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ω-Groundedness of argumentation and completeness of grounded dialectical proof procedures

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Abstract. Dialectical proof procedures in assumption-based argumentation are in general sound but not complete with respect to both the credulous and skeptical semantics (due to non-terminating loops). This raises the question of whether we could describe exactly what such procedures compute.

In a previous paper, we introduce infinite arguments to represent possibly non-terminating computations and present dialectical proof procedures that are both sound and complete with respect to the credulous semantics of assumption-based argumentation with infinite arguments.

In this paper, we study whether and under what conditions dialectical proof procedures are both sound and complete with respect to the grounded semantics of assumption-based argumentation with infinite arguments. We introduce the class of ω-grounded and finitary-defensible argumentation frameworks and show that finitary assumption-based argumentation is ω-grounded and finitary-defensible. We then present dialectical procedures that are sound and complete wrt finitary assumption-based argumentation.

Keywords: Dialectical proof procedure, infinite argument, grounded semantics, soundness and completeness

1. Introduction

Dialectical proof procedures for assumption-based argumentation ([12,13,15,17–19,22,24,32,33]) could be viewed as an integration of the dialectical procedures of abstract argumentation ([8,20,21,26,34,35]) with the process of argument constructions where the latter could get into a non-terminating loop leading to the incompleteness wrt both credulous and skeptical semantics.

A natural question here is: Can we give a precise semantical characterization of what dialectical proof procedures compute?

Representing possibly non-terminating loops as infinite arguments, we present dialectical proof procedures in [31], that are sound and complete wrt credulous semantics. Continuing this line of research, in this paper we present dialectical procedures that are sound and complete with respect to the grounded semantics.

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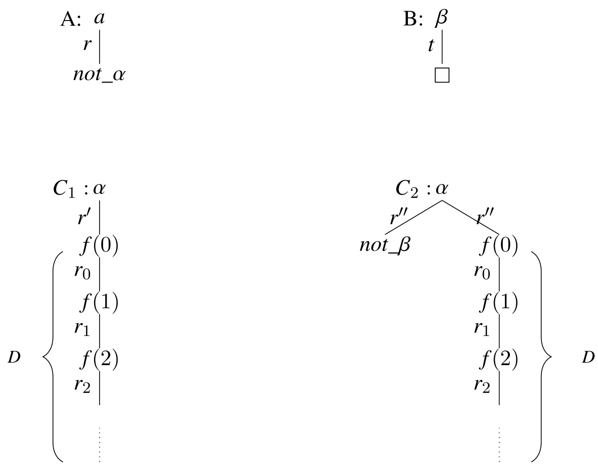

Fig. 1. Arguments of $\mathcal{F}_1$ and $\mathcal{F}_2$.

A detailed analysis of the need for infinite arguments in understanding the semantics of dialectical proof procedures is given in [31].

To illustrate how some practical systems of dialectical procedures (like SWI-Prolog ([36])) handle infinite arguments, we analyze the working of SWI-Prolog on two examples taken from [31] below.[1]

$$\mathcal{F}_1 : r : a \leftarrow not\_\alpha \qquad r' : \alpha \leftarrow f(0) \qquad r_n : f(n) \leftarrow f(n+1), \quad n \geqslant 0$$
$$t : \beta \leftarrow$$
$$\mathcal{F}_2 : r : a \leftarrow not\_\alpha \qquad r'' : \alpha \leftarrow not\_\beta, f(0) \qquad r_n : f(n) \leftarrow f(n+1), \quad n \geqslant 0$$
$$t : \beta \leftarrow$$

The behaviors of $\mathcal{F}_1$ and $\mathcal{F}_2$ could be explained by the proof trees (viewed arguments) in Fig. 1.

SWI-prolog does not deliver any answer to the queries $\leftarrow a$? and "$\leftarrow \alpha$?" because of infinite recursion in executing α.[2]

How could we interpret the outputs of SWI-prolog declaratively?

SWI-prolog could not overcome the non-termination of the process to construct an argument supporting α due to the "infinite-loop" represented by infinite argument C_1.

We could interpret this observation as indicating that *infinite arguments (representing "infinite-loop") do not support their conclusions the way finite arguments do.*

We could also say that the failure of SWI-prolog to deliver any answer to the query "$\leftarrow a$?" is due to the non-acceptability of argument A. It implies that *though infinite argument C_1 can not support its conclusion, it could still attack argument A.*

SWI-prolog delivers respectively the answers "True", "False", "True" to the queries $\leftarrow a$?, "$\leftarrow \alpha$?" and "$\leftarrow \beta$?" wrt program $\mathcal{F}_2$.[3] The answer to "$\leftarrow \alpha$?" is false because the only argument supporting it is C_2 that is based on assumption *not_β* that is attacked by argument B. We could hence say that *infinite arguments are attacked the same way finite arguments are attacked.*

[1]The written codes in SWI-prolog of both $\mathcal{F}_1$, $\mathcal{F}_2$ can be found in appendix A.2.
[2]See Fig. 10 in appendix A.2.
[3]See Fig. 11 in appendix A.2.

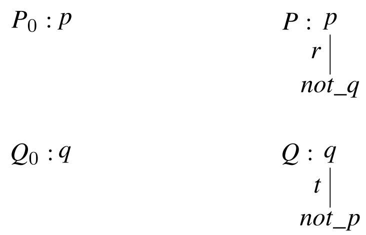

Fig. 2. Illustration of Example 1.

These observations suggest that infinite arguments still attack (or are attacked by) other arguments as finite arguments do though they do not support their own conclusions.

To capture this peculiar character of infinite arguments, [31] views infinite arguments as self-attacking arguments.

In [31], two proof procedures for credulous semantics are presented. In one, proof trees are constructed explicitly. A filtering mechanism to memorizing the attacked assumptions of both proponent and opponent is employed to prevent each player from constructing the same proof trees twice. The other procedure is simply a flattening of the first. The filtering mechanism in these procedures can not be applied for grounded semantics. The following example illustrates this point.

Example 1. Consider the argumentation framework $\mathcal{F}$ represented by the simple logic program:

$$r : p \leftarrow not\_q \qquad t : q \leftarrow not\_p$$

To support p, the proof procedures in [31] would correctly deliver argument P (See Fig. 2).

The proponent first constructs argument P to support p. The opponent responds by constructing argument Q to attack P at assumption not_q. The filtering mechanism memorizes assumption not_q after the opponent attacks it and does not allow the opponent to attack it again. The procedures hence stop and deliver P as an admissible argument supporting p.

As the grounded extension of $\mathcal{F}$ is empty, procedures for grounded extension need to drop this filtering mechanism.

A consequence of dropping the filtering mechanism in the procedures for credulous semantics in [31] is that an argument could be constructed repeatedly many times at different stages in the computation. To distinguish between these distinct "copies" of the same argument, we consider them together with their histories.

The grounded extension of an argumentation framework $AF = (Ar, att)$ coincides with the least fixed point of the characteristic function F_{AF}. It hence follows that the grounded extension of AF contains the set $F^{\omega}_{AF}(\emptyset) = \cup\{F^{i}_{AF}(\emptyset) | i \text{ is a natural number}\}$.[4]

[4] Readers who are familiar with theory of ordinals could recognize that the grounded extension could be "computed" as the least upper bound of the chain $(F^{\theta}_{AF}(\emptyset))_{\theta}$ where θ is an ordinal and

$$F^{0}_{AF}(\emptyset) = \emptyset,$$

$$F^{\theta+1}_{AF}(\emptyset) = F_{AF}\big(F^{\theta}_{AF}(\emptyset)\big),$$

$$F^{\theta}_{AF}(\emptyset) = \cup\big\{F^{\gamma}_{AF}(\emptyset) | \gamma < \theta\big\} \quad \text{if } \theta \text{ is a limit ordinal.}$$

Note that ω is the least limit ordinal.

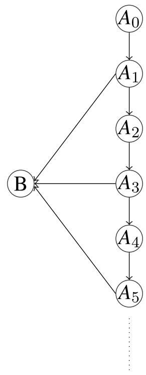

Fig. 3. An argumentation framework that is not ω-grounded.

To check whether an argument A is groundedly accepted, a dialectical proof procedure would compute the defenders of A (i.e. arguments attacking those attacking A) and then check whether such defenders could also be defended and so on. In cases where there are infinitely many arguments attacking A, such process may not succeed.

Example 2. Let $AF = (Ar, att)$ with $Ar = \{B, A_0, A_1, \ldots, A_i, A_{i+1}, \ldots\}$ and $att = \{(A_{2k+1}, B) | k \geqslant 0\} \cup \{(A_k, A_{k+1}) | k \geqslant 0\}$. (A graphical illustration of this argumentation framework is in Fig. 3).

It is easy to see that for each natural number $\geqslant 1$: $F^i_{AF}(\emptyset) = \{A_0, A_2, \ldots, A_{2(i-1)}\}$. Hence $F^{\omega}(\emptyset) = \cup\{F^i_{AF}(\emptyset) | i$ is a natural number$\} = \{A_0, A_2, \ldots, A_{2i}, \ldots\}$.

The grounded extension of AF is: $F_{AF}(F^{\omega}_{AF}(\emptyset)) = F^{\omega}_{AF}(\emptyset) \cup \{B\}$.

To check whether B is groundedly accepted, a dialectical procedure would need to compute the infinite set $\{A_{2k+1} | k \geqslant 0\}$ of arguments attacking B and find the infinite set $\{A_{2k} | k \geqslant 0\}$ of defenders of B. In general, no dialectical proof procedures could carry out such task successfully.

To address this issue, we introduce the class of ω-grounded argumentation frameworks where the grounded extension is "computed" after at most ω steps, i.e. the grounded extension of AF coincides with $F^{\omega}_{AF}(\emptyset) = \cup\{F^i_{AF}(\emptyset) | i$ is a natural number$\}$. We then present several key results:

- We show that an argumentation framework is ω-grounded iff it is finitary defensible (i.e. all minimal defences of non-selfattacking arguments are finite);
- We show that finitary assumption-based argumentation frameworks (see Definition 1) with infinite arguments are finitary defensible;
- We present two dialectical proof procedures that are both sound and complete wrt finitary assumption-based frameworks with infinite arguments where in the first procedure, proof trees together with their histories are fully and explicitly represented to shed light on the construction process of arguments and counter arguments during a derivation and hence providing key insights into the soundness and completeness of the procedure. We then present another procedure that is simply the result of flattening the first where most of the data structures representing the proof tree structures of the arguments are striped away. Consequently, the second procedure is both sound and complete but with much simpler data structure and hence much more amenable to possible implementation.

The rest of this paper is organized as follows. In Section 2, we recall the basic notions of abstract and assumption-based argumentation as well as the machinery of infinite arguments. Section 3 proposes the concept of ω-groundedness of argumentation framework and show that finitary assumption-based argument systems are ω-grounded. We then present a dialectical proof procedure wrt grounded semantics in Section 4. The soundness and completeness of the grounded proof procedure are discussed in Sections 5–6. Further we present the flatten version of the proof procedure in Section 7. We conclude and discuss possible expansions of our works in Section 8.

2. Preliminaries: Argumentation with infinite arguments

In this section, we recall key basic concepts of abstract argumentation from [16] and [14] and infinite arguments together with some illustrating examples in assumption-based argumentation from [31] and [30].

2.1. Abstract argumentation

Following [14,16], an argumentation framework is a pair $AF = (Ar, att)$ where Ar is a set of arguments, and $att \subseteq Ar \times Ar$ so that $(X, Y) \in att$ specifies that X attacks Y. A set of arguments S attacks an argument X if some argument in S attacks X. S attacks another set of arguments S' if S attacks some argument in S'. Moreover we say that S defends X iff S attacks all arguments attacking X. We also say that an argument X is *defensible* if it is defended by some set of arguments.

A set $S \subseteq Ar$ is said to be

- *conflict-free* iff S does not attack itself; and
- *admissible* iff S is conflict-free and S defends each argument in S; and
- a *preferred extension* if S is maximally (wt set inclusion) admissible;[5] and
- a *complete extension* if S is admissible and contains each argument it defends.

The *characteristic function* of AF, denoted by F_{AF}, is defined by

$$F_{AF} : 2^{Ar} \rightarrow 2^{Ar}$$

where

$$F_{AF}(S) = \{X \in Ar | S \text{ defends } X\}$$

It is straightforward to see that F_{AF} is monotonic (wrt set inclusion). Since 2^{Ar} is a complete partial order (wrt set inclusion),[6] F_{AF} has a least fixed point.[7]

The *grounded extension* of AF denoted by GE_{AF}, is defined as the least fixed point of F_{AF}.[8]

As the set of complete extensions of AF is a complete semilattice ([16]),[9] the grounded extension coincides with the least complete extension of AF.

[5] Assuming the axiom of choice (or equivalently the maximality principles) it follows immediately that for each admissible set S of arguments, there exists a preferred extensions E containing S (i.e $S \subseteq E$). More on this topic, see [9,29].

[6] A complete partial order is a partial order that has a bottom element and each directed subset has a least upper bound.

[7] More on fixpoints and least fixpoints, see Knaster–Tarski fixpoint theorem and fixpoint theorems for complete partial oder in [9].

[8] It is not difficult to show that grounded extension is also admissible (see [16] for more about semantics of argumentation frameworks).

[9] A partial order is a complete semilattice if each nonempty subset has a glb and each increasing sequence has a lub.

2.2. Assumption-based argumentation

Given a logical language $\mathcal{L}$, a *standard assumption-based argumentation (ABA) framework* ([6]) is a triple $\mathcal{F} = (\mathcal{R}, \mathcal{A}, \overline{})$ where $\mathcal{R}$ is a set of inference rules of the form $l_0 \leftarrow l_1, \ldots l_n$ ($n \geqslant 0$ and $l_0, \ldots, l_n \in \mathcal{L}$), and $\mathcal{A} \subseteq \mathcal{L}$ is a set of assumptions, and $\overline{}$ is a (total) one-one mapping from $\mathcal{A}$ into $\mathcal{L} \setminus \mathcal{A}$, where $\overline{x}$ is referred to as the *contrary* of x, and assumptions in $\mathcal{A}$ do not appear in the heads of rules.

Remark 1. In non-standard ABA frameworks ([24]), the contrary $\overline{\alpha}$ of an assumption α could be a set. Such non-standard frameworks could be translated into equivalent standard ones by introducing a new atom α' for each assumption α and i) define α' as the contrary of α; and ii) for each $\delta \in \overline{\alpha}$, add a new rule: $\alpha' \leftarrow \delta$ to $\mathcal{R}$.

Remark 2. Logic programming is a well-known instance of standard ABA where the contrary of a negation-as-failure assumption *not_p* is p.

Remark 3. For each rule r of the form $l_0 \leftarrow l_1, \ldots l_n$, l_0 and the set $\{l_1, \ldots, l_n\}$ are referred respectively as the head and the body of r and denoted by $hd(r)$, $bd(r)$.

Further the set of assumptions (resp non-assumptions) appearing in the body of r is denoted by $Ass(r)$ (resp. $NAss(r)$).

Definition 1 (Finitary ABA)**.** An ABA framework $\mathcal{F} = (\mathcal{R}, \mathcal{A}, \overline{})$ is finitary if for each sentence $\delta \in \mathcal{L}$, the set of rules with head δ is finite.[10]

Convention 1. *From now on until the end of the paper,*

- *we restrict our consideration to standard finitary ABA. Hence whenever we refer to an ABA framework, we mean a* standard finitary one*; and*
- *if not otherwise mentioned, we assume an arbitrary but fixed finitary standard assumption-based argumentation framework* $\mathcal{F} = (\mathcal{R}, \mathcal{A}, \overline{})$.

Definition 2 (Partial Proof)**.** Given an ABA $\mathcal{F}$, a partial proof supporting σ_0 (wrt $\mathcal{F}$) is a finite sequence of the form

$$(root, \sigma_0).(r_1, \sigma_1). \ldots .(r_n, \sigma_n)$$

where $r_i \in \mathcal{R}$, $i \geqslant 1$ such that $\sigma_{i-1} = hd(r_i)$ and $\sigma_i \in bd(r_i)$. If $bd(r_i) = \emptyset$ then $\sigma_i = true$.

