C.

Proof. Applying the Lagrangian divergence to the evolution equation (2.4) leads to

$$\Delta q = \partial_m\big(\big(\delta_{km} - a_i^m a_i^k\big)\partial_k q\big) + \partial_t a_i^m \partial_m v^i. \tag{3.34}$$

In order to obtain the boundary condition for q, we multiply the equation (2.4) with $a_i^m N_m$ and sum. We get

$$\frac{\partial q}{\partial N} = \big(\delta_{km} - a_i^m a_i^k\big)\partial_k q N_m - a_i^m \partial_t v^i N_m \tag{3.35}$$

which holds on $\Gamma_0 \cup \Gamma_1$. As in [17, Lemma 12.1, p. 866], we have a regularity estimate for

$$\begin{aligned}&\Delta q = f \quad \text{in } \Omega\\ &\nabla q \cdot N = g \quad \text{on } \partial\Omega\end{aligned} \tag{3.36}$$

which reads

$$\|q\|_{H^s} \leqslant C\|f\|_{H^{s-2}} + C\|g\|_{H^{s-1.5}(\partial\Omega)} + C\|q\|_{L^2} \tag{3.37}$$

and is valid for $s \geqslant 2$, with the constant C depending on s. Using (3.29), we then get

$$\|q\|_{H^s} \leqslant C\|f\|_{H^{s-2}} + C\|g\|_{H^{s-1.5}(\partial\Omega)} + C\|q\|_{L^2(\Gamma_1)} \tag{3.38}$$

for any $s \geqslant 2$. We use this estimate with

$$f = \partial_m\big(\big(\delta_{km} - a_i^m a_i^k\big)\partial_k q\big) + \partial_m\big(\partial_t a_i^m v^i\big) \tag{3.39}$$

and

$$g = \big(\delta_{km} - a_i^m a_i^k\big)\partial_k q N_m - a_i^m \partial_t v^i N_m \quad \text{on } \Gamma_0 \cup \Gamma_1. \tag{3.40}$$

In order to obtain (3.31), we apply the estimate (3.38) with $s = 3.5$. We thus have

$$\begin{aligned} \|f\|_{H^{1.5}} &\leqslant \big\|\big(I - a^T a\big)\nabla q\big\|_{H^{2.5}} + \|a_t v\|_{H^{2.5}} \\ &\leqslant \big\|I - a^T a\big\|_{H^{2.5}}\|q\|_{H^{3.5}} + \|a_t\|_{H^{2.5}}\|v\|_{H^{2.5}} \\ &\leqslant \epsilon\|q\|_{H^{3.5}} + C\|\nabla v\|_{H^{2.5}}\|v\|_{H^{2.5}} \end{aligned} \tag{3.41}$$

and, similarly,

$$\begin{aligned} \|g\|_{H^2(\Gamma_1)} &\leqslant \big\|I - a^T a\big\|_{H^{2.5}}\|q\|_{H^{3.5}} + \|a v_t\|_{H^2(\Gamma_1)} \\ &\leqslant \epsilon\|q\|_{H^{3.5}} + C\|v_t\|_{H^2(\Gamma_1)} \end{aligned} \tag{3.42}$$

by using (3.1), (3.2) and part (v) from Lemma 3.1. Also,

$$\|q\|_{L^2(\Gamma_1)} \leqslant C\|\eta\|_{H^2(\Gamma_1)} \leqslant C\|\eta\|_{H^{2.5}} \leqslant C. \tag{3.43}$$

Next, for q_t we apply (3.38) for the time differentiated problem (3.36) with $s = 2.5$. We get

$$\begin{aligned} \|f_t\|_{H^{0.5}} &\leqslant \big\|\big(I - a^T a\big)_t\nabla q\big\|_{H^{1.5}} + \big\|\big(I - a^T a\big)\nabla q_t\big\|_{H^{1.5}} + \|a_{tt} v\|_{H^{1.5}} + \|a_t v_t\|_{H^{1.5}} \\ &\leqslant C\|a_t\|_{H^{1.5+\epsilon}}\|q\|_{H^{2.5}} + \big\|I - a^T a\big\|_{H^{1.5+\epsilon}}\|q_t\|_{H^{2.5}} + \|a_{tt}\|_{H^{1.5}}\|v\|_{H^{1.5+\epsilon}} \\ &\quad + \|a_t\|_{H^{1.5+\epsilon}}\|v_t\|_{H^{1.5}} \\ &\leqslant C\|\nabla v\|_{H^{1.5+\epsilon}}\|q\|_{H^{2.5}} + \epsilon\|q_t\|_{H^{2.5}} \\ &\quad + C\big(\|\nabla v\|_{H^{1.5}}\|\nabla v\|_{L^\infty} + \|\nabla v_t\|_{H^{1.5}}\big)\|v\|_{H^{1.5+\epsilon}} + \|\nabla v\|_{H^{1.5+\epsilon}}\|v_t\|_{H^{1.5}} \end{aligned} \tag{3.44}$$

where we utilized the multiplicative Sobolev inequality and the parts (ii), (vi), and (viii) from Lemma 3.1. As in (3.43), we have

$$\|q_t\|_{L^2(\Gamma_1)} \leqslant C\|a_t\|_{L^4(\partial\Omega)}\|\eta\|_{H^{2.5}(\partial\Omega)} + C\|a\|_{L^\infty}\|\eta_t\|_{H^2(\partial\Omega)} \leqslant C\|v\|_{H^{2.5}}. \tag{3.45}$$

Lastly, we consider the twice differentiated in time system (3.34). First, we rewrite it as

$$\partial_m\big(\big(a_i^m a_i^k\big)\partial_k q\big) = \partial_t a_i^m \partial_m v^i \tag{3.46}$$

while the boundary condition (3.35) is

$$a_i^m a_i^k \partial_k q N_m = -a_i^m \partial_t v^i N_m. \tag{3.47}$$

The twice differentiated system then reads

$$\partial_m\big((a_i^m a_i^k)\partial_k q_{tt}\big) = -\partial_m\big(\partial_{tt}(a_i^m a_i^k)\partial_k q\big) - 2\partial_m\big(\partial_t(a_i^m a_i^k)\partial_k q_t\big) + \partial_m\big(\partial_{tt}(\partial_t a_i^m v^i)\big) \tag{3.48}$$

with the boundary condition

$$a_i^m a_i^k \partial_k q_{tt} N_m = -\partial_{tt}(a_i^m a_i^k)\partial_k q N_m - 2\partial_t(a_i^m a_i^k)\partial_k q_t N_m - \partial_{tt}(a_i^m \partial_t v^i N_m). \tag{3.49}$$

Applying the inequality (3.30), we obtain

$$\begin{aligned}\|q_{tt}\|_{H^1} &\leqslant C\sum_m \big\|\partial_{tt}(a_i^m a_i^k)\partial_k q\big\|_{L^2} + C\sum_m \big\|\partial_t(a_i^m a_i^k)\partial_k q_t\big\|_{L^2} + C\sum_m \big\|\partial_{tt}(\partial_t a_i^m v^i)\big\|_{L^2}\\ &\quad + C\big\|\partial_{tt}(a_i^m \partial_t v^i N_m + \partial_t a_i^m v^i N_m)\big\|_{H^{-1/2}(\partial\Omega)} + C\|q_{tt}\|_{L^2(\Gamma_1)}\\ &= I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}\tag{3.50}$$

In order to estimate the last term I_5 in (3.50), we use (2.11), which, when rewritten as

$$N_i q = (\delta_i^k - a_i^k) q N_k - \Delta_2 \eta^i \tag{3.51}$$

on Γ_1, leads to

$$q = (1 - a_3^3) q - \Delta_2 \eta^3 \quad \text{on } \Gamma_1. \tag{3.52}$$

Therefore,

$$\|q\|_{L^4(\Gamma_1)} \leqslant C\|\eta\|_{H^3} \leqslant C \tag{3.53}$$

by Lemma 3.1. Using (3.45) and (3.52)

$$\begin{aligned}\|q_{tt}\|_{L^2(\Gamma_1)} &\leqslant C\big\|\partial_{tt} a_3^3 q\big\|_{L^2(\Gamma_1)} + C\big\|\partial_t a_3^3 \partial_t q\big\|_{L^2(\Gamma_1)} + C\|v_t\|_{H^2(\Gamma_1)}\\ &\leqslant C\|a_{tt}\|_{L^4(\Gamma_1)}\|q\|_{L^4(\Gamma_1)} + C\big\|\partial_t a_3^3 \partial_t q\big\|_{L^2(\Gamma_1)} + C\|v_t\|_{H^2(\Gamma_1)}.\end{aligned}\tag{3.54}$$

In order to estimate the first term on the far right side, we use

$$\|a_{tt}\|_{L^4(\Gamma_1)} \leqslant C\|a_{tt}\|_{H^1(\Omega)} \leqslant C\|\nabla v\|_{H^{5/4}}^2 + C\|\nabla v_t\|_{H^1}. \tag{3.55}$$