Example 3. Let's consider an argumentation framework $\mathcal{F}_2$ in the introduction.

$$\mathcal{F}_2 : r : a \leftarrow not\_\alpha \qquad r'' : \alpha \leftarrow not\_\beta, f(0) \qquad r_n : f(n) \leftarrow f(n+1), \quad n \geqslant 0$$
$$t : \beta \leftarrow$$

[10] As we will see later, the finitariness guarantees that the corresponding argumentation framework is ω-grounded (see Theorem 2) and hence all possible attacks against proponent arguments could be considered (see Remark 7).

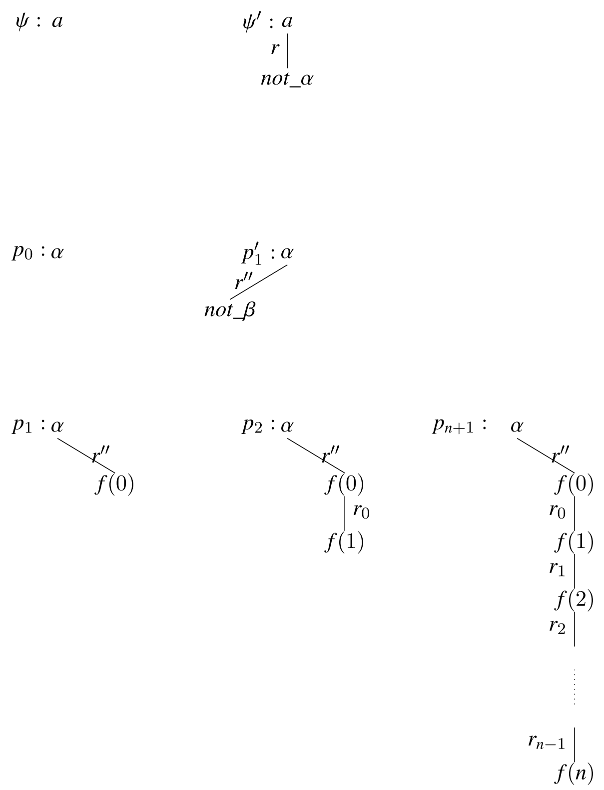

Fig. 4. A graphical representation of partial proofs of Example 3.

Some partial proofs supporting a and α are described below and illustrated in Fig. 4.

$$\psi = (root, a)$$
$$\psi' = (root, a).(r, not\_\alpha)$$
$$p_0 = (root, \alpha)$$
$$p'_1 = (root, \alpha).\big(r'', not\_\beta\big)$$
$$p_1 = (root, \alpha).\big(r'', f(0)\big)$$
$$p_2 = (root, \alpha).\big(r'', f(0)\big).\big(r_0, f(1)\big)$$
$$\ldots$$
$$p_{n+1} = (root, \alpha).\big(r'', f(0)\big).\big(r_0, f(1)\big) \ldots \big(r_{n-1}, f(n)\big)$$

We next define partial proof trees where we identify the nodes in such trees with the partial proofs representing the unique paths from the root to them.

Definition 3 (Partial Proof Trees)**.** A *partial proof tree* (or just *proof tree* for simplification) T for a sentence σ_0 wrt $\mathcal{F}$ is a non-empty set of partial proofs supporting σ_0 wrt $\mathcal{F}$ such that for each partial

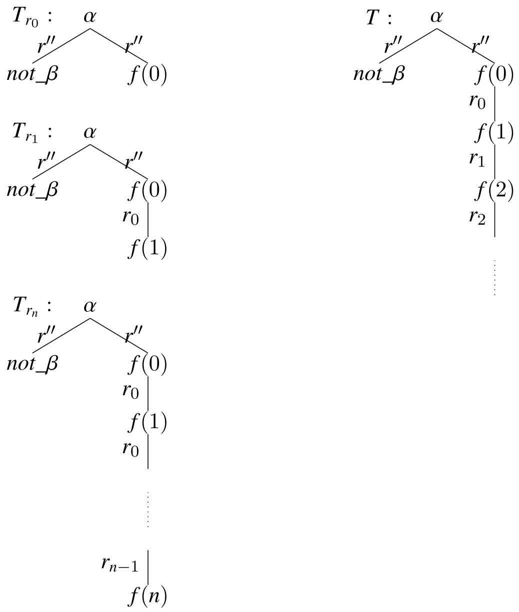

Fig. 5. Some proof trees of $\mathcal{F}_2$.

proof

$$p \equiv (root, \sigma_0).(r_1, \sigma_1). \ldots .(r_n, \sigma_n), \quad n > 0$$

from T, the following properties hold:

- The partial proof $p' \equiv (root, \sigma_0).(r_1, \sigma_1). \ldots .(r_{n-1}, \sigma_{n-1})$ also belongs to T and is referred to as the unique parent of p whereas p is referred to as a child of p';
- Every partial proof of the form $p'.(r_n, \sigma')$ with $\sigma' \in bd(r_n)$, also belongs to T and is a child of p';
- p' has no other children.

σ_0 is often referred to as the *conclusion* of T, denoted by $Cl(T)$ while the partial proof $(root, \sigma_0)$ is referred to as the *root* of T.

An example of partial proof trees is given in Fig. 5.

Remark 4. For convenience, we often refer to a partial proof tree without mentioning its conclusion if there is no possibility for misunderstanding.

Notation 1 (Nodes in Partial Proof Trees)**.** Abusing the notation for convenience, we often refer to a partial proof $(root, \sigma_0).(r_1, \sigma_1). \ldots .(r_n, \sigma_n)$ in a partial proof tree T as a *node labeled by* σ_n in T.

Notation 2. Let T be a partial proof tree and S be a set of partial proof trees,

- A node N in T is said to be a leaf of T if N has no children.
  A leaf N of T is said to be final if N is labeled by *true* or by an assumption. N is non-final if it is not final.

- The *support* of T, denoted by $Sp(T)$, is the set of all sentences labeling the leaves of T and different from *true*.
- The *set of all assumptions* appearing in T is denoted by $Ass(T)$.
  The *set of all assumptions appearing* in partial proof trees in S is denoted by $Ass(S)$.
- The set of conclusions of partial proof trees in S is denoted by $Cl(S)$.

Considering Fig. 5, the partial proof $(root, \alpha).(r'', f(0))$ is a leaf node of T_{r_0} while, $(root, \alpha).(r'', f(0))\ldots(r_{n-1}, f(n))$ is a leaf node of T_{r_n}.
Further, the support of T_{r_0}, i.e. $Sp(T_{r_0})$, is $\{not\_\beta, f(0)\}$ and $Ass(T_{r_0})$ is $\{not\_\beta\}$.

Definition 4 (Arguments)**.**

- A partial proof tree T is a *full proof tree* if all leaves of T are final.
- An *argument* for α is a full proof tree for α.
- The set of all arguments wrt the ABA framework $\mathcal{F}$ is denoted by $Ar_{\mathcal{F}}$.

Convention 2. *For short, we often simply say, proof trees instead of partial proof trees if there is no possibility of confusion.*

When a player in a dialectical computation is attempting to construct an argument, the attempt may not terminate. Declaratively, we model a non-terminating attempt to construct an argument as an attempt to construct an infinite argument.

As infinite arguments do not provide support for their conclusions, they can not be accepted as an admissible argument. We model this intuition as self-attacks of infinite arguments.

Definition 5 (Attacks)**.**

- An argument A *attacks* an argument B iff one of the following conditions holds:
  (1) The conclusion of A is the contrary of some assumption in the support of B.
  (2) A and B are identical and infinite.
- The attack relation between arguments in $Ar_{\mathcal{F}}$ is denoted by $att_{\mathcal{F}}$. Define

$$AF_{\mathcal{F}} = (Ar_{\mathcal{F}}, att_{\mathcal{F}})$$

Due to the fact that the infinite arguments always attack themselves, the following lemma holds obviously.

Lemma 1. *Let $S \subseteq Ar_{\mathcal{F}}$ be admissible wrt $AF_{\mathcal{F}} = (Ar_{\mathcal{F}}, att_{\mathcal{F}})$. Then S contains only finite arguments.*

Notation 3. Let T, T' be proof trees and N be a non-final leaf node in T labeled by a non-assumption σ.[11]

- T' is an *immediate expansion of T at N* if there is a rule r of the form $\sigma \leftarrow b_1, \ldots, b_m$ such that $T' = T \cup \{N.(r, b_1), \ldots, N.(r, b_m)\}$.
  Note that if $m = 0$, $T' = T \cup \{N.(r, true)\}$.[12]

[11] See Notations 1 and 2.

[12] I.e. T' is obtained from T by adding m children $N.(r, b_1), \ldots, N.(r, b_m)$ to N (for $m = 0$, a child node $N.(r, true)$ is added to N).

Fig. 6. Two compatible proof trees.

- We write $T' = \mathbf{exp}(\mathbf{T}, \mathbf{N}, \mathbf{r})$.
- We say T' is an *immediate expansion* of T if T' is an immediate expansion of T at some leaf node N of T.
- We define

$$\mathbf{CE}(\mathbf{T}, \mathbf{N}) = \{\exp(T, N, r') | r' \text{ is a rule s.t. } hd(r') = \sigma\}.$$

Considering Fig. 5, $T_{r_1} = \exp(T_{r_0}, N, r_0)$ where N is $(root, \alpha).(r'', f(0))$.
Further the set of all immediate expansion of T_{r_0} at node N, i.e. $CE(T_{r_0}, N)$, is $\{T_{r_1}\}$.

Notation 4. Let T_0, T_1 be proof trees for σ_0.

- We say T, T' are *compatible* iff $T \cup T'$ is also a proof tree.
- We say T_0 is a *prefix* of T_1 iff $T_0 \subseteq T_1$.[13] [14]
- We say T_0 is a *proper prefix* of T_1 if T_0 is a prefix of T_1 and $T_0 \neq T_1$.

Considering Fig. 5, T_{r_0} is a prefix of T_{r_1}.

Remark 5. It is worthwhile to note that two proof trees could be compatible without being in a prefix-relationship as illustrated below (see Fig. 6).

Consider an assumption-based framework with three rules:

$$r_1 : a \leftarrow b, c;$$

$$r_2 : b \leftarrow;$$

$$r_3 : c \leftarrow .$$

Consider two proof trees:

$$T_0 = \{p_0, p_1, p_2, p_3\}, \qquad T_1 = \{p_0, p_1, p_2, p_4\} \quad \text{where}$$

$$p_0 = (root, a), \qquad p_1 = (root, a).(r_1, b), \qquad p_2 = (root, a).(r_1, c) \quad \text{and}$$

$$p_3 = (root, a).(r_1, b).(r_2, true), \qquad p_4 = (root, a).(r_1, c).(r_3, true).$$

T_0, T_1 are compatible but neither is a prefix of the other.

[13] It may be worthwhile to note that if T_0 is a prefix of T_1 then obviously T_0, T_1 are compatible.

[14] One may wonder what is the prefix of a partial proof (nodes). Well, a partial proof (or node) is in essence a sequence. And so a prefix of a partial proof $(root, \sigma_0).(r_1, \sigma_1). \ldots .(r_n, \sigma_n)$ is a partial proof of the form $(root, \sigma_0).(r_1, \sigma_1). \ldots .(r_k, \sigma_k)$ where $k \leqslant n$.

Lemma 2. *Let T_0, T_1 be proof trees. The following statements hold:*

(1) *If T_1 is an immediate expansion of T_0 then T_0 is a proper prefix of T_1.*
(2) *Suppose T_0 is a prefix of T_1. It holds that*

  (a) *the roots of T_0, T_1 coincide; and*
  (b) *if N is a node in T_0 then the parent and children (if exist) of N in T_0 are respectively also the parent and children of N in T_1.*

Proof. Obvious. □

Lemma 3. *Let T be an argument, T_0 be a proof tree such that T_0 is a proper prefix of T. Furthermore, let N be a leaf node in T_0 such that $CE(T_0, N) \neq \emptyset$. Then there is an unique $T_1 \in CE(T_0, N)$ such that T_1 is a prefix of T.*

T_1 is often simply denoted by $\mathbf{exp(T_0, N, T)}$.

We also define $\mathbf{exp(T_0, T)} = \{\exp(T_0, N, T) | N$ *is a non-final leaf node of* $T_0\}$.[15]

Proof. Lemma 3 in [31] proves the existence of T_1. The uniqueness of T_1 should be obvious. □

Definition 6. An increasing sequence of proof trees $T_0 \subseteq T_1 \subseteq \dots T_i \subseteq \dots$ is said to be *fair* if for each T_i, for each non-final leaf node $N \in T_i$ there is a node $M \in T_j$, $j > i$, such that N is a proper prefix of M.

Lemma 4. *Let $sq \equiv T_0 \subseteq T_1 \subseteq \dots T_i \subseteq \dots$ be an increasing sequence of proof trees. The following statements hold:*

(1) *$T_0 \cup T_1 \cup \dots T_i \cup \dots$ is a proof tree.*
(2) *If the sequence sq is fair then $T_0 \cup T_1 \cup \dots T_i \cup \dots$ is an argument.*

Proof. This is Lemma 4 in [31]. □

Notation 5. Let T be a proof tree and $N \equiv (root, \sigma_0).(r_1, \sigma_1).\dots.(r_i, \sigma_i)$ be a node in T.

The *height* of N in T, denoted by $h(N, T)$, is defined by $h(N, T) = i$.[16]

The *minimum of the heights* of the non-final leaf nodes in T are denoted by $hi(T)$, i.e. $hi(T) = \min\{h(N, T) | N$ is a non-final leaf node in $T\}$

3. ω-Groundedness of argumentation frameworks

We first introduce the notion of ω-grounded argumentation frameworks. We then show that finitary ABA frameworks are ω-grounded.

[15] I.e. $\exp(T_0, T)$ is the set of all immediate expansions of T_0 that are prefixes of T.
[16] Hence the height of the root is 0.

3.1. ω-Grounded argumentation frameworks

Let $AF = (Ar, att)$ be an argumentation framework.

Definition 7 (ω-Groundedness)**.** An argumentation framework AF is ω-grounded if its grounded extension GE_{AF} can be "computed" after at most ω steps, i.e.

$$GE_{AF} = F^{\omega}_{AF}(\emptyset) = \bigcup_{i=0}^{\infty} F^{i}_{AF}(\emptyset)$$

It should be obvious that the argumentation framework in Fig. 1 is not ω-grounded.

We give now a sufficient condition for ω-groundedness.

Let $\mathcal{D}$ be a set of sets of arguments (i.e $\mathcal{D} \subseteq 2^{Ar}$). $\mathcal{D}$ is said to be *directed* iff for all $S, S' \in \mathcal{D}$, $S \cup S' \in \mathcal{D}$.[17]

Note that the least upper bound of $\mathcal{D}$, $lub(\mathcal{D})$, is $lub(\mathcal{D}) = \cup\mathcal{D}$.

Definition 8 (Admissibility Continuity)**.** A function $\Phi : 2^{Ar} \to 2^{Ar}$ is said to be admissibility-continuous (or just ad-continuous for short) if for each directed set of admissible sets $\mathcal{D}$

$$lub\{\Phi(S) | S \in \mathcal{D}\} = \Phi(lub(\mathcal{D}))$$

Lemma 5. *If the characteristic function F_{AF} is ad-continuous then AF is ω-grounded.*

Proof. Suppose $\mathcal{F}_{AF}$ is ad-continuous. Let $\mathcal{D} = \{F^{i}_{AF}(\emptyset) | i$ is a natural number$\}$. $\mathcal{D}$ is obviously directed and $lub(\mathcal{D}) = F^{\omega}_{AF}(\emptyset) = \bigcup_{i=0}^{\infty} F^{i}_{AF}(\emptyset)$.

Since F_{AF} is ad-continuous, $F^{\omega}_{AF}(\emptyset) = \bigcup_{i=0}^{\infty} F^{i}_{AF}(\emptyset) = lub\{F_{AF}(S) | S \in \mathcal{D}\} = F_{AF}(lub(\mathcal{D})) = F_{AF}(F^{\omega}_{AF}(\emptyset))$. Therefore $F^{\omega}_{AF}(\emptyset)$ coincides with the grounded extension of AF. □

Definition 9. A set S of arguments is said to be a *minimal defense* of an argument A if S defends A and no proper subset of S defends A.

Definition 10 (Finitary Defensible)**.**

- A defensible argument A is *finitary-defensible* iff all minimal defenses of A are finite.[18]
- An argumentation framework AF is said to be *finitary-defensible* if all defensible arguments in AF that are not self-attacking[19] are finitary defensible.

Theorem 1 (Finitary Defensibility implies ad-Continuity)**.** *Let AF be a finitary-defensible argumentation framework. Then F_{AF} is ad-continuous and hence AF is ω-grounded.*

Proof. Let AF be a finitary-defensible argumentation framework, $\mathcal{D}$ be a directed set of admissible sets of arguments. We show that $lub\{F_{AF}(S) | S \in \mathcal{D}\} = F_{AF}(lub(\mathcal{D}))$.

It is obvious that for each $S \in \mathcal{D}$: $S \subseteq lub(\mathcal{D})$. Since F_{AF} is monotonic, it holds for each $S \in \mathcal{D}$: $F_{AF}(S) \subseteq F_{AF}(lub(\mathcal{D}))$. Therefore $lub\{F_{AF}(S) | S \in \mathcal{D}\} \subseteq F_{AF}(lub(\mathcal{D}))$.

[17] The set 2^{Ar} is ordered wrt set inclusion.
[18] Note that A is *defensible* if it is defended by some set of arguments.
[19] An argument A is self-attacking if A attacks itself.

It remains to show that $lub\{F_{AF}(S)|S \in \mathcal{D}\} \supseteq F_{AF}(lub(\mathcal{D}))$.

Let $A \in F_{AF}(lub(\mathcal{D}))$. Hence $lub(\mathcal{D})$ is a defense of A. Since $\mathcal{D}$ is a set of admissible sets of arguments, $lub(\mathcal{D}) = \cup\mathcal{D}$ is also admissible. Therefore $F_{AF}(lub(\mathcal{D}))$ is admissible. Thus A is not self-attacking. Since AF is a finitary-defensible, there is a finite minimal defense $\mathcal{M} \subseteq lub(\mathcal{D})$ of A. Hence for each $X \in \mathcal{M}$ there is $S_X \in \mathcal{D}$ such that $X \in S_X$. Since $\mathcal{M}$ is finite, $S' = \cup\{S_X|X \in \mathcal{M}\} \in \mathcal{D}$. Thus $A \in F_{AF}(S') \subseteq lub\{F_{AF}(S)|S \in \mathcal{D}\}$.

From Lemma 5, it follows immediately that AF is ω-grounded. □

3.2. *Finitary ABA frameworks are finitary-defensible and ω-grounded*

We present a novel insight into the semantic structure of finitary ABA systems by showing that finitary ABA frameworks are finitary-defensible and hence ω-grounded. We begin with a lemma stating that finite arguments of finitary ABA framework are finitary-defensible.

Lemma 6 (Finite Arguments are Finitary Defensible). *Let $\mathcal{F}$ be a finitary ABA framework. Then defensible finite arguments in $AF_{\mathcal{F}}$ are finitary-defensible.*

Proof. Let B be a defensible finite argument in $Ar_{\mathcal{F}}$.

Let S be a minimal defense of B wrt $AF_{\mathcal{F}} = (Ar_{\mathcal{F}}, att_{\mathcal{F}})$. We show that S is finite.

Suppose the contrary that S is infinite. Hence $S = \{D_1, \ldots, D_i, \ldots\}$. Let $S_i = \{D_1, \ldots, D_i\}$.

Because S is a minimal defense of B, it follows that the set $\mathcal{A} \subseteq Ar_{\mathcal{F}}$ of attacks against B is infinite (if $\mathcal{A}$ is finite, pick for each $X \in \mathcal{A}$, an argument $D_X \in S$ s.t. D_X attacks X. Hence the set $\{D_X|X \in \mathcal{A}\} \subset S$ is a finite defense of B (wrt $AF_{\mathcal{F}}$). Contradiction to the fact that S is a infinite minimal defense of B).

Let PA_n be the set of balanced proof trees of height n[20] that are prefixes of arguments in $\mathcal{A}$ and PAN_n are those elements in PA_n that are not attacked by S.

Further let FA_n be the set of the full arguments of height n in $\mathcal{A}$.[21]

Since $\mathcal{F}$ is finitary, both PA_n and FA_n are finite[22] and so are PAN_n.

- We first show that for each n, $PAN_n \neq \emptyset$. Suppose the contrary that there is n such that $PAN_n = \emptyset$, i.e. each partial argument in PA_n is attacked by S. Since PA_n is finite, there is some i such that S_i attacks each partial argument in PA_n. Therefore each full argument of height $> n$ in $\mathcal{A}$ is attacked by S_i.
  Since the set $FA_0 \cup \cdots \cup FA_n$ is also finite, there is some j such that S_j attacks each argument in this set. Therefore $S_i \cup S_j$ attacks each (full) argument in $\mathcal{A}$. Hence $S_i \cup S_j$ is a finite defense of B, contrary to the assumption that S is a infinite minimal defense of B.
- We show now that there is an infinite sequence $T_0 \subset T_1 \subset \ldots$ such that for each $i \geqslant 0$: $T_i \in PAN_i$. It should be clear that each proof tree in PAN_{i+1} is obtained by expanding a proof tree in PAN_i at all non-final leaf nodes of the later.
  Let $\mathcal{T} = (V, E)$ be a tree defined as follows:

[20] A balanced proof tree of height n is a proof tree such that the heights of all non-final leaf nodes are n.

[21] The height of a full argument is the length of the longest path from the root to a leaf.

[22] We could show by induction. Since $PA_0 \cup FA_0$ consists of the contraries of the assumptions in B and B is a finite argument, both PA_0 and FA_0 are obviously finite. From the finitariness of $\mathcal{F}$ and the finiteness of PA_n, it should be clear that the set $PA_{n+1} \cup FA_{n+1}$ is finite since it is the set of the expansions of PA_n obtained by expanding each non-final leaf node in each proof tree in PA_n one step.

* The set of vertices of $\mathcal{T}$ is defined by: $V = \{\textbf{root}\} \cup \bigcup_{i=0}^{\infty} PAN_i$ where **root** is the root of $\mathcal{T}$.[23]
* The set of edges of $\mathcal{T}$ is defined by: $(T, T') \in E$ iff
  * $\exists i$ s.t. $T \in PAN_i$ and $T' \in PAN_{i+1}$ and $T \subseteq T'$; or
  * $T = \textbf{root}$ and $T' \in PAN_0$.

Since $PAN_i \neq \emptyset$ for all i, $\mathcal{T}$ is infinite and because $\mathcal{F}$ is finitary, each child of $\mathcal{T}$ has finitely many children. Thus there is an infinite path in $\mathcal{T}$. In other words, there is an infinite sequence $T_0 \subset T_1 \subset \ldots$ such that for each $i \geqslant 0$: $T_i \in PAN_i$. Since each T_i is a balanced tree of height i, the sequence $T_0 \subset T_1 \subset \ldots$ is obviously fair.

- Let $T = \bigcup_{i=0}^{\infty} T_i$. T is therefore a full proof tree and hence an argument. T is hence an attack against B and not attacked by S. Contradiction. □

Since the set of finite arguments in $AF_{\mathcal{F}}$ contains the set of all non-self-attacking arguments, it follows immediately from Theorem 1 and the above Lemma 6:

Theorem 2 (ω-Groundedness of finitary ABA frameworks)**.** *Let $\mathcal{F}$ be a finitary ABA framework. Then the following statements hold:*

- *$AF_{\mathcal{F}}$ is finitary-defensible;*
- *The characteristic function $F_{AF_{\mathcal{F}}}$ is ad-continuous;*
- *$AF_{\mathcal{F}}$ is ω-grounded.*

4. Grounded dispute derivations

In this section we will present a proof procedure for grounded semantics where proof trees together with their histories are fully and explicitly represented to shed light on the construction process of arguments and counter arguments during a derivation and hence providing key insights into the soundness and completeness of the procedure. Later, in Section 7, we present another procedure that is simply the result of flattening the one presented in this section. Consequently, the second procedure is both sound and complete but with much simpler data structure and hence much more amenable to possible implementation.

The purpose of the dispute derivations is to construct arguments. So it is kind of natural to refer to proof trees representing partly constructed arguments in a dispute derivation as partial arguments. It also makes it more intuitive to talk about attack between partial arguments.

Remark 6 (Partial Arguments)**.**

- Abusing the notation slightly for ease of reference, from now on until the end of the paper, we often refer to proof trees as partial arguments.
  To avoid any possibility of misunderstanding, we often refer to arguments as *full arguments*.
- We often say that a partial argument T attacks a partial argument T' if there is an assumption $\alpha \in Ass(T')$ such that $Cl(T) = \overline{\alpha}$.
- We also often refer to a proof tree consisting only of the root and supporting a sentence δ by $[\delta]$, i.e. $[\delta] = \{(root, \delta)\}$.

[23]Note that **root** $\notin \bigcup_{i=0}^{\infty} PAN_i$.

In a dispute derivation, an argument could be constructed repeatedly many times at different stages in the computation. To distinguish between these distinct "copies" of the same argument, we consider them together with their histories.

Definition 11 (Histories of Partial Arguments)**.** A possible history of a partial argument T is represented by a sequence $(T_0, t_0), \dots, (T_k, t_k)$ where $T_0, \dots, T_k$ are partial arguments such that

- $T_0 = [Cl(T)]$ and $T = T_k$; and
- T_i, $0 < i \leqslant k$, is an immediate expansion of T_{i-1}; and
- $t_0 < t_1 < \cdots < t_k$ are natural numbers representing the stages in the dispute when such expansions happen.

Definition 12 (Profiled Partial Arguments)**.** A profiled partial argument (ppa) is a pair $\pi = (T, h)$ where T is a partial argument and $h = (T_0, t_0), \dots, (T_k, t_k)$ is a history of T.

(1) We also often refer to T and h by $\mathbf{T}_\pi$ and $\mathbf{h}_\pi$ respectively;
(2) t_0 is often referred to as the starting time of π denoted by $\mathbf{st(\pi)}$ while k is referred to as the length of the history of π denoted by $\mathbf{lh(\pi)}$;
(3) An assumption α is often referred to as the target of π denoted by $\mathbf{target(\pi)}$, if $Cl(T) = \overline{\alpha}$;

Example 4. Considering again the argumentation framework $\mathcal{F}_2$ in the introduction

$$\mathcal{F}_2 : r : a \leftarrow not\_\alpha \qquad r'' : \alpha \leftarrow not\_\beta, f(0) \qquad r_n : f(n) \leftarrow f(n+1), \quad n \geqslant 0.$$
$$t : \beta \leftarrow$$

$\pi_0 = (T_0, h_0)$ is an example of a profiled partial argument where $T_0 = [a]$ and $h_0 = (T_0, 0)$ where h_0 says that T_0 is obtained at stage 0.

Suppose T_0 is expanded at stage 1 resulting in a new profiled partial argument $\pi_1 = (T_1, h_1)$ (see Fig. 7) where $h_1 = (T_0, 0), (T_1, 1)$. The history h_1 says that T_1 is obtained by expanding T_0 at stage 1 and T_0 is obtained a stage 0.

A dispute derivation is viewed as a game between a proponent and an opponent where the players take turn to develop their arguments. The goal of the proponent is to construct an argument to support some desired conclusion and other arguments to defend against the attacks from the opponent while the opponent's goal is to construct arguments to attack proponent's arguments. At each step, either player could choose to either expand their partly constructed arguments or start a new argument to attack the other's argument.

Definition 13 (Grounded Dispute Derivation)**.** A grounded dispute derivation for a sentence σ is a (possibly infinite) sequence of the form

$$\langle PPT_0, PTA_0, OPT_0\rangle, \dots, \langle PPT_n, PTA_n, OPT_n\rangle \dots$$

Fig. 7. Proof trees of Example 4.

where [24]

- (1) for each $i \leqslant n$, PPT_i, OPT_i are sets of profiled partial arguments; and
  (2) PTA_i is a set of assumptions appearing in the proponent partial arguments (up to stage i) that are to be attacked by the opponent; and
  (3) PPT_0 contains exactly one profiled partial argument consisting of only the root labeled by σ, i.e. $PPT_0 = \{(T_0, h_0)\}$ where $T_0 = [\sigma]$ and $h_0 = ([\sigma], 0)$, and
  (4) $PTA_0 = \emptyset$ if σ is not an assumption, otherwise $PTA_0 = \{\sigma\}$, and
  (5) $OPT_0 = \emptyset$.
- at stage $i > 0$, one of the dispute parties makes a move transforming the dispute from state $i - 1$ to state i as follows:

  (1) Suppose the proponent makes a move at stage i. The proponent can choose one of the following two options:

  (a) The proponent expands some profiled partial argument $\pi = (T, h) \in PPT_{i-1}$ by:

  * selecting a non-final leaf node N in T labeled by a non-assumption sentence δ and a rule r with $hd(r) = \delta$; and
  * expanding (T, h) at N by r.

  The result will be:

  * $PPT_i = (PPT_{i-1} \setminus \{\pi\}) \cup \{\pi'\}$ where $\pi' = (T', h')$ and $T' = \exp(T, N, r)$ and $h' = h.(T', i)$;[25]
  * $PTA_i = PTA_{i-1} \cup Ass(r)$
  * $OPT_i = OPT_{i-1}$.

  (b) The proponent attacks an opponent's profiled partial argument $\pi = (T, h) \in OPT_{i-1}$ at an assumption $\alpha \in Ass(T)$ resulting in:
  $PPT_i = PPT_{i-1} \cup \{(T', h')\}$ where $T' = [\overline{\alpha}]$ and $h' = ([\overline{\alpha}], i)$; and
  $PTA_i = PTA_{i-1}$
  $OPT_i = OPT_{i-1} \setminus \{\pi\}$

  (2) Suppose the opponent makes a move at stage i. The opponent can choose one of the following two options:

  (a) The opponent expands an opponent profiled partial argument $\pi = (T, h) \in OPT_{i-1}$ at a non-final leaf node $N \in T$ labeled by a non-assumption sentence δ resulting in:
  $PPT_i = PPT_{i-1}$
  $PTA_i = PTA_{i-1}$
  $OPT_i = (OPT_{i-1} \setminus \{\pi\}) \cup \{\pi' | \pi' = (T', h'), T' \in CE(T, N), h' = h.(T', i)\}$[26]
  (b) The opponent attacks an assumption $\alpha \in PTA_i$ resulting in:
  $PPT_i = PPT_{i-1}$
  $PTA_i = PTA_{i-1} \setminus \{\alpha\}$
  $OPT_i = OPT_{i-1} \cup \{\pi\}$ where $\pi = (T, h)$ and $T = [\overline{\alpha}]$ and $h = ([\overline{\alpha}], i)$

[24] PPT, PTA, OPT stand respectively for "proponent profiled trees", "proponent to-be-attacked assumptions "and "opponent profiled trees".
[25] See Notation 3 for the definition of $\exp(T, N, r)$.
[26] See Notation 3 for the definition of $CE(T, N)$.

Remark 7.

- When opponent carries out step (2.a), it is possible that $CE(T, N) = \emptyset$. We often refer to this case as a failed expansion of opponent profiled partial argument π.
- Note that if the assumption-based framework is not finitary, the set OPT_i in step (2a) could be infinite, and hence not implementable.

A dispute derivation is successful (for the proponent) if i) the proponent manages to construct in full her arguments to support her stated conclusion and to defend against the attacks from the opponent and ii) the opponent runs out of attacks against the proponent.

Definition 14 (Successful Grounded Dispute Derivation)**.** A grounded dispute derivation

$$\langle PPT_0, PTA_0, OPT_0\rangle, \ldots, \langle PPT_n, PTA_n, OPT_n\rangle$$

is successful (for the proponent) if $PTA_n = OPT_n = \emptyset$ and for each $(T, h) \in PPT_n$, T is a full argument.

Example 5. A successful grounded dispute derivation

$$\langle PPT_0, PTA_0, OPT_0\rangle, \ldots, \langle PPT_5, PTA_5, OPT_5\rangle$$

for sentence a (wrt the argumentation framework $\mathcal{F}_2$ in the introduction) is illustrated in Table 1 and explained in details below. For convenience we recall $\mathcal{F}_2$ below.

$$\mathcal{F}_2 : r : a \leftarrow not\_\alpha \qquad r'' : \alpha \leftarrow not\_\beta, f(0) \qquad r_n : f(n) \leftarrow f(n+1), \quad n \geqslant 0$$
$$t : \beta \leftarrow$$

At stage 0, we have $PPT_0 = \{\pi_0\}$, $PTA_0 = OPT_0 = \emptyset$ where $\pi_0 = (T_0, h_0)$ and $T_0 = [a]$ and $h_0 = (T_0, 0)$.