Replacing this inequality in (3.54), we get

$$\begin{aligned}\|q_{tt}\|_{L^2(\Gamma_1)} &\leqslant C\|\nabla v\|_{H^{5/4}}^2 + C\|\nabla v_t\|_{H^1} + C\|a_t\|_{L^\infty}\|v\|_{H^{2.5}} + C\|v_t\|_{H^{2.5}}\\ &\leqslant C\|\nabla v\|_{H^{5/4}}^2 + C\|\nabla v_t\|_{H^1} + C\|a_t\|_{L^\infty}\|v\|_{H^{2.5}} + C\|v_t\|_{H^{2.5}}\\ &\leqslant C\|v\|_{H^2}^{3/2}\|v\|_{H^3}^{1/2} + C\|\nabla v_t\|_{H^1} + C\|a_t\|_{L^\infty}\|v\|_{H^{2.5}} + C\|v_t\|_{H^{2.5}}.\end{aligned}\tag{3.56}$$

In order to bound I_4, we write

$$I_4 = C\big\|\partial_{ttt}\big(a_i^m v^i\big)N_m\big\|_{H^{-1/2}(\partial\Omega)} \leqslant C\sum_m \big\|\partial_{ttt}\big(a_i^m v^i\big)\big\|_{L^2(\Omega)} \tag{3.57}$$

the last inequality following from $\partial_m(a_i^m v^i) = 0$. Therefore,

$$\begin{aligned}
&I_1 + I_2 + I_3 + I_4\\
&\quad\leqslant C\big\|\big(a^T a\big)_{tt}\nabla q\big\|_{L^2} + C\big\|\big(a^T a\big)_t\nabla q_t\big\|_{L^2} + C\|a_{ttt}v\|_{L^2}\\
&\qquad + C\|a_{tt}v_t\|_{L^2} + C\|a_t v_{tt}\|_{L^2} + C\|a v_{ttt}\|_{L^2}\\
&\quad\leqslant C\big(\|\nabla v\|_{H^{0.5}}\|\nabla v\|_{L^\infty} + \|\nabla v_t\|_{H^{0.5}}\big)\|\nabla q\|_{H^1} + C\|\nabla v\|_{L^\infty}\|\nabla q_t\|_{L^2}\\
&\qquad + \|a_{ttt}\|_{L^3}\|v\|_{L^6} + C\big(\|\nabla v\|_{H^{0.5}}\|\nabla v\|_{L^\infty} + \|\nabla v_t\|_{H^{0.5}}\big)\|v_t\|_{H^1}\\
&\qquad + C\|\nabla v\|_{L^\infty}\|v_{tt}\|_{L^2} + C\|v_{ttt}\|_{L^2}.
\end{aligned} \tag{3.58}$$

The third term on the far right side is then estimated as

$$\|a_{ttt}\|_{L^3}\|v\|_{L^6} \leqslant C\big(\|\nabla v\|_{L^3}\|\nabla v\|_{L^\infty}^2 + \|\nabla v_t\|_{L^3}\|\nabla v\|_{L^\infty} + \|\nabla v_{tt}\|_{L^3}\big)\|v\|_{H^1} \tag{3.59}$$

and (3.33) follows. □

4. Local in time solutions

4.1. L^2 estimate on v_{ttt}

Applying ∂_t^3 to (2.4), multiplying the resulting equation by v_{ttt}, and integrating in space and time gives

$$\frac{1}{2}\big\|v_{ttt}(t)\big\|_{L^2}^2 = \frac{1}{2}\big\|v_{ttt}(0)\big\|_{L^2}^2 - \int_0^t\int\big(a_i^k\partial_k q\big)_{ttt}v_{ttt}^i = \frac{1}{2}\big\|v_{ttt}(0)\big\|_{L^2}^2 - \int_0^t\int\partial_k\big(a_i^k q\big)_{ttt}v_{ttt}^i \tag{4.1}$$

where we utilized the Piola identity

$$\partial_k a_i^k = 0, \quad i = 1, 2, 3. \tag{4.2}$$

In order to bound the integral on the right side, we integrate by parts,

$$-\int_0^t\int\partial_k\big(a_i^k q\big)_{ttt}v_{ttt}^i = -\int_0^t\int_{\partial\Omega}\big(a_i^k q\big)_{ttt}v_{ttt}^i N_k + \int_0^t\int\big(a_i^k q\big)_{ttt}\partial_k v_{ttt}^i = I_1 + I_2. \tag{4.3}$$

Since $v^3 = 0$ on Γ_0, we have $\bar\partial v^3 = 0$, where

$$\bar\partial = (\partial_1, \partial_2) \tag{4.4}$$

and $v^3_{ttt} = 0$ on Γ_0. Also, $\bar{\partial}\eta^3 = 0$ on Γ_0, which implies that $a^3_1 = a^3_2 = 0$ on Γ_0. As a consequence,

$$\int_0^t \int_{\Gamma_0} (a^k_i q)_{ttt} v^i_{ttt} N_k = \int_0^t \int_{\Gamma_0} (a^3_i q)_{ttt} v^i_{ttt} = 0. \tag{4.5}$$

Thus, for the boundary term in (4.3), we obtain

$$I_1 = -\int_0^t \int_{\Gamma_1} (a^k_i q N_k)_{ttt} v^i_{ttt} = \int_0^t \int_{\Gamma_1} \Delta_2 \eta^i_{ttt} v^i_{ttt} = -\frac{1}{2}\|\bar{\partial} v_{tt}(t)\|^2_{L^2(\Gamma_1)} + \frac{1}{2}\|\bar{\partial} v_{tt}(0)\|^2_{L^2(\Gamma_1)} \tag{4.6}$$

by using (2.7), (2.11), and integrating by parts in the tangential direction.

Now, we bound the second integral

$$\begin{aligned} I_2 &= \int_0^t \int a^k_i q_{ttt} \partial_k v^i_{ttt} + 3\int_0^t \int (a^k_i)_t q_{tt} \partial_k v^i_{ttt} + 3\int_0^t \int (a^k_i)_{tt} q_t \partial_k v^i_{ttt} + \int_0^t \int (a^k_i)_{ttt} q \partial_k v^i_{ttt} \\ &= I_{21} + I_{22} + I_{23} + I_{24}. \end{aligned} \tag{4.7}$$

Using the incompressibility condition to write

$$\begin{aligned} a^k_i \partial_k v^i_{ttt} &= (a^k_i \partial_k v^i)_{ttt} - 3(a^k_i)_t \partial_k v^i_{tt} - 3(a^k_i)_{tt} \partial_k v^i_t - (a^k_i)_{ttt} \partial_k v^i \\ &= -3(a^k_i)_t \partial_k v^i_{tt} - 3(a^k_i)_{tt} \partial_k v^i_t - (a^k_i)_{ttt} \partial_k v^i, \end{aligned} \tag{4.8}$$

we get

$$\begin{aligned} I_{21} &= -\int_0^t \int 3(a^k_i)_{tt} \partial_k v^i_t q_{ttt} - \int_0^t \int 3(a^k_i)_t \partial_k v^i_{tt} q_{ttt} - \int_0^t \int (a^k_i)_{ttt} \partial_k v^i q_{ttt} \\ &= I_{211} + I_{212} + I_{213}. \end{aligned} \tag{4.9}$$

For I_{211} we integrate by parts in time:

$$\begin{aligned} I_{211} &= -3\int (a^k_i)_{tt} \partial_k v^i_t q_{tt}|_0^t + 3\int_0^t \int \partial_t\big((a^k_i)_{tt} \partial_k v^i_t\big) q_{tt} \\ &\leqslant C\|a_{tt}(0)\|_{L^2} \|\nabla v_t(0)\|_{L^3} \|q_{tt}(0)\|_{L^6} + C\|a_{tt}(t)\|_{L^2} \|\nabla v_t(t)\|_{L^3} \|q_{tt}(t)\|_{L^6} \\ &\quad + C\int_0^t \|a_{ttt}\|_{L^3} \|\nabla v_t\|_{L^6} \|q_{tt}\|_{L^2} + C\int_0^t \|a_{tt}\|_{L^6} \|\nabla v_{tt}\|_{L^3} \|q_{tt}\|_{L^2} \\ &\leqslant P\big(\|v_0\|_{H^{3.5}}, \|v_0\|_{H^{3.5}(\Gamma_1)}\big) + C\|a_{tt}(t)\|_{L^2} \|\nabla v_t(t)\|_{L^3} \|q_{tt}(t)\|_{L^6} \\ &\quad + \int_0^t P\big(\|q_{tt}\|_{L^2}, \|v_{tt}\|_{H^{1.5}}, \|v_t\|_{H^2}, \|v\|_{H^{3.5}}\big). \end{aligned} \tag{4.10}$$