Table 1

A successful grounded dispute derivation of $\mathcal{F}_2$

Stage	Move	*PPT*	*PTA*	*OPT*
0		$\pi_0 = (T_0, h_0)$ $h_0 = (T_0, 0)$	$\emptyset$	$\emptyset$
1	1*a*	$\pi_1 = (T_1, h_1)$ $h_1 = (T_0, 0), (T_1, 1)$	*not_α*	$\emptyset$
2	2*b*	$\pi_1 = (T_1, h_1)$, $h_1 = (T_0, 0), (T_1, 1)$	$\emptyset$	$\pi'_0 = (T'_0, h'_0)$ $h'_0 = (T'_0, 2)$
3	2*a*	$\pi_1 = (T_1, h_1)$, $h_1 = (T_0, 0), (T_1, 1)$	$\emptyset$	$\pi'_1 = (T'_1, h'_1)$ $h'_1 = (T'_0, 2), (T'_1, 3)$
4	1*b*	$\pi_1 = (T_1, h_1), h_1 = (T_0, 0), (T_1, 1)$ $\pi''_0 = (T''_0, h''_0), h''_0 = (T''_0, 4)$	$\emptyset$	$\emptyset$
5	1*a*	$\pi_1 = (T_1, h_1), h_1 = (T_0, 0), (T_1, 1)$ $\pi''_1 = (T''_1, h''_1), h''_1 = (T''_0, 4), (T''_1, 5)$	$\emptyset$	$\emptyset$

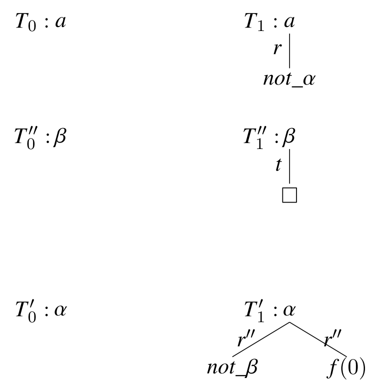

Fig. 8. Proof trees for Example 5.

- At stage 1, the proponent makes a move to expand the ppa π_0 by applying step (1a) using rule r. Hence $PPT_1 = (PPT_0 \setminus \{\pi_0\}) \cup \{\pi_1\} = \{\pi_1\}$ where $\pi_1 = (T_1, h_1)$, $T_1 = \exp(T_0, root, r)$ and $h_1 = (T_0, 0), (T_1, 1)$ (T_1 is illustrated in Fig. 8).
  Since rule r has the assumption not_α (i.e. $Ass(r) = \{not\_\alpha\}$), we have $PTA_1 = PTA_0 \cup Ass(r) = PTA_0 \cup \{not\_\alpha\} = \{not\_\alpha\}$.
  $OPT_1 = OPT_0 = \emptyset$.
- Next, at stage 2, the opponent attacks the assumption $not\_\alpha \in PTA_1$ by applying step (2b). Hence $OPT_2 = OPT_1 \cup \{\pi'_0\} = \{\pi'_0\}$ where $\pi'_0 = (T'_0, h'_0)$ with $T'_0 = [\alpha]$ and $h'_0 = (T'_0, 2)$ stating that T'_0 is created at stage 2 (See Fig. 8).
  $PPT_2 = PPT_1 = \{\pi_1\}$.
  $PTA_2 = PTA_1 \setminus \{not\_\alpha\} = \emptyset$.
- At stage 3, the opponent applies step (2a) to expand T'_0 to T'_1. Since rule r'' is the only one with head α, it follows $CE(T'_0, root) = \{T'_1\}$ with $T'_1 = \exp(T'_0, root, r'')$ (See Fig. 8). Therefore
  $OPT_3 = (OPT_2 \setminus \{\pi'_0\}) \cup \{\pi'_1\} = \{\pi'_1\}$ where $\pi'_1 = (T'_1, h'_1)$ and $h'_1 = (T'_0, 2), (T'_1, 3)$.
  $PPT_3 = PPT_2 = \{\pi_1\}$
  $PTA_3 = PTA_2 = \emptyset$
- At stage 4, the proponent attacks opponent's ppa π'_1 at assumption $not\_\beta \in T'_1$ by applying step (1b).
  Therefore $PPT_4 = PPT_3 \cup \{\pi''_0\} = \{\pi_1, \pi''_0\}$ where $\pi''_0 = (T''_0, h''_0)$ and $T''_0 = [\beta]$ and $h''_0 = (T''_0, 4)$ (See Fig. 8).
  $PTA_4 = PTA_3 = \emptyset$
  $OPT_4 = OPT_3 \setminus \{\pi'_1\} = \emptyset$.
- Finally at stage 5, the proponent applies step (1a), using rule t, to expand T''_0 to argument $T''_1 = \exp(T''_0, root, t)$ (See Fig. 8). Therefore
  $PPT_5 = (PPT_4 \setminus \{\pi''_0\}) \cup \{\pi''_1\} = \{\pi_1, \pi''_1\}$ where $\pi''_1 = (T''_1, h''_1)$ with $h''_1 = (T''_0, 4), (T''_1, 5)$.
  $PTA_5 = PTA_4 \cup Ass(t) = \emptyset$
  $OPT_5 = OPT_4 = \emptyset$

As there is no any other assumption in PTA_5 and OPT_5 is empty and all ppa in PPT_5 are full, the derivation is successful.

Table 2

A successful grounded dispute derivation of $\mathcal{F}_3$

Stage	Move	*PPT*	*PTA*	*OPT*
0		$\pi_0 = (T_0, h_0)$ $h_0 = (T_0, 0)$	$\emptyset$	$\emptyset$
1	1a	$\pi_1 = (T_1, h_1)$ $h_1 = (T_0, 0), (T_1, 1)$	*not_a*	$\emptyset$
2	2b	$\pi_1 = (T_1, h_1)$ $h_1 = (T_0, 0), (T_1, 1)$	$\emptyset$	$\pi'_0 = (T'_0, h'_0)$ $h'_0 = (T'_0, 2)$
3	2a	$\pi_1 = (T_1, h_1)$ $h_1 = (T_0, 0), (T_1, 1)$	$\emptyset$	$\emptyset$

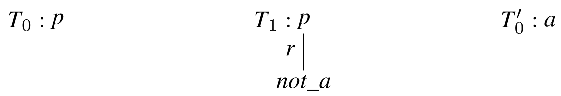

Fig. 9. Proof trees of the dispute derivation for sentence "p".

Example 6. Consider the following argumentation framework

$$\mathcal{F}_3 : r : p \leftarrow not\_a$$

A successful dispute derivation for sentence "p"

$$\langle PPT_0, PTA_0, OPT_0\rangle, \ldots, \langle PPT_3, PTA_3, OPT_3\rangle$$

is illustrated in Table 2 and is explained in details below.

At stage 0, we have $PPT_0 = \{\pi_0\}$, $PTA_0 = OPT_0 = \emptyset$ where $\pi_0 = (T_0, h_0)$ and $T_0 = [p]$ and $h_0 = (T_0, 0)$ (see Fig. 9).

- At stage 1, the proponent makes a move to expand the ppa π_0 by applying step (1a) using rule r. Hence $PPT_1 = (PPT_0 \setminus \{\pi_0\}) \cup \{\pi_1\} = \{\pi_1\}$ where $\pi_1 = (T_1, h_1)$, $T_1 = \exp(T_0, root, r)$ (see Fig. 9) and $h_1 = (T_0, 0), (T_1, 1)$.
  Since rule r has the assumption not_a (i.e. $Ass(r) = \{not\_a\}$), we have $PTA_1 = PTA_0 \cup Ass(r) = PTA_0 \cup \{not\_a\} = \{not\_a\}$.
  $OPT_1 = OPT_0 = \emptyset$.
- Next, at stage 2, the opponent attacks the assumption $not\_a \in PTA_1$ by applying step (2b). Hence $OPT_2 = OPT_1 \cup \{\pi'_0\} = \{\pi'_0\}$ where $\pi'_0 = (T'_0, h'_0)$ with $T'_0 = [a]$ and $h'_0 = (T'_0, 2)$ stating that T'_0 is created at stage 2 (see Fig. 9).
  $PPT_2 = PPT_1 = \{\pi_1\}$.
  $PTA_2 = PTA_1 \setminus \{not\_a\} = \emptyset$.
- At stage 3, the opponent applies step (2a) to expand T'_0. Since there is no rule with head a, it follows $CE(T'_0, root) = \emptyset$. Therefore
  $OPT_3 = (OPT_2 \setminus \{\pi'_0\}) \cup \emptyset = \emptyset$.
  $PPT_3 = PPT_2 = \{\pi_1\}$
  $PTA_3 = PTA_2 = \emptyset$

As there is no any other assumption in PTA_3 and OPT_3 is empty and all ppa in PPT_3 are full, the derivation is successful.

5. Soundness of grounded dispute derivation

We first introduce some new notations.

Notation 6.

- We say a ppa π' is a continuation of another ppa π iff h_π is a prefix of $h_{\pi'}$ (and hence $T_\pi \subseteq T_{\pi'}$ and $st(\pi) = st(\pi')$).
  π' is an *immediate expansion* of π if $h_\pi = (T_0, t_0), \ldots, (T_k, t_k)$ and $h_{\pi'} = h_\pi.(T_{k+1}, t_{k+1})$.[27]
- A ppa π is said to be full iff T_π is an argument.
- We say two ppas π, π' are compatible if $st(\pi) = st(\pi')$ and $T_\pi, T_{\pi'}$ are compatible.
- We often refer to a ppa appearing in some PPT_i (resp. OPT_i) in a grounded dispute derivation as a proponent ppa (resp opponent ppa).

The following lemma expresses the intuition that when the proponent has partially constructed an argument, she is expected to finish it. And she should not construct two distinct arguments for the same purpose at the same time, i.e. at any stage there should be no distinct continuations of the same proponent ppa at some previous stage.

Lemma 7. *Let $dd = \langle PPT_0, PTA_0, OPT_0\rangle, \ldots, \langle PPT_n, PTA_n, OPT_n\rangle$ be a grounded dispute derivation.*

(1) *Let π be some proponent ppa appearing in dd. The following statements hold:*

  (a) *For each $i : st(\pi) \leqslant i \leqslant n$, there is an unique continuation of π in PPT_i.*
  (b) *Let π_i, π_{i+1} ($st(\pi) \leqslant i < n$) be continuations of π in PPT_i, PPT_{i+1} respectively. Then π_{i+1} is either an immediate expansion of π_i or identical to π_i.*

(2) *Let π, π' be two proponent ppas in dd such that $st(\pi) = st(\pi')$. Then one is a continuation of the other.*
Consequently, if π, π' are full then they are identical.

Proof. See proof of Lemma 7 in Appendix A.3. □

The following lemma states intuitively that at any stage in a derivation, it is not possible for the opponent to construct the same argument in different ways.

Lemma 8. *Let $dd = \langle PPT_0, PTA_0, OPT_0\rangle, \ldots, \langle PPT_n, PTA_n, OPT_n\rangle$ be a grounded dispute derivation. It holds that for each $0 \leqslant i \leqslant n$, for all $\pi, \pi' \in OPT_i$, if π, π' are compatible then π, π' are identical.*

Proof. See proof of Lemma 8 in Appendix A.3. □

Remark 8. An opponent ppa π in a dispute derivation $dd = \langle PPT_0, PTA_0, OPT_0\rangle, \ldots, \langle PPT_n, PTA_n, OPT_n\rangle$ is said to be discontinued at step i iff $\pi \in OPT_{i-1}$ and there is no continuation of π afterwards, i.e. $\forall j \geqslant i$ there is no continuation of π in OPT_j.

It is obvious that in a successful grounded dispute derivation, every opponent ppa is discontinued at some stage. The following lemma sheds light on the structure of the evolution of opponent ppas in a grounded dispute derivation.

[27] I.e. π' is a continuation of π and $T_{\pi'}$ is an immediate expansion of T_π.

Lemma 9. *Let dd be a dispute derivation* $dd = \langle PPT_0, PTA_0, OPT_0\rangle, \ldots, \langle PPT_n, PTA_n, OPT_n\rangle$.

Further let A be an argument such that $Cl(A) = \overline{\alpha}$ *for some assumption* α *and* $\pi = ([\overline{\alpha}], ([\overline{\alpha}], i)) \in OPT_i$ *for some* $0 \leqslant i \leqslant n$.

Then there is an unique sequence $cont_{dd}(\pi) = \pi_0, \ldots, \pi_k$ $(i \leqslant i + k \leqslant n)$ *such that*

(1) $\pi_0 = \pi$*; and*
(2) *for each* $i < j \leqslant k$, $\pi_j \in OPT_{i+j}$ *and* π_j *is a continuation of* π_{j-1} *and* $T_{\pi_j} \subseteq A$*; and*
(3) *if* $i + k < n$ *then* π_k *is discontinued at* $i + k + 1$ *and there is no ppa at any* $OPT_{i+k+1}, \ldots, OPT_n$ *that is compatible with* π_k.

Proof. See proof of Lemma 9 in Appendix A.3. □

The following lemma states that each argument attacking a proponent argument in a successful dispute derivation is counter-attacked by some proponent argument.

Lemma 10. *Let dd be a successful grounded dispute derivation that terminates at* $\langle PPT_n, PTA_n, OPT_n\rangle$ *and* $\pi = (T, h) \in PPT_n$ *and* T' *be an argument attacking argument* T*. Then there is a opponent ppa* π' *such that* $st(\pi) < st(\pi')$ *and* $T_{\pi'} \subseteq T'$ *and* π' *is discontinued at some step between* $st(\pi')$ *and n.*

Proof. See proof of Lemma 10 in Appendix A.3. □

The following theorem shows that in a successful dispute derivation, each proponent argument whose construction started at some stage i is defended by the proponent arguments constructed after stage i.

Theorem 3. *Let* $dd = \langle PPT_0, PTA_0, OPT_0\rangle, \ldots, \langle PPT_n, PTA_n, OPT_n\rangle$ *be a successful grounded dispute derivation and* $\pi = (T, h) \in PPT_n$*. Further let* $\mathcal{T}_\pi = \{T_{\pi'} | \pi' \in PPT_n \text{ and } st(\pi') > st(\pi)\}$*. It holds:*

$$T \in F_{AF_{\mathcal{F}}}(\mathcal{T}_\pi)$$

Proof. Let $T' \in Ar_{\mathcal{F}}$ such that T' attacks T. From Lemma 10 there is opponent ppa π' (discontinued at stage i) such that $st(\pi) < st(\pi')$ and $T_{\pi'} \subseteq T'$.

There are two cases:

(1) **Case 1** The opponent makes a move at stage i. Because π' is discontinued at step i, the opponent expands π' at stage i. Because $T_{\pi'} \subseteq T'$, it follows from Lemma 3 that π' has a continuation at stage i. Contradiction to the fact that π' is discontinued at stage i. Hence this case is not possible.
(2) **Case 2** The proponent makes a move at stage i. Since π' is discontinued at stage i, it follows that the proponent attacks π' at some assumption α at this stage. Since $T_{\pi'} \subseteq T'$, it follows that α appears in T'.
Hence a new proponent ppa $\pi_1 = (T_1, h_1)$ is created in PPT_i where $T_1 = [\overline{\alpha}]$ and $h_1 = (T_1, i)$.
Since dd is successful, there is $\pi_2 \in PPT_n$ that is a continuation of π_1. Hence T_{π_2} attacks T'.
Since $st(\pi_2) = st(\pi_1) = i > st(\pi') > st(\pi)$, it follows that $T_{\pi_2} \in \mathcal{T}_\pi$. Thus $\mathcal{T}_\pi$ defends T against T'.
Since T' is arbitrary selected, it follows that $\mathcal{T}_\pi$ defends T against all attacks against T.
We have proved that $T \in F_{AF_{\mathcal{F}}}(\mathcal{T}_\pi)$. □

An immediate consequence of Theorem 3 is the soundness of the grounded dispute procedure as stated in the following theorem.