Integrating by parts $I_{212} = -3\int (a^k_i)_t \partial_k v^i_{tt} q_{ttt}$ in space, we have

$$I_{212} = -3\int_0^t \int_{\partial\Omega} (a^k_i)_t v^i_{tt} q_{ttt} N_k + 3\int_0^t \int (a^k_i)_t v^i_{tt} \partial_k q_{ttt} = I_{2121} + I_{2122} \tag{4.11}$$

where we used (4.2). Observe that

$$
\begin{aligned}
I_{2121} &= 3\int_0^t\int_{\Gamma_1} a_i^j\partial_j v^l a_l^k v_{tt}^i q_{ttt} N_k \\
&= -3\int_0^t\int_{\Gamma_1} a_i^j\partial_j v^l \Delta_2\eta_{ttt}^l v_{tt}^i \\
&\quad -\int_0^t\int_{\Gamma_1}\bigl(3\bigl(a_l^k\bigr)_t q_{tt}N_k + 3\bigl(a_l^k\bigr)_{tt} q_t N_k + \bigl(a_l^k\bigr)_{ttt} q N_k\bigr) a_i^j\partial_j v^l v_{tt}^i
\end{aligned} \tag{4.12}
$$

by using (2.6) in the first and (2.11) in the second equality; also note that the integral over Γ_0 vanishes. Integrating by parts in the tangential directions, we obtain

$$
\begin{aligned}
-3\int_0^t\int_{\Gamma_1} a_i^j\partial_j v^l\Delta_2\eta_{ttt}^l v_{tt}^i &= 3\sum_{k=1}^{2}\int_0^t\int \partial_k\eta_{ttt}^l\partial_k\bigl(a_i^j\partial_j v^l v_{tt}^i\bigr) \\
&\leqslant C\int_0^t P\bigl(\|v_{tt}\|_{H^{1.5}}, \|v\|_{H^3}\bigr),
\end{aligned} \tag{4.13}
$$

while the lower order terms are bounded as

$$
\begin{aligned}
&-\int_0^t\int_{\Gamma_1}\bigl(3\bigl(a_l^k\bigr)_t q_{tt}N_k + 3\bigl(a_l^k\bigr)_{tt} q_t N_k + \bigl(a_l^k\bigr)_{ttt} q N_k\bigr) a_i^j\partial_j v^l v_{tt}^i \\
&\quad\leqslant \int_0^t P\bigl(\|q_{tt}\|_{H^1}, \|v_{tt}\|_{H^{1.5}}, \|v\|_{H^{3.5}}, \|q_t\|_{H^{1.5}}, \|v_t\|_{H^2}, \|q\|_{H^{2.5}}\bigr).
\end{aligned} \tag{4.14}
$$

Next, integrating by parts in time gives

$$
\begin{aligned}
I_{2122} &= 3\int\bigl(a_i^k\bigr)_t v_{tt}^i\partial_k q_{tt}\Big|_0^t - 3\int_0^t\int\bigl(a_i^k\bigr)_{tt} v_{tt}^i\partial_k q_{tt} - 3\int_0^t\int\bigl(a_i^k\bigr)_t v_{ttt}^i\partial_k q_{tt} \\
&\leqslant P\bigl(\|v_0\|_{H^{3.5}}\bigr) + C\bigl\|q_{tt}(t)\bigr\|_{H^1}\bigl\|v_{tt}(t)\bigr\|_{L^3}\bigl\|\nabla v(t)\bigr\|_{L^6} \\
&\quad + \int_0^t P\bigl(\|q_{tt}\|_{H^1}, \|v_{ttt}\|_{L^2}, \|v_{tt}\|_{H^1}, \|v_t\|_{H^{1.5}}, \|v\|_{H^3}\bigr).
\end{aligned} \tag{4.15}
$$

By (2.6) and integrating by parts in time we have

$$
I_{213} = \int_0^t\int a_i^j\partial_j v_{tt}^l a_l^k\partial_k v^i q_{ttt} + R. \tag{4.16}
$$

Until the end of this paper, we denote by R the remainder terms. In (4.16), the lower order terms are of the form

$$
\int_0^t\int (a_{tt}\nabla v a + a_t\nabla v a_t + a_t\nabla v_t a)\nabla v q_{ttt}, \tag{4.17}
$$

(written in a symbolic way, omitting all the indices) which can be bounded by

$$
\begin{aligned}
R &\leqslant P\bigl(\|v_0\|_{H^{3.5}}\bigr) + P\bigl(\|v(t)\|_{eH^3}, \|q_{tt}(t)\|_{H^1}\bigr)\|v_t(t)\|_{H^{1.5}} \\
&\quad + \int_0^t P\bigl(\|q_{tt}\|_{H^1}, \|v\|_{H^3}, \|v_t\|_{H^{1.5}}, \|v_{tt}\|_{H^{1.5}}\bigr)\,dt
\end{aligned}
\tag{4.18}
$$

after integrating by parts in time. Integrating by parts in space, the leading term of (4.16) becomes

$$
\begin{aligned}
&\int_0^t \int a_i^j \partial_j v_{tt}^l a_l^k \partial_k v^i q_{ttt} \\
&\quad = -\int_0^t \int a_i^j v_{tt}^l a_l^k \partial_{jk} v^i q_{ttt} - \int_0^t \int a_i^j v_{tt}^l a_l^k \partial_k v^i \partial_j q_{ttt} \\
&\qquad - \int_0^t \int a_i^j v_{tt}^l \partial_j a_l^k \partial_k v^i q_{ttt} + \int_0^t \int_{\Gamma_1} a_i^j v_{tt}^l a_l^k \partial_k v^i q_{ttt} N_j \\
&\quad = I_{2131} + I_{2132} + I_{2133} + I_{2134},
\end{aligned}
\tag{4.19}
$$

where we have omitted the term when the j-th derivatives fall on a_i^j which equals to zero by (4.2). First, observe that the boundary term I_{2134} can be treated exactly as I_{2121} above. Now, using $a_i^j \partial_{jk} v^i = -\partial_k a_i^j \partial_j v^i$ for $k = 1, 2, 3$, we write

$$
\begin{aligned}
I_{2131} &= -\int_0^t \int \partial_k a_i^j v_{tt}^l a_l^k \partial_j v^i q_{ttt} \\
&= -\int \partial_k a_i^j v_{tt}^l a_l^k \partial_j v^i q_{tt}\Big|_0^t + \int_0^t \int \bigl(\partial_k a_i^j v_{tt}^l a_l^k \partial_j v^i\bigr)_t q_{tt} \\
&\leqslant P\bigl(\|v_0\|_{H^{3.5}}, \|v_0\|_{H^{3.5}(\Gamma_1)}\bigr) + C\|q_{tt}(t)\|_{L^6}\|v_{tt}(t)\|_{L^2}\|\nabla v(t)\|_{L^3}, \\
&\quad + \int_0^t P\bigl(\|q_{tt}\|_{H^1}, \|v_{ttt}\|_{L^2}, \|v_{tt}\|_{H^1}, \|v_t\|_{H^{1.5}}, \|v\|_{H^3}\bigr),
\end{aligned}
\tag{4.20}
$$

while

$$
\begin{aligned}
I_{2132} &= -\int a_i^j v_{tt}^l a_l^k \partial_k v^i \partial_j q_{tt}\Big|_0^t + \int_0^t \int \bigl(a_i^j v_{tt}^l a_l^k \partial_k v^i\bigr)_t \partial_j q_{tt} \\
&\leqslant P\bigl(\|v_0\|_{H^{3.5}}, \|v_0\|_{H^{3.5}(\Gamma_1)}\bigr) + C\|q_{tt}(t)\|_{H^1}\|v_{tt}(t)\|_{L^3}\|\nabla v(t)\|_{L^6} \\
&\quad + \int_0^t P\bigl(\|q_{tt}\|_{H^1}, \|v_{ttt}\|_{L^2}, \|v_{tt}\|_{H^1}, \|v_t\|_{H^{1.5}}, \|v\|_{H^3}\bigr).
\end{aligned}
\tag{4.21}
$$

Note that the lower order term I_{2133} is also bounded by the right side of (4.21). For I_{22} we integrate by parts in space

$$
\begin{aligned}
I_{22} &= 3\int_0^t \int_{\Gamma_1} \partial_t a_i^k q_{tt} v_{ttt}^i N_k - 3\int_0^t \int \left(a_i^k\right)_t \partial_k q_{tt} v_{ttt}^i \\
&= -3\int_0^t \int_{\Gamma_1} \left(a_i^j \partial_j v^l a_l^k\right) q_{tt} v_{ttt}^i N_k - 3\int_0^t \int \left(a_i^k\right)_t \partial_k q_{tt} v_{ttt}^i \\
&= 3\int_0^t \int_{\Gamma_1} \left(a_i^j \partial_j v^l \Delta_2 \eta_{tt}^l v_{ttt}^i + a_i^j \partial_j v^l \left(\left(a_l^k\right)_{tt} q + 2\left(a_l^k\right)_t q_t\right) N_k v_{ttt}^i\right) - 3\int_0^t \int \left(a_i^k\right)_t \partial_k q_{tt} v_{ttt}^i \\
&= I_{221} + I_{222}.
\end{aligned}
\tag{4.22}
$$