Theorem 4 (Soundness Theorem)**.** *Let* $dd = \langle PPT_0, PTA_0, OPT_0\rangle, \dots, \langle PPT_n, PTA_n, OPT_n\rangle$ *be a successful grounded dispute derivation. It holds*

$$\{T_\pi | \pi \in PPT_n\} \subseteq GE_{AF_\mathcal{F}}$$

Proof. Let $m = |PPT_n|$. Let $\pi_m, \dots, \pi_1$ be an increasing (according to the starting times) listing of ppas in PPT_n, i.e. for $m \geqslant i > 1$, $st(\pi_m) = 0$ and $st(\pi_i) < st(\pi_{i-1})$.

Let $T_i = T_{\pi_i}$. We first prove by induction on i, $T_i \in F^i_{AF_\mathcal{F}}(\emptyset)$.

Basic Step $i = 1$.

From Theorem 3, it follows immediately that $T_1 \in F_{AF_\mathcal{F}}(\emptyset)$ (because $\mathcal{T}_{\pi_1} = \emptyset$).

Inductive Step Suppose for all j $(i > j \geqslant 1)$, $T_j \in F^j_{AF_\mathcal{F}}(\emptyset)$.

We prove that $T_i \in F^i_{AF_\mathcal{F}}(\emptyset)$.

From $F_{AF_\mathcal{F}}(\emptyset) \subseteq F^2_{AF_\mathcal{F}}(\emptyset) \subseteq \cdots \subseteq F^{i-1}_{AF_\mathcal{F}}(\emptyset)$ and for each j $(i > j \geqslant 1)$, $T_j \in F^j_{AF_\mathcal{F}}(\emptyset)$, it follows that $\mathcal{T}_{\pi_i} = \{T_{i-1}, \dots, T_1\} \subseteq F_{AF_\mathcal{F}}(\emptyset) \cup F^2_{AF_\mathcal{F}}(\emptyset) \cup \cdots \cup F^{i-1}_{AF_\mathcal{F}}(\emptyset) \subseteq F^{i-1}_{AF_\mathcal{F}}(\emptyset)$.

Thus from Theorem 3, $T_i \in F_{AF}(\mathcal{T}_{\pi_i}) \subseteq F_{AF}(F^{i-1}_{AF_\mathcal{F}}(\emptyset)) = F^i_{AF_\mathcal{F}}(\emptyset)$.

Therefore $\{T_\pi | \pi \in PPT_n\} = \{T_m, \dots, T_0\} \subseteq F^m_{AF_\mathcal{F}}(\emptyset) \subseteq GE_{AF_\mathcal{F}}$. □

6. Completeness of grounded proof procedure

It should be clear that for each sentence $\sigma \in Cl(GE_{AF_\mathcal{F}})$, there are always some arguments in $GE_{AF_\mathcal{F}}$ supporting σ. The completeness of the grounded proof procedure means that at least one of such arguments could be derived in a successful grounded dispute derivation.

In other words, completeness of grounded procedure is not about verifying whether a sentence is supported by an argument in the grounded extension. It is rather about showing that for each sentence in the grounded extension, an argument supporting it could be derived by a grounded derivation.

The proof is constructive by first constructing (according to the definition of strongly grounded dd) for each sentence $\sigma \in Cl(GE_{AF_\mathcal{F}})$, a special kind of grounded dispute derivation, and then show that each of such special grounded derivation is always successful.

For each argument $A \in GE_{AF_\mathcal{F}}$ and sentence $\delta \in Cl(GE_{AF_\mathcal{F}})$, define

$$rank(A) = \min\{i | A \in F^i_{AF_\mathcal{F}}(\emptyset)\} \quad \text{and}$$

$$rank(\delta) = \min\{i | \delta \in Cl(F^i_{AF_\mathcal{F}}(\emptyset))\}$$

Remark 9.

(1) Since $AF_\mathcal{F}$ is ω-grounded (because $\mathcal{F}$ is finitary (see Convention 1)), both $rank(A)$ and $rank(\delta)$ are natural numbers.

(2) It is not difficult to see that for each $A \in GE_{AF_\mathcal{F}}$, $rank(Cl(A)) \leqslant rank(A)$.

(3) For each $\delta \in Cl(GE_{AF_{\mathcal{F}}})$, an argument $A \in GE_{AF_{\mathcal{F}}}$ with $Cl(A) = \delta$ and $rank(\delta) = rank(A)$, is often referred to as a *ground support* of δ.
It is obvious that each sentence $\delta \in Cl(GE_{AF_{\mathcal{F}}})$ has a ground support.
(4) A mapping λ assigning to each sentence in $\delta \in Cl(GE_{AF_{\mathcal{F}}})$ a ground support $\lambda(\delta) \in GE_{AF_{\mathcal{F}}}$ of δ is often referred to as a *ground map* of $\mathcal{F}$.[28]

Lemma 11. *Let $A \in GE_{AF_{\mathcal{F}}}$ and $\alpha \in Ass(A)$. Further let B be an argument attacking α. The following statements hold:*

(1) *$\alpha \in Cl(GE_{AF_{\mathcal{F}}})$ and $rank(\alpha) \leqslant rank(A)$; and*
(2) *There is an assumption $\beta \in Ass(B)$ such that $\overline{\beta} \in Cl(GE_{AF_{\mathcal{F}}})$ and $rank(\overline{\beta}) < rank(\alpha)$.*

Proof. Since $A \in GE_{AF_{\mathcal{F}}}$, it follows from Theorem 2 that $A \in F^{k}_{AF_{\mathcal{F}}}(\emptyset)$ where $k = rank(A)$ and k is a natural number. Hence, every attack against A is attacked by some argument in $F^{k-1}_{AF_{\mathcal{F}}}(\emptyset)$. Thus every attack against α is attacked by some argument in $F^{k-1}_{AF_{\mathcal{F}}}(\emptyset)$ implying that $\alpha \in Cl(F^{k}_{AF_{\mathcal{F}}}(\emptyset))$. Hence $rank(\alpha) \leqslant k = rank(A)$.

Let $rank(\alpha) = m$. Hence $\alpha \in Cl(F^{m}_{AF_{\mathcal{F}}}(\emptyset))$. From Theorem 2, m is a natural number. Thus every attack against α is attacked by some argument in $F^{m-1}_{AF_{\mathcal{F}}}(\emptyset)$. It follows that B is attacked by some argument $C \in F^{m-1}_{AF_{\mathcal{F}}}(\emptyset)$ at some assumption $\beta \in Ass(B)$. Thus $Cl(C) = \overline{\beta}$ and $rank(\overline{\beta}) \leqslant rank(C) \leqslant m - 1 < m = rank(\alpha) \leqslant rank(A)$. □

We introduce the concept of strongly grounded dispute derivation below where the proponent arguments are grounded according to some ground map λ. This concept is inspired by the idea of the H-constrainted dispute derivations for admissibility semantics in [31].

The construction of strongly grounded dispute derivations mimics such derivations in abstract argumentation when arguments are assumed to be given and a key basic step is like: *"pick some argument…"*.

This step is elaborated in the strongly grounded dispute derivation by the proponents and opponents as follows:

- Once the proponent has started to build up an argument, she will continue until getting the full argument constructed without being bothered to attack the other's arguments. The opponent is not allowed to disrupt her even if he could attack her still partly constructed argument at some stage.
- On the contrary, the opponent's construction of his arguments will be disrupted by attacks from the proponent whenever she could launch such attacks. But the opponent is not allowed to interrupt the constructions of his own arguments to launch an attack against a proponent argument even if such attack is possible.

Definition 15 (Strongly Grounded Dispute Derivation)**.** A *strongly grounded dispute derivation* for a sentence δ wrt a ground map λ is a grounded dispute derivation for δ (as defined in Definition 13) such that the following extra constraints are satisfied:

- The proponent executes step (1.a) (to expand proponent ppa $\pi = (T, h)$) with an extra condition that $\exp(T, N, r) \subseteq \lambda(Cl(T))$;

[28] We assume the axiom of choice or equivalently the maximality principles (see [9] for more details). Hence such mapping always exists.

- The proponent executes step (1.b) (to attack an opponent ppa $\pi = (T, h) \in OPT_i$ at an assumption $\alpha \in Ass(T)$) with two extra conditions:
  - $\overline{\alpha} \in Cl(GE_{AF_{\mathcal{F}}})$ and $rank(\overline{\alpha}) < rank(target(\pi))$; and
  - step (1.a) is not possible;
- The opponent executes step (2.a) (to expand an opponent ppa $\pi = (T, h) \in OPT_i$) with two extra conditions:
  - $h(N, T) = hi(T)$;[29]
  - It is not possible for the proponent to perform any of steps (1.a) or (1.b).
- The opponent executes step (2.b) (to attack a proponent assumption $\alpha \in PTA_i$) with two extra conditions:
  - It is not possible for the proponent to perform any of steps (1.a) or (1.b);
  - It is not possible for the opponent to perform step (2.a).

Remark 10. We often simply refer to a strongly grounded dispute derivation without explicitly mentioning the associated ground map if there is no possibilities for misunderstanding.

Definition 16. A strongly grounded dispute derivation $\langle PPT_0, PTA_0, OPT_0\rangle, \ldots, \langle PPT_n, PTA_n, OPT_n\rangle$ is said to be *terminated* if neither the proponent nor the opponent could make a move at stage n.

Theorem 5 (Completeness Theorem)**.** *Let $\mathcal{F}$ be a finitary ABA framework and $\sigma \in Cl(GE_{AF_{\mathcal{F}}})$. Then there is a successful strongly grounded dispute derivation for σ $\langle PPT_0, PTA_0, OPT_0\rangle, \ldots, \langle PPT_n, PTA_n, OPT_n\rangle$ such that $\sigma \in Cl(T_\pi)$ for some $\pi \in PPT_n$.*

Proof. See Section 6.1. □

The proof of the completeness theorem follows directly from two insights:

i) each terminated strongly grounded dispute derivation for $\sigma \in Cl(GE_{AF_{\mathcal{F}}})$ is successful; and
ii) there is no infinite strongly grounded dispute derivation.

We first show below that each terminated strongly grounded dispute derivation for $\sigma \in Cl(GE_{AF_{\mathcal{F}}})$ is successful.

Theorem 6. *Let $sdd = \langle PPT_0, PTA_0, OPT_0\rangle, \ldots, \langle PPT_n, PTA_n, OPT_n\rangle$ be a terminated strongly grounded dispute derivation for $\delta \in Cl(GE_{AF_{\mathcal{F}}})$ wrt a ground map λ. Then sdd is successful.*

Proof. Let $\pi \in PPT_n$ and $h_\pi = (T_0, i_0) \ldots (T_m, i_m)$ with $T_0 = [\delta]$. From the design of the proof procedure in definitions 13, 15, it holds: $T_0 \subset T_1 \subset \cdots \subset T_m \subseteq \lambda(\delta)$. Since sdd is terminated, it is not possible to expand T_m further into $\lambda(\delta)$. Hence $\lambda(\delta) = T_m = T_\pi$. Hence T_π is a full argument.

Since the opponent can not carry out step (2.b) (attack proponent assumptions in PTA_n), even though all other steps are not possible, it follows that $PTA_n = \emptyset$.

Since the opponent can not carry out step (2.a) (expand opponent partial arguments) even though none of steps (1.a, 1.b, 2.b) can be executed, it follows that for each $\pi \in OPT_n$: T_π is a full argument and there is $p \in PPT_n$ s.t. T_π attacks T_p. Since $T_p = \lambda(Cl(T_p))$, $T_p \in GE_{AF_{\mathcal{F}}}$.

[29] See Notation 5 for the introduction of $hi(T)$.

From Lemma 11, it follows there is an assumption $\alpha \in Ass(T_\pi)$ such that $\overline{\alpha} \in Cl(GE_{AF_{\mathcal{F}}})$ and $rank(\overline{\alpha}) < rank(target(\pi))$. Thus it is possible for the proponent to execute step (1b) at stage n. Contradiction. Therefore π does not exist. Hence $OPT_n = \emptyset$.

We have proved that *sdd* is successful. □

To show the completeness theorem, it remains for us to show that there is no infinite strongly grounded dispute derivation.

Lemma 12. *Let* $sdd = \langle PPT_0, PTA_0, OPT_0\rangle, \ldots, \langle PPT_n, PTA_n, OPT_n\rangle$ *be a strongly grounded dispute derivation wrt a ground map* λ. *The following statements hold:*

(1) *Each* PPT_i *contains at most one not-full ppa;*
(2) *Let* $\pi \in PPT_i$, $0 \leqslant i \leqslant n$. *Then it holds that* π *is not full iff there is an unique immediate expansion of* π *in* PPT_{i+1};
(3) *Let* $\pi = (T, h) \in PPT_i$, $0 \leqslant i \leqslant n$. *Then* h *is of the form*

$$h = (T_0, i_0), (T_1, i_0 + 1), \ldots, (T_k, i_0 + k)$$

such that

- $i_0 + k \leqslant i$; *and*
- $T_j \subseteq \lambda(Cl(T))$ *for* $k \geqslant j \geqslant 1$; *and*
- *if* π *is not full then* $i_0 + k = i$.

(4) *Let* $\pi \in PPT_i$ $(0 \leqslant i \leqslant n)$ *such that the proponent does not make a move to expand* π *at some stage* k *with* $i \geqslant k > st(\pi)$. *Then* π *is full.*

Proof.

(1) By induction on i. The statement holds obviously for $i = 0$.
Suppose the statement holds for i. We show that it also holds for $i + 1$. If the opponent makes a move at stage $i + 1$ then $PPT_{i+1} = PPT_i$. The statement holds obviously.
If the proponent executes step (1b) at stage $i + 1$ meaning that step (1a) is not possible at this stage. Hence all ppas in PPT_i are full. The statement holds obviously.
Suppose the proponent executes step (1a) at stage $i + 1$ (to expand a ppa $\pi \in PPT_i$ into a new ppa $\pi' \in PPT_{i+1}$). Hence π is the only non-full ppa in PPT_i. Therefore, if π' is full, there is no non-full ppa in PPT_{i+1}. If π' is not full, π' is the only non-full ppa in PPT_{i+1}. The statement holds.
(2) Suppose π is not full. It follows immediately from the definition of sdd that the proponent will execute step (1a) at stage $i + 1$ to expand π into $\pi' \in PPT_{i+1}$. The uniqueness of π' follows from Lemma 7.
Suppose $\pi' \in PPT_{i+1}$ is the unique immediate expansion of π. Hence π is no-full obviously.
(3) By induction on i with $i \geqslant i_0$. The statement holds obviously for $i = i_0$. Suppose the statement holds for $i \geqslant i_0$. We show that it also holds for $i + 1$.
Let $\pi \in PPT_{i+1}$. There are two cases:
Case 1 $\pi \in PPT_i$. From statements (1,2) and Lemma 7, it follows that π is full. The statement follows directly from the induction hypothesis.

Case 2 $\pi \notin PPT_i$. Hence the proponent executes step (1a) at stage $i+1$ to expand some $p \in PPT_i$ into π with $h_p = (T_0, i_0), (T_1, i_0+1), \ldots, (T_k, i_0+k)$ and $T_p = T_k$. Hence p is not full. From the induction hypothesis, we have $i_0 + k = i$. Therefore $h_\pi = h_p.(T_\pi, i+1)$ and $T_\pi \subseteq Cl(T_\pi)$. Hence the statement holds.

(4) Suppose π is not full. From statement (3), $h_\pi = (T_0, i_0), (T_1, i_0+1), \ldots, (T_k, i_0+k)$ with $i_0+k = i$ where $i_0 = st(\pi)$. It follows that the proponent makes the move (1a) at every stage $j : i \geqslant j > 0$. Contradiction. We have proved that π is full. □

Lemma 13. *Let* $sdd = \langle PPT_0, PTA_0, OPT_0\rangle, \ldots, \langle PPT_n, PTA_n, OPT_n\rangle$ *be a strongly grounded dispute derivation for* $\delta \in Cl(GE_{AF_\mathcal{F}})$ *wrt a ground map* λ.

It holds that for all i: $PTA_i \subseteq Cl(GE_{AF_\mathcal{F}})$.

Proof. Let $\alpha \in PTA_i$, $0 \leqslant i \leqslant n$. We show by induction on i that $\alpha \in Cl(GE_{AF_\mathcal{F}})$.

Base Step: "$i = 0$". If δ is not an assumption then $PTA_0 = \emptyset$. The lemma holds vacuously. Suppose δ is an assumption. The lemma holds since it is assumed that $\delta \in Cl(GE_{AF_\mathcal{F}})$.

Inductive Step: Suppose the lemma holds for $i-1$. We show that it also holds for i.

The lemma follows directly from the inductive hypothesis if at step i, the opponent makes a move or the proponent attacks an opponent ppa.

Suppose now that the proponent expands some ppa π at step i. If $\alpha \in PTA_{i-1}$, the lemma follows from the inductive hypothesis.

Suppose now $\alpha \in PTA_i \setminus PTA_{i-1}$. Therefore $\alpha \in Ass(\lambda(Cl(T_\pi))$. Since $\lambda(Cl(T_\pi)) \in GE_{AF_\mathcal{F}}$, it follows $\alpha \in Cl(GE_{AF_\mathcal{F}})$. □

Lemma 14. *Let* $sdd = \langle PPT_0, PTA_0, OPT_0\rangle, \ldots, \langle PPT_n, PTA_n, OPT_n\rangle$ *be a strongly grounded dispute derivation for* $\delta \in Cl(GE_{AF_\mathcal{F}})$ *wrt a ground map* λ.

For each $0 \leqslant i \leqslant n$, *for all* $\pi, \pi' \in OPT_i$, *it holds that* $st(\pi) = st(\pi')$.

Proof. Suppose there are π, π' s.t. $st(\pi) \neq st(\pi')$; Without loss of generality, we assume that $st(\pi) < st(\pi')$. We first show that π is full.

Suppose π is not full. That means that at stage $st(\pi')$, the opponent attacks a proponent assumption (step 2.b) even though the opponent could have perform step (2a) (expanding a prefix of π). Hence *sdd* is not a strongly grounded dispute derivation. Contradiction.

We have proved that π is full.

Since step (2a) has higher priority than step (2b) in the definition of sdd, it follows that $st(\pi) + lh(\pi) < st(\pi')$.

Since π is full, T_π is an argument. Let α be an assumption such that $Cl(T_\pi) = \overline{\alpha}$. From Lemma 13, it follows $\alpha \in Cl(GE_{AF_\mathcal{F}})$. From Lemma 11, there is $\beta \in Ass(T_\pi)$ such that $rank(\overline{\beta}) < rank(\alpha)$ and $\overline{\beta} \in Cl(GE_{AF_\mathcal{F}})$. It is thus possible for the proponent to execute step (1b) before $st(\pi')$ to attack β. Hence $\pi \notin OPT_i$. Contradiction.

Therefore the assumption that $st(\pi) \neq st(\pi')$ is wrong. □

Definition 17. Let π be an opponent ppa and π', π'' be proponent ppas in a strongly grounded dispute derivation *sdd*.

(1) We say π hits π' at an assumption $\alpha' \in Ass(T_{\pi'})$ if

- π' is full and $\overline{\alpha'} = Cl(T_\pi)$; and

- $st(\pi') < st(\pi)$; and
- there is no opponent ppa p in *sdd* such that $\overline{\alpha'} = Cl(T_p)$ and $st(\pi') < st(p) < st(\pi)$.

We say π hits π' if π hits π' at some assumption.

(2) We say π' hits π (at an assumption $\alpha \in Ass(T_\pi)$) if the proponent attacks π at assumption α at stage $st(\pi')$.

(3) We say π'' supports π' if there is an opponent ppa p such that p hits π' and π'' hits p.

It is not difficult to see that the following lemma holds.

Lemma 15. *Let π be an opponent ppa and π' be proponent ppa in a strongly grounded dispute derivation sdd.*

(1) *for each assumption $\alpha \in T_\pi$, there is at most one proponent ppa hitting π at α.*
(2) *for each assumption $\alpha' \in T_{\pi'}$, there is at most one opponent ppa hitting π' at α'.*
(3) *if π' hits π then π is discontinued at stage $st(\pi')$;*[30]

Lemma 16. *Let π, π' be two proponent ppas in a strongly grounded dispute derivation sdd for $\delta \in Cl(GE_{AF_\mathcal{F}})$ such that π' supports π. It holds $rank(Cl(T_\pi)) > rank(Cl(T'_\pi))$.*

Proof. Let p be an opponent ppa such that p hits π and π' hits p.

From Lemma 11, it follows $rank(Cl(T_\pi)) \geqslant rank(target(p))$. From the selection criteria for step (1b) in sdd, it follows $rank(target(p)) > rank(Cl(T_{\pi'}))$.

Hence $rank(Cl(T_\pi)) > rank(Cl(T_{\pi'}))$. □

Theorem 7. *Let $sdd = \langle PPT_0, PTA_0, OPT_0\rangle, \ldots, \langle PPT_n, PTA_n, OPT_n\rangle, \ldots$ be an infinite strongly grounded dispute derivation for $\delta \in Cl(GE_{AF_\mathcal{F}})$ wrt ground map λ.*

Then for each k, the set $OPT^k = \{\pi \in \bigcup_{i=0}^{\infty} OPT_i | st(\pi) = k\}$ is finite.

Proof. Suppose the contrary that there is k, such that OPT^k is infinite. From Definition 13 (of grounded dispute derivation), it follows that the opponent attacks a proponent assumption α at stage k. From Lemma 14, it follows $OPT_k = \{\pi_0\}$ where $\pi_0 = (T_0, h_0)$ with $T_0 = [\overline{\alpha}]$ and $h_0 = ([\overline{\alpha}], k)$.

It follows that OPT^k has the structure of a tree $\mathcal{T}_k$ as follows:

The root of $\mathcal{T}_k$ is π_0. A ppa $\pi' \in OPT^k$ is a child of $\pi \in OPT^k$ iff π' is an immediate expansion of π.

Because $\mathcal{F}$ is finitary, each ppa has only finitely many immediate expansions. Hence each node in $\mathcal{T}_k$ has finitely many children. Since $\mathcal{T}_k$ is infinite (because OPT^k is infinite), there is an infinite path labeled by $\pi_0, \pi_1, \ldots,$ in $\mathcal{T}_k$ such that π_{i+1} is an immediate expansion of π_i.

We show that $T = \bigcup_{i=0}^{\infty} T_{\pi_i}$ is a full argument. Suppose T is not a full argument. Then there is a non-final leaf node N in T. Therefore there is m such that for each $j \geqslant m$, N is a non-final leaf node in T_{π_j}.

For each $j \geqslant m$, let N_j be the non-final leaf node in T_{π_j} selected for expansion to $T_{\pi_{j+1}}$. It follows from the extra condition for step (2a) in Definition 15 of sdd, that $h(N_j, T) = h(N_j, T_{\pi_j}) \leqslant h(N, T_{\pi_j}) = h(N, T)$. The set of N_js is therefore finite implying that the sequence $\pi_0, \pi_1, \ldots$ is finite. Contradiction.

We have proved that T is a full argument.

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[30]Since π is discontinued at stage i, there are two possibilities: π is attacked by the proponent or the expansion of π by the opponent fails. Since π' starts at i, it is the proponent who makes a move at stage i. Hence π is attacked by the proponent.

It is clear that T attacks α. Since $\alpha \in Cl(GE_{AF_{\mathcal{F}}})$ (Lemma 13), T is attacked by some $X \in GE_{AF_{\mathcal{F}}}$ such that $rank(X) < rank(\alpha)$. It follows there is $\beta \in Ass(T)$ such that $\overline{\beta} = Cl(X)$. Hence $rank(\overline{\beta}) < rank(X) < rank(\alpha)$.

Let $j = \min\{i | \beta \in Ass(T_{\pi_i})\}$. From Definition 15 of sdd, it is possible for the proponent to execute step (1b) at stage j or some later stage $> j$.[31] As step (1b) has higher priority than step (2a), the infinite path $\pi_0, \pi_1, \ldots$, in $\mathcal{T}_k$ does not exist. Contradiction

Hence tree $\mathcal{T}_k$ is finite. Contradiction. Hence we have proved that OPT^k is finite. □

Lemma 17. *Let sdd* $= \langle PPT_0, PTA_0, OPT_0\rangle, \ldots, \langle PPT_n, PTA_n, OPT_n\rangle, \ldots$ *be an infinite strongly grounded dispute derivation for* $\delta \in Cl(GE_{AF_{\mathcal{F}}})$ *wrt ground map* λ.

Let $\pi \in \cup\{PPT_i | i \geqslant 0\}$ *and* π *be a full ppa. Then the set of proponent ppas in sdd supporting* π *is finite.*

Proof. It is obvious that for each assumption $\alpha \in T_\pi$ there is at most one opponent ppa hitting π at α. Let P be the set of opponent ppas in sdd that hit π. Since $Ass(T_\pi)$ is finite, it follows directly from Lemma 15 that P is finite. Hence it also follows directly from Lemma 15 that there are only finite number of proponent ppas hitting the ppas in P. Therefore the lemma holds. □

Theorem 8. *Let sdd* $= \langle PPT_0, PTA_0, OPT_0\rangle, \ldots, \langle PPT_n, PTA_n, OPT_n\rangle, \ldots$ *be an infinite strongly grounded dispute derivation wrt ground map* λ.

The set of proponent profiled full arguments in sdd is finite.

Proof. We first prove two relevant properties.

Property 1.

(1) Each opponent ppa in sdd hits some proponent ppa in sdd;
(2) Each proponent ppa in sdd that does not start at stage 0 hits some opponent ppa in sdd.
(3) Each full proponent ppa in sdd that does not start at stage 0 supports some other full proponent ppa in sdd.

Proof of Property 1. Statement 2 holds obviously. Statement 3 follows directly from statements 1,2. We prove statement 1. Let π be an opponent ppa in sdd and $i = st(\pi)$ and $\alpha \in PTA_{i-1}$ such that $\overline{\alpha} = Cl(T_\pi)$.

Let π' be a proponent full ppa in sdd satisfying the following condition:

- $\alpha \in Ass(T_{\pi'})$;
- $st(\pi') = \max\{st(q) | \alpha \in Ass(T_q) \text{ and } q \in PPT_{i-1}\}$

We show that π hits π'. Suppose π does not hit π'. Hence there is opponent ppa p such that $st(\pi') < st(p) < st(\pi)$ s.t. $\overline{\alpha} = Cl(T_p)$. Let $j = st(p)$. It follows $j < i$ and $\alpha \notin PTA_j$. Since $\alpha \in PTA_{i-1}$, it follows that there is proponent full ppa q such that $st(\pi') < j < st(q) < i$ and $\alpha \in T_q$. Because $st(q) < i$, it follows $q \in PPT_{i-1}$. Contradiction to the construction of π'. □

[31] From statements (1,2,3) in Lemma 12, it follows that at any point during a sdd, the proponent has at most one ppa to expand and such ppa is expanded in consecutively stages and hence finished after finitely many steps (1a). Afterwards, if step (1b) is executable, it will be executed.

Let π_0 be the unique proponent profiled full argument such that $st(\pi_0) = 0$ (the existence and uniqueness of π_0 follows from Lemma 12).

A support path in *sdd* is a sequence of profiled full arguments $\pi_0, \ldots, \pi_n$ ($n \geqslant 0$) in $\bigcup_{i=0}^{\infty} PPT_i$ such that π_{i+1} supports π_i ($0 \leqslant i < n$).

Property 2. Each full proponent ppa π in sdd appears as the last element of a support path.

The proof is by induction on $st(\pi)$. The property holds obviously for $st(\pi) = 0$. Suppose that the property holds for all full proponent ppas that start before π. Therefore π supports some full proponent ppa π' (from above Property 1). Hence $st(\pi') < st(\pi)$. From the induction hypothesis, it follows that π' is the last element of some support path *sq*. Hence $sq.\pi$ is a support path.

The set of all support paths in *sdd* forms a tree $\mathcal{T}$ where the root is π_0 and the nodes in $\mathcal{T}$ are support paths and a support path $\pi_0, \ldots, \pi_n, \pi_{n+1}$ is a child of $\pi_0, \ldots, \pi_n$.

Suppose the set of proponent profiled full argument in *sdd* is infinite. Therefore $\mathcal{T}$ is infinite.

From Lemma 17, each node in $\mathcal{T}$ has only finitely many children. Hence there is an infinite support path in $\mathcal{T}$. Contrary to Lemma 16.

Hence the set of proponent profiled full arguments in *sdd* is finite. □

6.1. *Proof of completeness theorem*

We only need to show that there exists no infinite *sdd* for δ wrt λ.

Assume that there exists an infinite strongly grounded dispute derivation for $\delta \in Cl(GE_{AF_{\mathcal{F}}})$ wrt ground map λ

$$sdd = \langle PPT_0, PTA_0, OPT_0\rangle, \ldots, \langle PPT_i, PTA_i, OPT_i\rangle, \ldots$$

From Theorem 8, it follows that there is n such that PPT_n contains all proponent profiled full arguments in *sdd*. Therefore the opponent makes a move at all stages $m > n$ in *sdd*.

Let K_0 be the set of starting times before n of opponent ppas in *sdd* while K_1 be the set of starting times of all opponent ppas in *sdd*, i.e.

$$K_0 = \left\{ j \mid \exists \pi \in \bigcup_{i=0}^{\infty} OPT_i \text{ s.t. } j = st(\pi) < n \right\}, \quad \text{and}$$

$$K_1 = \left\{ j \mid \exists \pi \in \bigcup_{i=0}^{\infty} OPT_i \text{ s.t. } j = st(\pi) \right\}.$$

From the Definition 15 (of strongly grounded dispute derivation), it follows that $|K_1| \leqslant |K_0| + |PTA_n|$.

From $\bigcup_{i=0}^{\infty} OPT_i = \bigcup_{k \in K_1} OPT^k$, it follows immediately from Theorem 7 that $\bigcup_{i=0}^{\infty} OPT_i$ is finite. Hence *sdd* is finite. Contradiction.

We have proved that there is no infinite *sdd*.

The completeness theorem follows directly from Theorem 6. □

7. Flatten dispute derivation

We have presented the grounded dispute derivations that are both sound and complete wrt finitary assumption-based frameworks where proof trees together with their histories are fully and explicitly represented to shed light on the construction process of arguments and counter arguments during a derivation and hence providing key structural insights into the soundness and completeness of the procedure.

In this section, we present another procedure that is simply the result of flattening the first where instead of the whole proof trees, only their supports are recorded. Consequently, the second procedure is both sound and complete but with much simpler date structure and hence much more amenable to possible implementation. The disadvantage of the flatten version (like Prolog for logic programming) is that it does not give the reasons how a returned answer is supported and defended.

In this section we propose the flatten grounded dispute derivations which focus on the supports of proof trees instead of the whole trees like other proposals including [17–19,24].

The representation of flatten grounded dispute derivations is based on multisets. To keep the paper self-contained, we recall in Appendix A.1 a brief but formal introduction of multisets from [31].[32]

Definition 18 (Flatten Grounded Dispute Derivation)**.** A *flatten grounded dispute derivation* for a sentence δ is a sequence of the form

$$\langle PS_0, OS_0\rangle, \dots, \langle PS_n, OS_n\rangle$$

where

- for each i, PS_i is a multiset of sentences while OS_i is a multisets of multisets of sentences; and
- $PS_0 = \{\delta\}$, and $OS_0 = \emptyset$, and
- at stage i $(i > 0)$, one of the dispute parties makes a move satisfying the following properties:

  (1) Suppose the proponent makes a move at stage i. The proponent has two options:

    (a) The proponent selects a non-assumption $\sigma \in PS_{i-1}$, a rule r with $hd(r) = \sigma$ and,
    $PS_i = (PS_{i-1} \setminus \{\sigma\}) \oplus bd(r)$
    $OS_i = OS_{i-1}$.

    (b) The proponent selects $S \in OS_{i-1}$ and an assumption $\alpha \in S$ and,
    $PS_i = PS_{i-1} \oplus \{\overline{\alpha}\}$
    $OS_i = OS_{i-1} \setminus \{S\}$

  (2) Suppose the opponent makes a move at stage i. The opponent has two options:

    (a) The opponent selects $S \in OS_{i-1}$ and a non-assumption $\sigma \in S$ and proceeds as follows:
    $PS_i = PS_{i-1}$
    $OS_i = (OS_{i-1} \setminus \{S\}) \oplus \{(S \setminus \{\sigma\}) \oplus bd(r) | hd(r) = \sigma\}$.