We denote the first boundary term in I_{221} by I_{2211}. The other two terms in I_{221} are easy to bound. Integrating by parts in the tangential directions, we get

$$
I_{2211} = -3\int_0^t \int_{\Gamma_1} a_i^j \partial_j v^l \bar{\partial} v_t^l \bar{\partial} v_{ttt}^i - 3\int_0^t \int_{\Gamma_1} \bar{\partial}\left(a_i^j \partial_j v^l\right) \bar{\partial} v_t^l v_{ttt}^i \tag{4.23}
$$

which after an additional integration by parts in time leads to

$$
\begin{aligned}
I_{2211} &= -3\int_{\Gamma_1} \left(a_i^j \partial_j v^l \bar{\partial} v_t^l \bar{\partial} v_{tt}^i + \bar{\partial}\left(a_i^j \partial_j v^l\right) \bar{\partial} v_t^l v_{tt}^i\right)\Big|_0^t \\
&\quad + 3\int_0^t \int_{\Gamma_1} \left(a_i^j \partial_j v^l \bar{\partial} v_t^l\right)_t \bar{\partial} v_{tt}^i + \left(\bar{\partial}\left(a_i^j \partial_j v^l\right) \bar{\partial} v_t^l\right)_t v_{tt}^i.
\end{aligned}
\tag{4.24}
$$

Thus,

$$
\begin{aligned}
I_{2211} &\leqslant P\left(\|v_0\|_{H^{3.5}}, \|v_0\|_{H^{3.5}(\Gamma_1)}\right) + C\left\|v_{tt}(t)\right\|_{H^{1.5}} \left\|v_t(t)\right\|_{H^2} \left\|v(t)\right\|_{H^{2.5}} \\
&\quad + \int_0^t P\left(\|v_{tt}\|_{H^{1.5}}, \|v_t\|_{H^{2.5}}, \|v\|_{H^3}\right).
\end{aligned}
\tag{4.25}
$$

Next, for I_{23} we proceed as in I_{22} by first integrating by parts in space

$$
\begin{aligned}
I_{23} &= -3\int_0^t \int_{\Gamma_1} \left(a_i^j \partial_j v^l a_l^k\right)_t q_t v_{ttt}^i N_k - 3\int_0^t \int \left(a_i^k\right)_{tt} \partial_k q_t v_{ttt}^i \\
&= 3\int_0^t \int_{\Gamma_1} a_i^j \partial_j v_t^l \Delta_2 \eta_t^l v_{ttt}^i + R - 3\int_0^t \int \left(a_i^k\right)_{tt} \partial_k q_t v_{ttt}^i,
\end{aligned}
\tag{4.26}
$$

where the remainder term

$$
\begin{aligned}
R &= -3\int_0^t \int_{\Gamma_1} \left(a_i^j\right)_t \partial_j v^l a_l^k q_t v_{ttt}^i N_k - 3\int_0^t \int_{\Gamma_1} a_i^j \partial_j v^l \left(a_l^k\right)_t q_t v_{ttt}^i N_k \\
&\quad + 3\int_0^t \int_{\Gamma_1} a_i^j \partial_j v_t^l \left(a_l^k\right)_t q v_{ttt}^i N_k
\end{aligned}
\tag{4.27}
$$

is bounded by

$$
\begin{aligned}
R \leqslant\ & P\big(\|v_0\|_{H^{3.5}}, \|v_0\|_{H^{3.5}(\Gamma_1)}\big) + C\big\|q_t(t)\big\|_{H^{2.5}}\big\|v_{tt}(t)\big\|_{H^{1.5}}\big\|v(t)\big\|^2_{H^{2.5}} \\
&+ C\big\|q(t)\big\|_{H^{1.5}}\big\|v_{tt}(t)\big\|_{H^{1.5}}\big\|v_t(t)\big\|_{H^{1.5}}\big\|v(t)\big\|_{H^{2.5}} \\
&+ \int_0^t P\big(\|q_{tt}\|_{H^1}, \|q_t\|_{H^{1.5}}, \|q\|_{H^{2.5}}, \|v_{tt}\|_{H^{1.5}}, \|v_t\|_{H^{2.5}}, \|v\|_{H^3}\big).
\end{aligned}
\tag{4.28}
$$

The first boundary term on the far right sides in (4.26) can be bounded similarly as I_{2211} above, by integrating by parts in time. We omit further details.

Lastly, we consider I_{24}. We use that $(a_i^k)_{ttt} = (a_i^j \partial_j v^l a_l^k)_{tt} = a_i^j \partial_j v_{tt}^l a_l^k + \text{l.o.t.}$, where the lower order terms are of the form $a_{tt}\nabla v a$, $a_t \nabla v_t a$, $a_t \nabla v a_t$ (and the resulting integrals are clearly easy to bound). Thus, we estimate only the leading term in I_{24}. We have

$$
I_{24} = \int_0^t \int a_i^j \partial_j v_{tt}^l a_l^k \partial_k v_{ttt}^i q + R
\tag{4.29}
$$

and observe that

$$
\begin{aligned}
\partial_t\big(a_i^j \partial_j v_{tt}^l a_l^k \partial_k v_{tt}^i\big) &= a_i^j \partial_j v_{tt}^l a_l^k \partial_k v_{ttt}^i + a_i^j \partial_j v_{ttt}^l a_l^k \partial_k v_{tt}^i \\
&\quad + \big(a_i^j\big)_t \partial_j v_{tt}^l a_l^k \partial_k v_{tt}^i + a_i^j \partial_j v_{tt}^l \big(a_l^k\big)_t \partial_k v_{tt}^i \\
&= 2a_i^j \partial_j v_{tt}^l a_l^k \partial_k v_{ttt}^i + 2\big(a_i^j\big)_t \partial_j v_{tt}^l a_l^k \partial_k v_{tt}^i.
\end{aligned}
\tag{4.30}
$$

Hence,

$$
\begin{aligned}
I_{24} &= \frac{1}{2}\int a_i^j \partial_j v_{tt}^l a_l^k \partial_k v_{tt}^i q\Big|_0^t - \frac{1}{2}\int_0^t \int a_i^j \partial_j v_{tt}^l a_l^k \partial_k v_{tt}^i q_t - \int_0^t \int \big(a_i^j\big)_t \partial_j v_{tt}^l a_l^k \partial_k v_{tt}^i q + \text{l.o.t.} \\
&\leqslant P\big(\|v_0\|_{H^{3.5}}, \|v_0\|_{H^{3.5}(\Gamma_1)}\big) + C\big\|v_{tt}(t)\big\|_{H^{1.5}}\big\|v_{tt}(t)\big\|_{H^1}\big\|q(t)\big\|_{H^1} \\
&\quad + \int_0^t P\big(\|v_{tt}\|_{H^1}, \|v_t\|_{H^2}, \|v\|_{H^3}, \|q_t\|_{H^2}, \|q\|_{H^2}\big).
\end{aligned}
\tag{4.31}
$$

Therefore, we conclude

$$
\begin{aligned}
&\big\|v_{ttt}(t)\big\|^2_{L^2} + \big\|\bar\partial v_{tt}(t)\big\|^2_{L^2(\Gamma_1)} \\
&\quad \leqslant P\big(\|v_0\|_{H^{3.5}}, \|v_0\|_{H^{3.5}(\Gamma_1)}\big) + \epsilon\big(\big\|v_{tt}(t)\big\|^2_{H^{1.5}} + \big\|q_{tt}(t)\big\|^2_{H^1}\big) \\
&\qquad + \int_0^t P\big(\|v_{ttt}\|_{L^2}, \|v_{tt}\|_{H^{1.5}}, \|v_t\|_{H^{2.5}}, \|v\|_{H^3}, \|q_{tt}\|_{H^1}, \|q_t\|_{H^2}, \|q\|_{H^2}\big).
\end{aligned}
\tag{4.32}
$$

4.2. Tangential H^1 estimate on v_{tt}

Applying $\partial_m \partial_t^2$ to the equation (2.4), multiplying by $\partial_m v_{tt}$, summing for $m = 1, 2$, and integrating in space and time, we get

$$\frac{1}{2}\|\bar\partial v_{tt}(t)\|_{L^2}^2 = \frac{1}{2}\|\bar\partial v_{tt}(0)\|_{L^2}^2 - \int_0^t \int \partial_m \big(a_i^k \partial_k q\big)_{tt} \partial_m v_{tt}^i$$
$$= \frac{1}{2}\|\bar\partial v_{tt}(0)\|_{L^2}^2 - \int_0^t \int a_i^k \partial_{km} q_{tt} \partial_m v_{tt}^i + R. \tag{4.33}$$

Here and in the next section, for simplicity of notation, we modify the summation convention for repeated indices in m with $m = 1, 2$ (while other indices are still summed for $1, 2, 3$). Note that the remainder term R on the right of (4.33) is bounded by

$$R = -\int_0^t \int \big(\partial_m \big(a_i^k\big)_{tt} \partial_k q + \big(a_i^k\big)_{tt} \partial_{km} q + 2\partial_m \big(a_i^k\big)_t \partial_k q_t + 2\big(a_i^k\big)_t \partial_{km} q_t + \partial_m a_i^k \partial_k q_{tt}\big) \partial_m v_{tt}^i$$
$$\leqslant \int_0^t P\big(\|v_{tt}\|_{H^1}, \|v_t\|_{H^{2.5}}, \|v\|_{H^3}, \|q_{tt}\|_{H^1}, \|q_t\|_{H^2}, \|q\|_{H^{2.5}}\big). \tag{4.34}$$

Now, we integrate by parts in the higher order term

$$-\int_0^t \int a_i^k \partial_{km} q_{tt} \partial_m v_{tt}^i = -\int_0^t \int_{\partial\Omega} a_i^k \partial_m q_{tt} \partial_m v_{tt}^i N_k + \int_0^t \int a_i^k \partial_m q_{tt} \partial_{km} v_{tt}^i = I_1 + I_2. \tag{4.35}$$

For I_1, the integral over Γ_0 vanishes, while on Γ_1 we use

$$a_i^k \partial_m q_{tt} N_k = \partial_m \partial_{tt} \big(a_i^k q N_k\big) - \partial_m \big(\big(a_i^k\big)_{tt} q N_k\big) - 2\partial_m \big(\big(a_i^k\big)_t q_t N_k\big) - \partial_m a_i^k q_{tt} N_k \tag{4.36}$$

(to check this, write $\partial_m (a_i^k q_{tt} N_k) = a_i^k \partial_m q_{tt} N_k + \partial_m a_i^k q_{tt} N_k$ and rewrite the second term) and get

$$I_1 = \int_0^t \int_{\Gamma_1} \Delta_2 \partial_m \eta_{tt} \partial_m v_{tt}^i + \int_0^t \int_{\Gamma_1} \big(\partial_m \big(\big(a_i^k\big)_{tt} q\big) + 2\partial_m \big(\big(a_i^k\big)_t q_t\big) + \partial_m a_i^k q_{tt}\big) \partial_m v_{tt}^i N_k$$
$$= -\frac{1}{2}\|\bar\partial^2 v_t(t)\|_{L^2(\Gamma_1)}^2 + \frac{1}{2}\|\bar\partial^2 v_t(0)\|_{L^2(\Gamma_1)}^2$$
$$+ \int_0^t \int_{\Gamma_1} \big(\partial_m \big(\big(a_i^k\big)_{tt} q\big) + 2\partial_m \big(\big(a_i^k\big)_t q_t\big) + \partial_m a_i^k q_{tt}\big) \partial_m v_{tt}^i N_k, \tag{4.37}$$

and the last term on the right side can be bounded by

$$\int_0^t P\big(\|v_{tt}\|_{H^{1.5}}, \|q_{tt}\|_{H^1}, \|q_t\|_{H^{1.5}}, \|v_t\|_{H^{2.5}}, \|q\|_{H^{2.5}}, \|v\|_{H^{3.5}}\big). \tag{4.38}$$

For I_2, we use the divergence free condition to write

$$a_i^k \partial_{km} v_{tt}^i = -\partial_m a_i^k \partial_k v_{tt}^i - \partial_m \big(2\big(a_i^k\big)_t \partial_k v_t^i + \big(a_i^k\big)_{tt} \partial_k v^i\big). \tag{4.39}$$

Thus, we obtain

$$
\begin{aligned}
I_2 &= -\int_0^t \int \left(\partial_m a_i^k \partial_k v_{tt}^i + \partial_m \left(2\left(a_i^k\right)_t \partial_k v_t^i + \left(a_i^k\right)_{tt} \partial_k v^i\right)\right) \partial_m q_{tt} \\
&\leqslant \int_0^t P\left(\|q_{tt}\|_{H^1}, \|v_{tt}\|_{H^{1.5}}, \|v_t\|_{H^2}, \|v\|_{H^3}\right).
\end{aligned}
\tag{4.40}
$$

We conclude

$$
\begin{aligned}
&\left\|\bar\partial v_{tt}(t)\right\|_{L^2}^2 + \left\|\bar\partial^2 v_t(t)\right\|_{L^2(\Gamma_1)}^2 \\
&\quad\leqslant \left\|\bar\partial^2 v_t(0)\right\|_{L^2(\Gamma_1)}^2 + \int_0^t P\left(\|v_{tt}\|_{H^{1.5}}, \|q_{tt}\|_{H^1}, \|q_t\|_{H^2}, \|v_t\|_{H^{2.5}}, \|q\|_{H^{2.5}}, \|v\|_{H^{3.5}}\right).
\end{aligned}
\tag{4.41}
$$

4.3. Tangential H^2 estimate on v_t

Applying $\partial_{lm}\partial_t$ to (2.4), multiplying by $\partial_{lm} v_t$, summing for $l, m = 1, 2$, and integrating in space and time, we get

$$
\begin{aligned}
\frac{1}{2}\left\|\bar\partial^2 v_t(t)\right\|_{L^2}^2 &= \frac{1}{2}\left\|\bar\partial^2 v_t(0)\right\|_{L^2}^2 - \int_0^t \int \partial_{lm}\left(a_i^k \partial_k q\right)_t \partial_{lm} v_t^i \\
&= \frac{1}{2}\left\|\bar\partial^2 v_t(0)\right\|_{L^2}^2 - \int_0^t \int a_i^k \partial_{klm} q_t \partial_{lm} v_t^i - \text{l.o.t.},
\end{aligned}
\tag{4.42}
$$

where the lower order terms on the right are bounded by $\int_0^t P(\|v_t\|_{H^2}, \|q\|_{H^3}, \|v\|_{H^{3.5}}, \|q_t\|_{H^{2.5}})$. Next, integrating by parts, we get similarly as in the previous section

$$
-\int_0^t \int a_i^k \partial_{klm} q_t \partial_{lm} v_t^i = -\int_0^t \int_{\partial\Omega} a_i^k \partial_{lm} q_t \partial_{lm} v_t^i N_k + \int_0^t \int a_i^k \partial_{lm} q_t \partial_{klm} v_t^i = I_1 + I_2,
\tag{4.43}
$$

where

$$
\begin{aligned}
I_1 &= \int_0^t \int_{\partial\Omega} \Delta_2 \partial_{lm} \eta_t \partial_{lm} v_t^i - \text{l.o.t.} \\
&= -\frac{1}{2}\left\|\bar\partial^3 v(t)\right\|_{L^2(\Gamma_1)}^2 + \frac{1}{2}\left\|\bar\partial^3 v(0)\right\|_{L^2(\Gamma_1)}^2 - \text{l.o.t.}
\end{aligned}
\tag{4.44}
$$

and, by using the divergence free condition,

$$
I_2 \leqslant \int_0^t P\left(\|q_t\|_{H^2}, \|v_t\|_{H^2}, \|v\|_{H^3}\right).
\tag{4.45}
$$

Therefore, we conclude

$$
\begin{aligned}
&\left\|\bar\partial^2 v_t(t)\right\|_{L^2}^2 + \left\|\bar\partial^3 v(t)\right\|_{L^2(\Gamma_1)}^2 \\
&\quad\leqslant \left\|\bar\partial^2 v_t(0)\right\|_{L^2}^2 + \left\|\bar\partial^3 v(0)\right\|_{L^2(\Gamma_1)}^2 + \int_0^t P\left(\|v_t\|_{H^2}, \|q\|_{H^3}, \|v\|_{H^{3.5}}, \|q_t\|_{H^{2.5}}\right).
\end{aligned}
\tag{4.46}
$$

4.4. Div-curl estimates

We use the elliptic estimate (cf. [12,17])

$$\|f\|_{H^s} \leqslant C\|f\|_{L^2} + C\|\operatorname{curl} f\|_{H^{s-1}} + C\|\operatorname{div} f\|_{H^{s-1}} + C\|f \cdot N\|_{H^{s-0.5}(\partial\Omega)} \tag{4.47}$$

for $s \geqslant 1$.