[32] Intuitively, a multiset is like a set where its member may occur multiple times. For example, $A = \{2, 2, 2, 5\}$ is a multiset with 2 occurs 3 times and 5 occurs one time. Some readers may find it sensible to imagine A as a bag of three 2-dollars notes and one 5-dollar note (if such notes are ever in circulation). So if you have another bag $B = \{2, 5, 5\}$ of money containing one 2-dollar note and two 5-dollar notes then putting them together (their union) gives you a bag $C = A \oplus B = \{2, 2, 2, 2, 5, 5, 5\}$ of four 2-dollar notes and 3 5-dollars notes. If you removes one 2-dollar note from C $(C \setminus \{2\})$, you would get a bag of 3 2-dollar notes and 3 5-dollar notes ($C \setminus \{2\} = \{2, 2, 2, 5, 5, 5\}$. But if you remove all 2-dollar note from C $(C - \{2\})$, what remains is a bag of 3-dollar notes ($C - \{2\} = \{5, 5, 5\}$.

Table 3

A successful flatten grounded dispute derivation

Stage	Move	*PS*	*OS*
0		$\{a\}$	$\emptyset$
1	1*a*	$\{not\_\alpha\}$	$\emptyset$
2	2*b*	$\emptyset$	$\{\{\alpha\}\}$
3	2*a*	$\emptyset$	$\{\{not\_\beta, f(0)\}\}$
4	1*b*	$\{\beta\}$	$\emptyset$
5	1*a*	$\emptyset$	$\emptyset$

(b) The opponent selects an assumption $\alpha \in PS_{i-1}$ and,
$PS_i = PS_{i-1} - \{\alpha\}$
$OS_i = OS_{i-1} \oplus \{\{\overline{\alpha}\}\}$.

Definition 19. A flatten grounded dispute derivation $\langle PS_0, OS_0\rangle, \dots, \langle PS_n, OS_n\rangle$ is successful if $PS_n = OS_n = \emptyset$.

Example 7. A successful flatten grounded dispute derivation $\langle PS_0, OS_0\rangle, \dots, \langle PS_5, OS_5\rangle$ for sentence a (wrt the argumentation framework $\mathcal{F}_2$ in the introduction) is illustrated in Table 3 and explained in details below. For convenience we recall $\mathcal{F}_2$ below.

$$\mathcal{F}_2 : r : a \leftarrow not\_\alpha \qquad r'' : \alpha \leftarrow not\_\beta, f(0) \qquad r_n : f(n) \leftarrow f(n+1), \quad n \geqslant 0$$
$$t : \beta \leftarrow$$

At stage 0, we have $PS_0 = \{a\}$ and $OS_0 = \emptyset$.

- At stage 1, the proponent makes a move to expand the non-assumption a by applying step (1a) using rule r.
  Hence $PS_1 = (PS_0 \setminus \{a\}) \oplus bd(r) = \{not\_\alpha\}$ where $bd(r) = \{not\_\alpha\}$.
  $OS_1 = OS_0 = \emptyset$.
- Next, at stage 2, the opponent attacks the assumption $not\_\alpha \in PS_1$ by applying step (2b). Hence
  $OS_2 = OS_1 \oplus \{\{\alpha\}\} = \{\{\alpha\}\}$.
  $PS_2 = PS_1 - \{not\_\alpha\} = \emptyset$.
- At stage 3, the opponent applies step (2a), selecting $S = \{\alpha\}$ from OS_2 and expanding α. Since rule r'' is the only one with head α, it follows
  $OS_3 = (OS_2 \setminus \{S\}) \oplus \{(S \setminus \{\alpha\}) \oplus bd(r'')\} = \{\{not\_\beta, f(0)\}\}$.
  $PS_3 = PS_2 = \emptyset$
- At stage 4, the proponent applies step (1b) by selecting $S = \{not\_\beta, f(0)\} \in OS_3$ and assumption $not\_\beta \in S$ for attack. Therefore
  $PS_4 = PS_3 \oplus \{\beta\} = \{\beta\}$.
  $OS_4 = OS_3 \setminus \{S\} = \emptyset$.
- Finally at stage 5, the proponent applies step (1a), using rule t, to expand β. Therefore
  $PS_5 = (PS_4 \setminus \{\beta\}) \oplus bd(r) = \emptyset$ (since $bd(r) = \emptyset$).
  $OS_5 = OS_4 = \emptyset$

As both PS and OS are empty, the derivation is successful.

Definition 20.

(1) Let *dd* be a grounded dispute derivation of length n for a sentence δ (as defined in Definition 13). Define a sequence of multisets of assumptions

$$\mathcal{M}(dd) = MPTA_0, \dots, MPTA_n$$

by:

- $MPTA_0 = PTA_0$.
- Suppose $MPTA_{i-1}$ is defined. Then
  - ∗ $MPTA_i = MPTA_{i-1}$ if the move at stage i is (1b) or (2a);
  - ∗ $MPTA_i = MPTA_{i-1} \oplus Ass(r)$ if the move at stage i is (1a);
  - ∗ $MPTA_i = MPTA_{i-1} - \{\alpha\}$ if the move at stage i is (2b);

(2) For any proof tree T, $NSpm(T)$ and $Spm(T)$ denote respectively the multiset of the non-assumptions and the multiset of sentences labeling the leaf nodes of T and different to *true*. $NSpm(T)$ and $Spm(T)$ are often referred to as the *multiset non-assumption support* or *multiset support* of T, respectively.
For a set S of ppas, define

$$NSpm(S) = \oplus\big\{NSpm(T_\pi) | \pi \in S\big\};$$

Lemma 18. *Let dd be a grounded dispute derivation of length n for a sentence δ and $\mathcal{M}(dd) = MPTA_0, \dots, MPTA_n$. It holds that for all $0 \leqslant i \leqslant n$, $MPTA_i$ and PTA_i are set-equivalent.*

Proof. We prove by induction that for each i, $MPTA_i$ and PTA_i are set-equivalent.

It is obvious that $PTA_0 = MPTA_0$.

Let $i > 0$. Suppose for each $j < i$, PTA_j and $MPTA_j$ are set-equivalent. We show PTA_i and $MPTA_i$ are set-equivalent. There are four cases according to four possible moves of the players at stage i.

- The move at stage i is (1a). Hence $PTA_i = PTA_{i-1} \cup Ass(r)$ and $MPTA_i = MPTA_{i-1} \oplus Ass(r)$. From the induction hypothesis, PTA_{i-1} and $MPTA_{i-1}$ are set-equivalent, it follows obviously that PTA_i and $MPTA_i$ are set-equivalent.
- The move at stage i is (1b) or (2a). From the induction hypothesis, PTA_{i-1} and $MPTA_{i-1}$ are set-equivalent and $MPTA_i = MPTA_{i-1}$ and $PTA_i = PTA_{i-1}$, it follows obviously that PTA_i and $MPTA_i$ are set-equivalent.
- The move at stage i is (2b). Hence $PTA_i = PTA_{i-1} \setminus \{\alpha\}$ and $MPTA_i = MPTA_{i-1} - \{\alpha\}$. From the induction hypothesis, PTA_{i-1} and $MPTA_{i-1}$ are set-equivalent, it follows obviously that PTA_i and $MPTA_i$ are set-equivalent. □

The following lemmas show that each grounded dispute derivation could be translated into an equivalent flatten one and vice versa.

Lemma 19. *Let $dd = \langle PPT_0, PTA_0, OPT_0\rangle, \dots, \langle PPT_n, PTA_n, OPT_n\rangle$ be a grounded dispute derivation for δ.*

Further let $\mathcal{M}(dd) = MPTA_0, \dots, MPTA_n$.

Then the sequence $\langle PS_0, OS_0\rangle, \dots, \langle PS_n, OS_n\rangle$ where for each $0 \leqslant i \leqslant n$,

- $PS_i = NSpm(PPT_i) \oplus MPTA_i$; *and*
- $OS_i = \{Spm(T_\pi) | \pi \in OPT_i\}$;

is a flatten grounded dispute derivation for δ.

Proof. See Appendix A.4 □

Lemma 20. *Let* $fdd = \langle PS_0, OS_0\rangle, \ldots, \langle PS_n, OS_n\rangle$ *be a flatten grounded dispute derivation for* δ.

There is a grounded dispute derivation $dd = \langle PPT_0, PTA_0, OPT_0\rangle, \ldots, \langle PPT_n, PTA_n, OPT_n\rangle$ *for* δ *such that for all* $0 \leqslant i \leqslant n$*, the following properties hold:*

- $PS_i = NSpm(PPT_i) \oplus MPTA_i$; *and*
- $OS_i = \{Spm(T_p) | p \in OPT_i\}$;

where $\mathcal{M}(dd) = MPTA_0, \ldots, MPTA_n$.

Proof. See Appendix A.5. □

Theorem 9 (Soundness Theorem for Flatten Grounded Dispute Derivation)**.** *Let* $dd = \langle PS_0, OS_0\rangle, \ldots, \langle PS_n, OS_n\rangle$ *be a successful flatten grounded dispute derivation for* δ*. Then* $\delta \in Cl(GE_{AF_\mathcal{F}})$.

Proof. From Lemma 20, there exists a grounded dispute derivation $dd' = \langle PPT_0, PTA_0, OPT_0\rangle, \ldots, \langle PPT_n, PTA_n, OPT_n\rangle$ for δ such that $PS_i = NSpm(PPT_i) \oplus MPTA_i$; and $OS_i = \{Spm(T_p) | p \in OPT_i\}$;

Since $PS_n = \emptyset$, the sets $NSpm(PPT_n)$ are empty. Therefore all ppas in PPT_n are full.

As $OS_n = \{Spm(T_p) | p \in OPT_n\}$ and $OS_n = \emptyset$, it follows $OPT_n = \emptyset$.

Because $PS_n = \emptyset$ and $PS_n = NSpm(PPT_n) \oplus MPTA_n$, $MPTA_n$ is empty.

Since PTA_n and $MPTA_n$ are set-equivalent (see Lemma 18) and $MPTA_n$ is empty, PTA_n is empty.

Therefore dd' is a successful grounded dispute derivation. The theorem follows from Theorem 4. □

Theorem 10 (Completeness Theorem for Flatten Grounded Dispute Derivation)**.** *Let* $\mathcal{F}$ *be a finitary ABA framework and* $\sigma \in Cl(GE_\mathcal{F})$*. Then there is a successful flatten dispute derivation* $\langle PS_0, OS_0\rangle, \ldots, \langle PS_n, OS_n\rangle$ *for* σ.

Proof. From the completeness Theorem 5, there is a successful grounded dispute derivation $sdd = \langle PPT_0, PTA_0, OPT_0\rangle, \ldots, \langle PPT_n, PTA_n, OPT_n\rangle$ for σ.

It holds that $OPT_n = \emptyset$ and all ppas in PPT_n are full.

Let $sdd' = \langle PS_0, OS_0\rangle, \ldots, \langle PS_n, OS_n\rangle$ be the flatten dispute derivation for σ as defined in Lemma 19.

Since $PTA_n = \emptyset$, $MPTA_n = \emptyset$ (see Lemma 18).

Since all ppas in PPT_n are full, $NSpm(PPT_n) = \emptyset$. It hence follows $PS_n = NSpm(PPT_n) \oplus MPTA_n = \emptyset$.

Since $OPT_n = \emptyset$, $OS_n = \{Spm(T_p) | p \in OPT_n\} = \emptyset$, sdd' is successful. □

8. Discussion

We study the soundness and completeness of dialectical proof procedures for assumption-based argumentation with respect to the skeptical semantics. We have presented two proof procedures. In one, proof trees together with their histories are fully and explicitly represented. The other procedure is simply the result of flattening the first. We show the soundness and completeness of both proof procedures wrt grounded semantics of assumption-based argumentation frameworks where possibly non-terminating computation is represented by infinite arguments.

Assumption-based argumentation is an instance of abstract argumentation. Another well-known instance of abstract argumentation is the ASPIC+ system [27,28]. It would be interesting to see how to apply our proof procedures to ASPIC+.

In many applications, a sentence is (universally) accepted if it is accepted in every preferred extensions (or stable extensions) [2]. It would be interesting to see how our proof systems in this paper as well as in [31] could be extended for the universal acceptance.

Attacks may have different strength [16]. [7] proposed a dialectical proof procedure for abstract argumentation frameworks with attacks having different strength. It would be interesting to see how to integrate the ideas of [7]'s into our proof system for assumption-based argumentation with attacks of different strength.

Dialectical proof procedures are not the only approach to proof theory of logical argumentation. [1] proposes dynamic proof systems for logical argumentation. It would be intriguing to see the relationship between these rather distinct approaches.

Grounded semantics of an assumption-based argumentation could be viewed as a form of nonmonotonic inductive definition of the ABA framework [10]. In that sense, our proof system can be viewed as an example of a dialectical proof procedure for a nonmonotonic inductive definition. It may be worth exploring further whether dialectical proof systems could be developed for other nonmonotonic inductive definition systems as defined in [10].

There are recently several interesting works on the semantics of infinite argumentation frameworks [3,4]. It is worth noting that the inclusion of infinite arguments into the argumentation frameworks of assumption-based argumentation does not transform it into infinite argumentation frameworks as illustrated in the introduction. This leads to an interesting question of how to extend the results in [3,4] to frameworks with "two kinds of infinity": infinite number of arguments and arguments of infinite size.

Dialectical procedures could be viewed as a form of dialogue games where for both players the sets of rules as well as assumptions are given and fixed at the beginning. A more general form of dialogue games is proposed in [23] where rules and assumptions are introduced during the dialogue. While the soundness of such games has been studied in [23], their completeness is not. The completeness results in this paper, especially the completeness of the flatten procedure could shed light on the completeness of such dialogue games.

Appendix

A.1. Appendix: Multisets

This section recalls a brief introduction of multisets from [31].[33]

Intuitively, a multiset is like a set but allowing each element to have many instances.

Formally, a multiset is a pair $A = (B, \mu)$ where B is a set referred to as the base set of A, and μ is a function from B into the set of positive integers. The function μ is referred to as the *multiplicity function* of the multiset A and often denoted referred to by μ_A.

Two multisets $M = (B, \mu)$ and $M' = (B', \mu)$ are *set-equivalent* if B, B' are identical.

[33] Further details can be seen in [5,11,25].

For simplicity we represent a multiset as a set where its member may occur multiple times. For example, the prime factorization of 40 can be represented as a multiset $A = \{2, 2, 2, 5\}$. Another representation of the prime factorization of 40 is $A = (\{2, 5\}, \mu_A)$ where $\mu_A = \{(2, 3), (5, 1)\}$.[34]

We introduce the major types of the multisets operations performing on two multisets as follows.

Definition 21. The union and intersection of two multisets $M = (B, \mu)$, $M' = (B', \mu')$ are defined by:

(1) $M \oplus M' = (B \cup B', \mu + \mu')$ where for $x \in B \cup B'$, $(\mu + \mu')(x) = \mu(x) + \mu'(x)$.
(2) $M \cap M' = (B \cap B', \mu'')$ where $\mu''(x) = \min\{\mu(x), \mu'(x)\}$.

We also introduce two notions of difference and strong difference between multisets and sets in the definition below.

Definition 22. Let $M = (B, \mu)$, $M' = (B', \mu')$ be multisets and S be a set.

(1) The difference between M, M' is defined by:
$M \setminus M' = (B'', \mu'')$ where the following conditions are satisfied:
    (a) $B'' = (B \setminus B') \cup \{x \in B \cap B' | \mu(x) > \mu'(x)\}$.
    (b)

$$\mu''(x) = \begin{cases} \mu(x) & \text{if } x \in B \setminus B' \\ \mu(x) - \mu'(x) & \text{if } x \in B \cap B' \end{cases}$$

(2) The strong difference between M and S is defined by

$$M - S = \big(B \setminus S, \mu'\big)$$

where for each $x \in B \setminus S$, $\mu'(x) = \mu(x)$.

A.2. Appendix: Execution of programs on SWI-prolog

See Fig. 10 and Fig. 11 .

A.3. Appendix: Proofs of lemmas supporting soundness theorem

A.3.1. Proof of Lemma 7

Lemma. *Let* $dd = \langle PPT_0, PTA_0, OPT_0\rangle, \dots, \langle PPT_n, PTA_n, OPT_n\rangle$ *be a grounded dispute derivation.*

(1) *Let* π *be some proponent ppa appearing in dd. The following statements hold:*
    (a) *For each* $i : st(\pi) \leqslant i \leqslant n$*, there is an unique continuation of* π *in* PPT_i*.*
    (b) *Let* π_i, π_{i+1} $(st(\pi) \leqslant i < n)$ *be continuations of* π *in* PPT_i, PPT_{i+1} *respectively. Then* π_{i+1} *is either an immediate expansion of* π_i *or identical to* π_i*.*

(2) *Let* π, π' *be two proponent ppas in dd such that* $st(\pi) = st(\pi')$*. Then one is a continuation of the other.*
Consequently, if π, π' *are full then they are identical.*

[34] $\{(2, 3), (5, 1)\}$ represents the function assigning 3 to 2 and 1 to 5.