We recall that $\bar{\partial} v_{tt} \in L^2(\Gamma_1)$, so in particular $v_{tt}^3 \in H^1(\Gamma_1)$. By (4.47) with $s = 1.5$, we have

$$\|v_{tt}\|_{H^{1.5}} \leqslant C\|v_{tt}\|_{L^2} + C\|\operatorname{curl} v_{tt}\|_{H^{0.5}} + C\|\operatorname{div} v_{tt}\|_{H^{0.5}} + C\|v_{tt}^3\|_{H^1(\Gamma_1)}, \tag{4.48}$$

where we also used $v_{tt}^3 = 0$ on Γ_0. Similarly, applying (4.47) with $s = 2.5$ and $s = 3.5$ respectively, we have

$$\|v_t\|_{H^{2.5}} \leqslant C\|v_t\|_{L^2} + C\|\operatorname{curl} v_t\|_{H^{1.5}} + C\|\operatorname{div} v_t\|_{H^{1.5}} + C\left\|v_t^3\right\|_{H^2(\Gamma_1)} \tag{4.49}$$

and

$$\|v\|_{H^{3.5}} \leqslant C\|v\|_{L^2} + C\|\operatorname{curl} v\|_{H^{2.5}} + C\|\operatorname{div} v\|_{H^{2.5}} + C\left\|v^3\right\|_{H^3(\Gamma_1)}. \tag{4.50}$$

The first term on the right side of (4.48) (same for (4.49) and (4.50)) is of lower order and can be written as

$$\left\|v_{tt}(t)\right\|_{L^2} = \left\|v_{tt}(0)\right\|_{L^2} + t^{1/2}\int_0^t \|v_{ttt}\|_{L^2}. \tag{4.51}$$

By the multiplicative Sobolev inequality, for div v we have

$$\|\operatorname{div} v\|_{H^{2.5}} = \left\|\left(\delta_{ik} - a_i^k\right)\partial_k v^i\right\|_{H^{2.5}} \leqslant \epsilon\|v\|_{H^{3.5}}, \tag{4.52}$$

as well as

$$\|\operatorname{div} v_t\|_{H^{1.5}} = \left\|\left(\delta_{ik} - a_i^k\right)\partial_k v_t^i - \left(a_i^k\right)_t\partial_k v^i\right\|_{H^{1.5}} \leqslant \epsilon\|v_t\|_{H^{2.5}} + C\|a_t\|_{H^{1.5+\delta}}\|v\|_{H^{2.5}} \tag{4.53}$$

and

$$\begin{aligned}\|\operatorname{div} v_{tt}\|_{H^{0.5}} &= \left\|\left(\delta_{ik} - a_i^k\right)\partial_k v_{tt}^i - 2\left(a_i^k\right)_t\partial_k v_t^i - \left(a_i^k\right)_{tt}\partial_k v^i\right\|_{H^{0.5}} \\ &\leqslant \epsilon\|v_{tt}\|_{H^{1.5}} + C\|a_t\|_{H^{1.5+\delta}}\|v_t\|_{H^{1.5}} + C\|a_{tt}\|_{H^{0.5}}\|v\|_{H^{2.5+\delta}}.\end{aligned} \tag{4.54}$$

Recall the Cauchy invariance (cf. [32] for instance)

$$\epsilon_{ijk}\partial_j v^l \partial_k \eta^l = \operatorname{curl} v_0^i, \quad i = 1, 2, 3, \tag{4.55}$$

for $t \geqslant 0$, where ϵ_{ijk} is the antisymmetric tensor defined by $\epsilon_{123} = 1$ with $\epsilon_{ijk} = -\epsilon_{jik}$ and $\epsilon_{ijk} = \epsilon_{jki}$. Thus, we have

$$(\operatorname{curl} v)^i = \epsilon_{ijk}\partial_j v^k = \epsilon_{ijk}\partial_j v^l\left(\delta_{lk} - \partial_k\eta^l\right) + \operatorname{curl} v_0^i, \tag{4.56}$$

where

$$\delta_{lk} - \partial_k \eta^l = -\int_0^t \partial_k \eta_t^l = -\int_0^t \partial_k v^l, \quad k, l = 1, 2, 3, \tag{4.57}$$

which implies

$$\|\operatorname{curl} v\|_{H^{2.5}} \leqslant C\|v\|_{H^{3.5}} \int_0^t \|v\|_{H^{3.5}} + \|\operatorname{curl} v_0\|_{H^{2.5}}. \tag{4.58}$$

Differentiating (4.55) in time, we have

$$0 = \left(\epsilon_{ijk}\partial_j v^l \partial_k \eta^l\right)_t = \epsilon_{ijk}\partial_j v_t^l \partial_k \eta^l + \epsilon_{ijk}\partial_j v^l \partial_k \eta_t^l, \tag{4.59}$$

where the second term on the right vanishes because it is equal to $\epsilon_{ijk}\partial_j v^l \partial_k v^l$ and $\epsilon_{ijk} = -\epsilon_{ikj}$. Thus, we also get

$$\epsilon_{ijk}\partial_j v_t^l \partial_k \eta^l = 0 \tag{4.60}$$

from where

$$(\operatorname{curl} v_t)^i = \epsilon_{ijk}\partial_j v_t^k = \epsilon_{ijk}\partial_j v_t^l\left(\delta_{lk} - \partial_k \eta^l\right). \tag{4.61}$$

Therefore,

$$\|\operatorname{curl} v_t\|_{H^{1.5}} \leqslant C\|\nabla v_t\|_{H^{1.5}}\|I - \nabla\eta\|_{H^{1.5+\delta}} \leqslant C\|v_t\|_{H^{2.5}} \int_0^t \|v\|_{H^{2.5+\delta}}. \tag{4.62}$$

Differentiating (4.60), using (2.7), and rearranging the terms in the equality, we obtain

$$\epsilon_{ijk}\partial_j v_{tt}^l \partial_k \eta^l = -\epsilon_{ijk}\partial_j v_t^l \partial_k v^l. \tag{4.63}$$

Then, we may write

$$(\operatorname{curl} v_{tt})^i = \epsilon_{ijk}\partial_j v_{tt}^k = \epsilon_{ijk}\partial_j v_{tt}^l\left(\delta_{lk} - \partial_k \eta^l\right) - \epsilon_{ijk}\partial_j v_t^l \partial_k v^l, \tag{4.64}$$

from where

$$\|\operatorname{curl} v_{tt}\|_{H^{0.5}} \leqslant C\|v_{tt}\|_{H^{1.5}} \int_0^t \|v\|_{H^{2.5+\delta}} + C\|v_t\|_{H^{1.5}}\|v\|_{H^{2.5+\delta}}. \tag{4.65}$$

Now, we gather the div-curl inequalities to obtain Sobolev estimates on v, v_t, and v_{tt}. Namely, we have

$$\begin{aligned}\|v\|_{H^{3.5}} \leqslant{} & C\left(\left\|v(0)\right\|_{L^2} + \|\operatorname{curl} v_0\|_{H^{2.5}}\right) + t^{1/2}\int_0^t \|v_t\|_{L^2} \\ & + C\|v\|_{H^{3.5}} \int_0^t \|v\|_{H^{3.5}} + C\left\|v^3\right\|_{H^3(\Gamma_1)}\end{aligned} \tag{4.66}$$

and

$$\|v_t\|_{H^{2.5}} \leqslant C\|v_t(0)\|_{L^2} + Ct^{1/2}\int_0^t \|v_{tt}\|_{L^2} + C\|v_t\|_{H^{2.5}}\int_0^t \|v\|_{H^{2.5+\delta}} + C\|a_t\|_{H^{1.5+\delta}}\|v\|_{H^{2.5}} + C\|v_t^3\|_{H^2(\Gamma_1)}. \tag{4.67}$$

Finally,

$$\|v_{tt}\|_{H^{1.5}} \leqslant C\|v_{tt}(0)\|_{L^2} + Ct^{1/2}\int \|v_{ttt}\|_{L^2} + C\|v_{tt}\|_{H^{1.5}}\int_0^t \|v\|_{H^{2.5+\delta}} + C\|v_t\|_{H^{1.5}}\|v\|_{H^{2.5+\delta}} + C\|a_t\|_{H^{1.5+\delta}}\|v_t\|_{H^{1.5}} + C\|a_{tt}\|_{H^{0.5}}\|v\|_{H^{2.5+\delta}} + C\|v_{tt}^3\|_{H^1(\Gamma_1)}. \tag{4.68}$$

5. Closing the estimates

Squaring the estimate (4.66) and using (4.46) for the bound of $\|v^3\|_{H^3(\Gamma_1)}$, we have

$$\|v\|_{H^{3.5}}^2 \leqslant P(\|v_0\|_{H^{3.5}}, \|v_0\|_{H^{3.5}(\Gamma_1)}) + C\|v\|_{H^{3.5}}^2\int_0^t \|v\|_{H^{3.5}}^2 + \int_0^t P(\|v_t\|_{H^2}, \|q\|_{H^3}, \|v\|_{H^{3.5}}, \|q_t\|_{H^{2.5}}). \tag{5.1}$$

By the pressure estimate (3.31),

$$\|q\|_{H^{3.5}} \leqslant C(\|v\|_{H^{3.5}} + 1)\left(\|v_0\|_{H^{2.5}} + C\int_0^t \|v_t\|_{H^{2.5}}\right) + C\|v_t\|_{H^2(\Gamma_1)}. \tag{5.2}$$

This, combined with (4.41) for the bound of $\|v_t\|_{H^2(\Gamma_1)}$, gives

$$\|q\|_{H^{3.5}}^2 \leqslant P(\|v_0\|_{H^{3.5}}, \|v_0\|_{H^{3.5}(\Gamma_1)}) + C\|v\|_{H^{3.5}}^2\left(\|v_0\|_{H^{2.5}}^2 + C\int_0^t \|v_t\|_{H^{2.5}}^2\right) + \int_0^t P(\|v_{tt}\|_{H^{1.5}}, \|q_{tt}\|_{H^1}, \|q_t\|_{H^2}, \|v_t\|_{H^{2.5}}, \|q\|_{H^{2.5}}, \|v\|_{H^{3.5}}). \tag{5.3}$$

Similarly, squaring (4.67) and using (4.41),

$$\|v_t\|_{H^{2.5}}^2 \leqslant P(\|v_0\|_{H^{3.5}}, \|v_0\|_{H^{3.5}(\Gamma_1)}) + C\|v_t\|_{H^{2.5}}^2\int_0^t \|v\|_{H^{2.5+\delta}}^2 + C\|v\|_{H^{2.5+\delta}}^2\left(\|v_0\|_{H^{2.5}}^2 + \int_0^t \|v_t\|_{H^{2.5}}^2\right) + \int_0^t P(\|v_{tt}\|_{H^{1.5}}, \|q_{tt}\|_{H^1}, \|q_t\|_{H^2}, \|v_t\|_{H^{2.5}}, \|q\|_{H^{2.5}}, \|v\|_{H^{3.5}}), \tag{5.4}$$

while combining the square of the estimate (3.32),

$$\|q_t\|_{H^{2.5}}^2 \leqslant C\|v\|_{H^{2.5+\delta}}^2\biggl(P\bigl(\|v_0\|_{H^{3.5}}, \|v_0\|_{H^{3.5}(\Gamma_1)}\bigr) + C\int_0^t \bigl(\|v_{tt}\|_{H^{1.5}}^2 + \|q_t\|_{H^{2.5}}^2\bigr)\biggr)$$
$$+ C\|v\|_{H^{2.5}}^2 + C\|v_t\|_{H^{2.5}}^2$$
$$+ C\bigl(\|v\|_{H^{2.5+\delta}}^4 + \|v_t\|_{H^{2.5}}^2\bigr)\biggl(\|v_0\|_{H^{1.5+\delta}}^2 + \int_0^t \|v_t\|_{H^{1.5+\delta}}^2\biggr) + C\|v_{tt}\|_{H^1(\Gamma_1)}^2 \tag{5.5}$$

(4.32) and (5.4), we obtain

$$\|q_t\|_{H^{2.5}}^2 \leqslant P\bigl(\|v_0\|_{H^{3.5}}, \|v_0\|_{H^{3.5}(\Gamma_1)}\bigr) + C\epsilon\bigl(\|v_{tt}\|_{H^{1.5}}^2 + \|q_{tt}\|_{H^1}^2 + \|v\|_{H^{3.5}}^2\bigr)$$
$$+ P\bigl(\|v_t\|_{H^{2.5}}, \|v\|_{H^{2.5+\delta}}\bigr)\int_0^t P\bigl(\|v\|_{H^{2.5+\delta}}, \|v_t\|_{H^{2.5}}\bigr)$$
$$+ \int_0^t P\bigl(\|q_{tt}\|_{H^1}, \|v_{ttt}\|_{L^2}, \|v\|_{H^{3.5}}, \|v_{tt}\|_{H^{1.5}}, \|v_t\|_{H^{2.5}}, \|q_t\|_{H^2}, \|q\|_{H^2}\bigr). \tag{5.6}$$

Lastly, squaring (4.68) and using (4.32),

$$\|v_{tt}\|_{H^{1.5}}^2 \leqslant P\bigl(\|v_0\|_{H^{3.5}}, \|v_0\|_{H^{3.5}(\Gamma_1)}\bigr) + C\epsilon\bigl(\|v_{tt}\|_{H^{1.5}}^2 + \|q_{tt}\|_{H^1}^2 + \|v\|_{H^{3.5}}^2\bigr)$$
$$+ C\|v_{tt}\|_{H^{1.5}}^2\int_0^t \|v\|_{H^{2.5+\delta}}^2 + C\|v\|_{H^{2.5+\delta}}^2\int_0^t P\bigl(\|v_{tt}\|_{H^{1.5}}, \|v_t\|_{H^{2.5}}\bigr)$$
$$+ \int_0^t P\bigl(\|q_{tt}\|_{H^1}, \|v_{ttt}\|_{L^2}, \|v\|_{H^{3.5}}, \|v_{tt}\|_{H^{1.5}}, \|v_t\|_{H^{2.5}}, \|q_t\|_{H^2}, \|q\|_{H^2}\bigr), \tag{5.7}$$

while squaring (3.33) and (4.32) give

$$\|q_{tt}\|_{H^1}^2 \leqslant P\bigl(\|v_0\|_{H^{3.5}}, \|v_0\|_{H^{3.5}(\Gamma_1)}\bigr) + C\epsilon\bigl(\|v_{tt}\|_{H^{1.5}}^2 + \|q_{tt}\|_{H^1}^2 + \|v\|_{H^{3.5}}^2\bigr)$$
$$+ C\|v_{tt}\|_{H^{1.5}}^2\biggl(\|v_0\|_{H^1}^2 + \int_0^t \|v_t\|_{H^1}^2\biggr)$$
$$+ P\bigl(\|v\|_{H^{3.5}}, \|v_t\|_{H^{2.5}}\bigr)\int_0^t P\bigl(\|q_{tt}\|_{H^1}, \|v_{ttt}\|_{L^2}, \|q_t\|_{H^2}, \|v_{tt}\|_{H^1}, \|v_t\|_{H^{2.5}}, \|v\|_{H^{3.5}}\bigr)$$
$$+ \int_0^t P\bigl(\|q_{tt}\|_{H^1}, \|v_{ttt}\|_{L^2}, \|v\|_{H^{3.5}}, \|v_{tt}\|_{H^{1.5}}, \|v_t\|_{H^{2.5}}, \|q_t\|_{H^2}, \|q\|_{H^2}\bigr). \tag{5.8}$$

Combining all the estimates, we obtain a Gronwall type inequality yielding the a priori estimates for the local in time existence.

Acknowledgements

I.K. was supported in part by the NSF grant DMS-1311943 and DMS-1615239. We would like to thank Peter Constantin for useful discussions, and especially for Lemma 3.2 and the inequality (3.30). We also thank Marcelo Disconzi and the referee for useful suggestions.

References

[1] T. Alazard, N. Burq and C. Zuily, On the water-wave equations with surface tension, *Duke Math. J.* **158**(3) (2011), 413–499. doi:10.1215/00127094-1345653.
[2] T. Alazard, N. Burq and C. Zuily, Low regularity Cauchy theory for the water–waves problem: canals and swimming pools, Journeés Équations aux Dérivées Partielles, Biarritz 6 Juin–10 Juin, Exposé no. III (2011), 20 p.
[3] T. Alazard and J.M. Delort, Global solutions and asymptotic behavior for two dimensional gravity water waves, 2013 (arXiv:1305.4090).
[4] D.M. Ambrose and N. Masmoudi, The zero surface tension limit of two-dimensional water waves, *Comm. Pure Appl. Math.* **58**(10) (2005), 1287–1315. doi:10.1002/cpa.20085.
[5] D.M. Ambrose and N. Masmoudi, The zero surface tension limit of three-dimensional water waves, *Indiana Univ. Math. J.* **58**(2) (2009), 479–521. doi:10.1512/iumj.2009.58.3450.
[6] C. Amrouche, V. Girault and J. Giroire, Dirichlet and Neumann exterior problems for the n-dimensional Laplace operator: An approach in weighted Sobolev spaces, *J. Math. Pures Appl. (9)* **76**(1) (1997), 55–81. doi:10.1016/S0021-7824(97)89945-X.
[7] J.T. Beale, The initial value problem for the Navier–Stokes equations with a free surface, *Comm. Pure Appl. Math.* **34**(3) (1981), 359–392. doi:10.1002/cpa.3160340305.
[8] J.T. Beale, T.Y. Hou and J.S. Lowengrub, Growth rates for the linearized motion of fluid interfaces away from equilibrium, *Comm. Pure Appl. Math.* **46**(9) (1993), 1269–1301. doi:10.1002/cpa.3160460903.
[9] A. Castro, D. Cordoba, C. Fefferman, F. Gancedo and J. Gomez-Serrano, Splash singularity for water waves, *Proceedings of the National Academy of Sciences* **109**(3) (2012), 733–738. doi:10.1073/pnas.1115948108.
[10] A. Castro and D. Lannes, Well-posedness and shallow-water stability for a new Hamiltonian formulation of the water waves equations with vorticity, 2014 (arXiv:1402.0464 [math.AP]).
[11] A.L. Cauchy, Sur l'état du fluide à une époque quelconque du mouvement, *Mémoires extraits des recueils de l'Académie des sciences de l'Institut de France, Sciences mathématiques et physiques*, Tome I, Seconde Partie, 1827, pp. 33–73.
[12] C.H.A. Cheng and S. Shkoller, Solvability and regularity for an elliptic system prescribing the curl, divergence, and partial trace of a vector field on Sobolev-class domains (arXiv:1408.2469v1).
[13] D. Christodoulou and H. Lindblad, On the motion of the free surface of a liquid, *Comm. Pure Appl. Math.* **53**(12) (2000), 1536–1602. doi:10.1002/1097-0312(200012)53:12<1536::AID-CPA2>3.0.CO;2-Q.
[14] P. Constantin, An Eulerian-Lagrangian approach for incompressible fluids: Local theory, *J. Amer. Math. Soc.* **14** (2000), 263–278. doi:10.1090/S0894-0347-00-00364-7.
[15] P. Constantin, Euler and Navier–Stokes equations, *Publ. Mat.* **52**(2) (2008), 235–265. doi:10.5565/PUBLMAT_52208_01.
[16] P. Constantin, Private communication.
[17] D. Coutand and S. Shkoller, Well-posedness of the free-surface incompressible Euler equations with or without surface tension, *J. Amer. Math. Soc.* **20**(3) (2007), 829–930. doi:10.1090/S0894-0347-07-00556-5.
[18] D. Coutand and S. Shkoller, A simple proof of well-posedness for the free-surface incompressible Euler equations, *Discrete Contin. Dyn. Syst. Ser* **3**(3) (2010), 429–449. doi:10.3934/dcdss.2010.3.429.
[19] W. Craig, An existence theory for water waves and the Boussinesq and Korteweg-de Vries scaling limits, *Comm. Partial Differential Equations* **10**(8) (1985), 787–1003. doi:10.1080/03605308508820396.
[20] M.M. Disconzi and D.G. Ebin, On the limit of large surface tension for a fluid motion with free boundary, *Comm. Partial Differential Equations* **39**(4) (2014), 740–779. doi:10.1080/03605302.2013.865058.
[21] M.M. Disconzi and D.G. Ebin, The free boundary Euler equations with large surface tension, *J. Differential Equations* **261**(2) (2016), 821–889. doi:10.1016/j.jde.2016.03.029.
[22] D.G. Ebin, The equations of motion of a perfect fluid with free boundary are not well posed, *Comm. Partial Differential Equations* **12**(10) (1987), 1175–1201. doi:10.1080/03605308708820523.
[23] U. Frisch and B. Villone, Cauchy's almost forgotten Lagrangian formulation of the Euler equation for 3D incompressible flow, *Eur. Phys. J. H* **39** (2014), 325–351. doi:10.1140/epjh/e2014-50016-6.
[24] P. Germain, N. Masmoudi and J. Shatah, Global solutions for the gravity surface water waves equation in dimension 3, *Ann. of Math.* **175**(2) (2012), 691–754. doi:10.4007/annals.2012.175.2.6.

[25] J. Hunter, M. Ifrim and D. Tataru, Two dimensional water waves in holomorphic coordinates, 2014 (arXiv:1401.1252).

[26] M. Ifrim and D. Tataru, Two dimensional water waves in holomorphic coordinates II: global solutions, 2014 (arXiv:1404.7583).

[27] T. Iguchi, Well-posedness of the initial value problem for capillary-gravity waves, *Funkcial. Ekvac.* **44**(2) (2001), 219–241.

[28] A.D. Ionescu and F. Pusateri, Global solutions for the gravity water waves system in 2d, *Invent. Math.* **199**(3) (2015), 653–804. doi:10.1007/s00222-014-0521-4.

[29] M. Köhne, J. Prüss and M. Wilke, Qualitative behaviour of solutions for the two-phase Navier–Stokes equations with surface tension, *Math. Ann.* **356**(2) (2013), 737–792. doi:10.1007/s00208-012-0860-7.

[30] I. Kukavica and A. Tuffaha, On the 2D free boundary Euler equation, *Evolution Equations and Control Theory* **1** (2012), 297–314. doi:10.3934/eect.2012.1.297.

[31] I. Kukavica and A. Tuffaha, A regularity result for the incompressible Euler equation with a free interface, *Applied Mathematics and Optimization, Appl. Math. Optim.* **69** (2014), 337–358. doi:10.1007/s00245-013-9221-5.

[32] I. Kukavica, A. Tuffaha and V. Vicol, On the local existence and uniqueness for the 3D Euler equation with a free interface, *Appl. Math. Optim.* (to appear). doi:10.1007/s00245-016-9360-6.

[33] I. Kukavica, A. Tuffaha, V. Vicol and F. Wang, On the existence for the free interface 2D Euler equation with a localized vorticity condition, *Appl. Math. Optim.* **73** (2016), 523–544. doi:10.1007/s00245-016-9346-4.

[34] D. Lannes, Well-posedness of the water-waves equations, *J. Amer. Math. Soc.* **18**(3) (2005), 605–654 (electronic). doi:10.1090/S0894-0347-05-00484-4.

[35] L. Lichtenstein, Über einige Existenzprobleme der Hydrodynamik homogener, unzusammendrückbarer, reibungsloser Flüssigkeiten und die Helmholtzschen Wirbelsätze, *Math. Z.* **23**(1) (1925), 89–154 (in German). doi:10.1007/BF01506223.

[36] H. Lindblad, Well-posedness for the linearized motion of an incompressible liquid with free surface boundary, *Comm. Pure Appl. Math.* **56**(2) (2003), 153–197. doi:10.1002/cpa.10055.

[37] H. Lindblad, Well-posedness for the motion of an incompressible liquid with free surface boundary, *Ann. of Math. (2)* **162**(1) (2005), 109–194. doi:10.4007/annals.2005.162.109.

[38] V.I. Nalimov, The Cauchy-Poisson problem, *Dinamika Splošn. Sredy* Vyp. 18 Dinamika Zidkost. so Svobod. Granicami (1974), 104–210, 254.

[39] M. Ogawa and A. Tani, Free boundary problem for an incompressible ideal fluid with surface tension, *Math. Models Methods Appl. Sci.* **12**(12) (2002), 1725–1740. doi:10.1142/S0218202502002306.

[40] J. Prüss, S. Shimizu and M. Wilke, Qualitative behaviour of incompressible two-phase flows with phase transitions: The case of non-equal densities, *Comm. Partial Differential Equations* **39**(7) (2014), 1236–1283. doi:10.1080/03605302.2013.821131.

[41] J. Prüss, G. Simonett and R. Zacher, On the qualitative behaviour of incompressible two-phase flows with phase transitions: The case of equal densities, *Interfaces Free Bound.* **15**(4) (2013), 405–428. doi:10.4171/IFB/308.

[42] B. Schweizer, On the three-dimensional Euler equations with a free boundary subject to surface tension, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **22**(6) (2005), 753–781. doi:10.1016/j.anihpc.2004.11.001.

[43] J. Shatah and C. Zeng, Geometry and a priori estimates for free boundary problems of the Euler equation, *Comm. Pure Appl. Math.* **61**(5) (2008), 698–744. doi:10.1002/cpa.20213.

[44] J. Shatah and C. Zeng, Local well-posedness for fluid interface problems, *Arch. Ration. Mech. Anal.* **199**(2) (2011), 653–705. doi:10.1007/s00205-010-0335-5.

[45] M. Shinbrot, The initial value problem for surface waves under gravity. I. The simplest case, *Indiana Univ. Math. J.* **25**(3) (1976), 281–300. doi:10.1512/iumj.1976.25.25023.

[46] A. Tani, Small-time existence for the three-dimensional Navier-Stokes equations for an incompressible fluid with a free surface, *Arch. Rational Mech. Anal.* **133**(4) (1996), 299–331. doi:10.1007/BF00375146.

[47] S. Wu, Well-posedness in Sobolev spaces of the full water wave problem in 2-D, *Invent. Math.* **130**(1) (1997), 39–72. doi:10.1007/s002220050177.

[48] S. Wu, Well-posedness in Sobolev spaces of the full water wave problem in 3-D, *J. Amer. Math. Soc.* **12**(2) (1999), 445–495. doi:10.1090/S0894-0347-99-00290-8.

[49] S. Wu, Global well posedness of the 3-D full water wave problem, *Invent. Math.* **184**(1) (2011), 125–220. doi:10.1007/s00222-010-0288-1.

[50] L. Xu and Z. Zhang, On the free boundary problem to the two viscous immiscible fluids, *J. Differential Equations* **248**(5) (2010), 1044–1111. doi:10.1016/j.jde.2009.11.001.

[51] H. Yosihara, Gravity waves on the free surface of an incompressible perfect fluid of finite depth, *Publ. Res. Inst. Math. Sci.* **18**(1) (1982), 49–96. doi:10.2977/prims/1195184016.

[52] H. Yosihara, Capillary-gravity waves for an incompressible ideal fluid, *J. Math. Kyoto Univ.* **23**(4) (1983), 649–694.

[53] P. Zhang and Z. Zhang, On the free boundary problem of three-dimensional incompressible Euler equations, *Comm. Pure Appl. Math.* **61**(7) (2008), 877–940. doi:10.1002/cpa.20226.