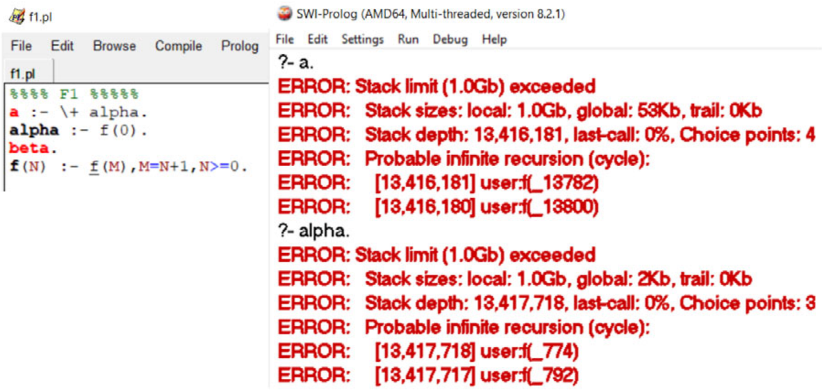

Fig. 10. The execution of $\mathcal{F}_1$ on SWI-prolog.

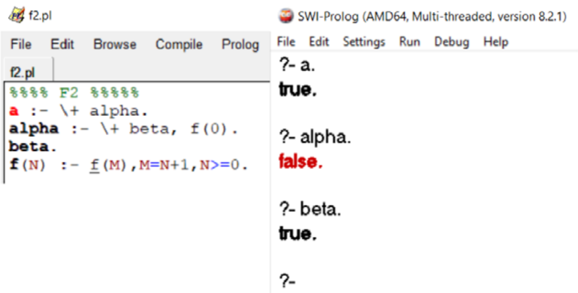

Fig. 11. The execution of $\mathcal{F}_2$ on SWI-prolog.

Proof. Statement 2 follows directly from statements (1a,1b) and the fact that at any stage, at most one new proponent ppa is created (according to Definition 13 of dispute derivation).

Let $k = st(\pi)$. We prove statements (1a,1b) by induction on $i > st(\pi)$.

Base case: $i = k$. It follows directly from the fact that at any stage, at most one new proponent ppa is created (according to Definition 13 of dispute derivation).

Inductive step. Suppose the lemma holds for $i : k \leqslant i < n$. We show that it also holds for $i + 1$.

If it is the opponent who makes a move at stage $i + 1$. Then $PPT_i = PPT_{i+1}$. The lemma follows directly from the induction hypothesis.

If the proponent makes a move according to step (1b) (in Definition 13 of dispute derivation) at stage $i + 1$ or step (1a) where π_i is not selected. Then $\pi_i \in PPT_{i+1}$. The lemma follows directly from the induction hypothesis.

Suppose the proponent makes a move according to step (1a) (in Definition 13 of dispute derivation) by expanding π_i at stage $i + 1$. The lemma follows directly from the definition of step (1a) of dispute derivation. □

A.3.2. Proof of Lemma 8

Lemma. *Let* $dd = \langle PPT_0, PTA_0, OPT_0\rangle, \ldots, \langle PPT_n, PTA_n, OPT_n\rangle$ *be a grounded dispute derivation. It holds that for each* $0 \leqslant i \leqslant n$*, for all* $\pi, \pi' \in OPT_i$*, if* π, π' *are compatible then* π, π' *are identical.*

Proof. By induction on i.

Basic step: $i = 0$. Since $OPT_0 = \emptyset$, the lemma holds obviously.

Inductive step: Suppose the lemma holds for $i-1$. Let π, π' be compatible ppas from OPT_i. We show that π, π' are identical.

There are two cases:

- Both π, π' belong to OPT_{i-1}. The lemma follows immediately from the induction hypothesis.
- Suppose $\pi \in OPT_i \setminus OPT_{i-1}$.
  It follows immediately that the opponent moves at stage i. Hence there are two cases:
  - ∗ The opponent makes a move according to step (2a). Hence there exists $\pi_0 \in OPT_{i-1}$ such that π is an immediate expansion of π_0 and $\pi_0 \notin OPT_i$.
    There are two cases:
    - ∗ $\pi' \in OPT_{i-1}$. Since π, π' are compatible, it follows that π', π_0 are also compatible. From the induction hypothesis, it is obvious that $\pi' = \pi_0$. Hence $\pi' \notin OPT_i$. Contradiction. This case hence can not happen.
    - ∗ $\pi' \in OPT_i \setminus OPT_{i-1}$. Hence π' is also an immediate expansion of π_0. Since π, π' are compatible, $T_\pi, T_{\pi'}$ are compatible. From Lemma 3, $T_\pi = T_{\pi'}$. Hence $\pi = \pi'$.
  - ∗ The opponent makes a move according to step (2b). Therefore the starting time of π is i. Since π, π' are compatible, the starting time of π' is also i. Hence $\pi = \pi'$. □

A.3.3. Proof of Lemma 9

Lemma. *Let dd be a dispute derivation*

$$dd = \langle PPT_0, PTA_0, OPT_0\rangle, \ldots, \langle PPT_n, PTA_n, OPT_n\rangle$$

Further let A be an argument such that $Cl(A) = \overline{\alpha}$ *for some assumption* α *and* $\pi = ([\overline{\alpha}], ([\overline{\alpha}], i)) \in OPT_i$ *for some* $0 \leqslant i \leqslant n$.

Then there is an unique sequence $cont_{dd}(\pi) = \pi_0, \ldots, \pi_k$ $(i \leqslant i + k \leqslant n)$ *such that*

(1) $\pi_0 = \pi$; *and*
(2) *for each* $i < j \leqslant k$, $\pi_j \in OPT_{i+j}$ *and* π_j *is a continuation of* π_{j-1} *and* $T_{\pi_j} \subseteq A$; *and*
(3) *if* $i + k < n$ *then* π_k *is discontinued at* $i + k + 1$ *and there is no ppa at any* $OPT_{i+k+1}, \ldots, OPT_n$ *that is compatible with* π_k.

Proof. Let $m = n - i$. We prove by induction on m. Note that i is fixed.

Base Step: $m = 0$. The lemma holds obviously (as the statements 2,3 do not apply).

Inductive Step: Suppose the lemma holds for $m \geqslant 0$. We show that it also holds for $m + 1$ (i.e. for $n + 1$).

Let $dd' = dd, \langle PPT_{n+1}, PTA_{n+1}, OPT_{n+1}\rangle$, and $\pi \in OPT_i$ and $cont_{dd} = \pi_0, \ldots, \pi_k$ where $\pi_0 = \pi$.

There are two cases:

- $i + k < n$. Let $cont_{dd'}(\pi) = cont_{dd}(\pi)$.
  From the induction hypothesis, it follows that π_k is discontinued at $i + k + 1$ and there is no ppa at any $OPT_{i+k+1}, \ldots, OPT_n$ that is compatible with π_k.
  We show that there is no ppa in OPT_{n+1} that is compatible with π_k. Suppose there is $\pi' \in OPT_{n+1}$ that is compatible with π_k. Therefore $\pi' \in OPT_{n+1} \setminus OPT_n$. Hence it is the opponent who makes a move at stage $n + 1$. There are two cases:

- ∗ The opponent makes a move according to step (2a) to expand some of her ppa $p \in OPT_n$. Since π', π_k are compatible, p, π_k are hence compatible. Contradiction!
- ∗ The opponent makes a move according to step (2b). Therefore the starting time of π' is $n+1$. π' is hence not compatible with π_k. Contradiction.

The uniqueness of $cont_{dd'}(\pi)$ comes from the uniqueness of $cont_{dd}(\pi)$.

- $i + k = n$. There are six cases.

(1) At stage $n+1$, the proponent makes a move to expand some proponent ppa. Hence $OPT_{n+1} = OPT_n$. The lemma holds obviously for $cont_{dd'}(\pi) = cont_{dd}(\pi).\pi_k$.
(2) At stage $n+1$, the proponent makes a move to attacks some opponent ppa that is different to π_k. Hence $\pi_k \in OPT_{n+1}$. The lemma holds obviously for $cont_{dd'}(\pi) = cont_{dd}(\pi).\pi_k$.
(3) At stage $n+1$, the proponent makes a move to attacks π_k. Hence $\pi_k \notin OPT_{n+1}$. The lemma holds obviously for $cont_{dd'}(\pi) = cont_{dd}(\pi)$.
(4) At stage $n+1$, the opponent moves to attack some proponent ppa and create a new ppa π'. It is obvious that $st(\pi') > st(\pi_k)$. Hence the lemma holds obviously for $cont_{dd'}(\pi) = cont_{dd}(\pi).\pi_k$.
(5) At stage $n+1$, the opponent moves to expand some opponent ppa, and the selected ppa is not π_k. Hence $\pi_k \in OPT_{n+1}$. Thus the lemma holds obviously for $cont_{dd'}(\pi) = cont_{dd}(\pi).\pi_k$.
(6) At stage $n+1$, the opponent moves to expand π_k at a non-final leaf node $N \in T_{\pi_k}$ labeled by a non-assumption sentence δ and $OPT_{n+1} = (OPT_n \setminus \{\pi_k\}) \cup \{(T', h') | T' \in CE(T_{\pi_k}, N), h' = h_{\pi_k}.(T', n+1)\}$
If the expansion fails at stage $n+1$ (i.e. $CE(T_{\pi_k}, N) = \emptyset$), the lemma holds obviously for $cont_{dd'}(\pi) = cont_{dd}(\pi)$.
Suppose now that $CE(T_{\pi_k}, N) \neq \emptyset$. Hence from Lemma 3, there is an unique immediate expansion of $\pi_{k+1} \in OPT_{n+1}$ of π_k and $T_{\pi_{k+1}} \subseteq A$.
Let $cont_{dd'}(\pi) = cont_{dd}(\pi).\pi_{k+1}$. It remains to show that $cont_{dd'}(\pi)$ is unique.
Suppose the contrary that $cont_{dd'}(\pi)$ is not unique. From the uniqueness of $cont_{dd}(\pi)$, it follows that there is $\pi' \in OPT_{n+1}$ such that π' is a continuation of π_k and $T_{\pi'} \subseteq A$ and $\pi' \neq \pi_{k+1}$.
There are two cases:

- ∗ $\pi' \in OPT_n$. From Lemma 8, it follows that $\pi' = \pi_k$. Hence $\pi_k \in OPT_{n+1}$ Contradiction. This case can not happen.
- ∗ $\pi' \in OPT_{n+1} \setminus OPT_n$. Hence π' is an immediate expansion of π_k. Since $T_{\pi'} \subseteq A$, it follows from Lemma 3 that $\pi' = \pi_{k+1}$. Contradiction. Hence this case is also not possible.

Thus $cont_{dd'}(\pi_{k+1})$ is unique. □

A.3.4. *Proof of Lemma 10*

The following lemma states that each argument attacking a proponent argument in a successful dispute derivation is counter-attacked by some proponent argument.

Lemma. *Let dd be a successful grounded dispute derivation that terminates at* $\langle PPT_n, PTA_n, OPT_n\rangle$ *and* $\pi = (T, h) \in PPT_n$ *and* T' *be an argument attacking argument* T. *Then there is a opponent ppa* π' *such that* $st(\pi) < st(\pi')$ *and* $T_{\pi'} \subseteq T'$ *and* π' *is discontinued at some step between* $st(\pi')$ *and* n.

Proof. Let $\alpha \in Ass(T)$ such that $\overline{\alpha} = Cl(T')$. Therefore there is $m \geqslant st(\pi)$ s.t. $\alpha \in PTA_m$ and there is $i \geqslant m$ s.t. the opponent attack α at stage i. Hence the ppa $\pi_0 = ([\overline{\alpha}], ([\overline{\alpha}], i)) \in OPT_i$.

From Lemma 9 and the fact that $OPT_n = \emptyset$, it follows that there is an unique $cont_{dd}(p) = \pi_0, \ldots, \pi_k$, $i + k < n$, such that for each j $(0 \leqslant j \leqslant k)$, $\pi_j \in OPT_{i+j}$ and π_j is a continuation of π_{j-1} and $T_{\pi_j} \subseteq A$; and

π_k is discontinued at $i + k + 1$ and there is no ppa at any $OPT_{i+k+1}, \ldots, OPT_n$ that is compatible with π_k.

The lemma holds obviously for $\pi' = \pi_k$. □

A.4. Appendix: Proof of Lemma 19

Lemma. *Let $dd = \langle PPT_0, PTA_0, OPT_0\rangle, \ldots, \langle PPT_n, PTA_n, OPT_n\rangle$ be a grounded dispute derivation for δ.*

Further let $\mathcal{M}(dd) = MPTA_0, \ldots, MPTA_n$.

Then the sequence $\langle PS_0, OS_0\rangle, \ldots, \langle PS_n, OS_n\rangle$ where for each $0 \leqslant i \leqslant n$,

- *$PS_i = NSpm(PPT_i) \oplus MPTA_i$; and*
- *$OS_i = \{Spm(T_\pi) | \pi \in OPT_i\}$;*

is a flatten grounded dispute derivation for δ.

Proof. We prove by induction on n. The base case $(n = 0)$ holds obviously.

Suppose the lemma holds for $n - 1$. We show that it also holds for n.

Let $dd' = \langle PS_0, OS_0\rangle, \ldots, \langle PS_{n-1}, OS_{n-1}\rangle$ be a flatten grounded dispute derivation such that $0 \leqslant i \leqslant n - 1$,

$PS_i = NSpm(PPT_i) \oplus MPTA_i$;

$OS_i = \{Spm(T_\pi) | \pi \in OPT_i\}$;

There are four cases:

(1) The proponent executes step (1a) at stage n by expanding some profiled partial argument $\pi = (T, h) \in PPT_{n-1}$ by expanding T at a non-final leaf node $N \in T$ labeled by a non-assumption sentence σ and a rule r with $hd(r) = \sigma$ such that
$PPT_n = (PPT_{n-1} \setminus \{\pi\}) \cup \{\pi'\}$ where $\pi' = (T', h')$ and $T' = \exp(T, N, r)$ and $h' = h.(T', n)$;
$MPTA_n = MPTA_{n-1} \oplus Ass(r)$
$OPT_n = OPT_{n-1}$.
It is clear that $NSpm(T_{\pi'}) = (NSpm(T_\pi) \setminus \{\sigma\}) \oplus (bd(r) \setminus Ass(r))$.
From the induction hypothesis, we have
$PS_{n-1} = NSpm(PPT_{n-1}) \oplus MPTA_{n-1}$;
$OS_{n-1} = \{Spm(T_p) | \pi \in OPT_{n-1}\}$;
It is clear that $\sigma \in PS_{n-1}$.
Let $OS_n = OS_{n-1}$ and $PS_n = (PS_{n-1} \setminus \{\sigma\}) \oplus bd(r)$.
It is obvious that the sequence dd', $\langle PS_n, OS_n\rangle$ is a flatten grounded dispute derivation.
We now proceed to show that it satisfies the lemma.
Since $OPT_n = OPT_{n-1}$ and $OS_n = OS_{n-1}$, it holds obviously that $OS_n = \{Spm(T_\pi) | \pi \in OPT_n\}$;
Let $PPTX = (PPT_{n-1} \setminus \{\pi\})$
It follows:

$$\begin{aligned} PS_n &= \big(PS_{n-1} \setminus \{\sigma\}\big) \oplus bd(r) \\ &= \big(\big(NSpm(PPT_{n-1}) \oplus MPTA_{n-1}\big) \setminus \{\sigma\}\big) \oplus bd(r) \quad \text{(from induction hypothesis)} \end{aligned}$$

$$= \big(\big(NSpm(PPTX) \oplus NSpm(T_\pi) \oplus MPTA_{n-1}\big) \setminus \{\sigma\}\big) \oplus bd(r)$$
$$= \big(\big(NSpm(PPTX) \oplus MPTA_{n-1}\big) \oplus NSpm(T_\pi)\big) \setminus \{\sigma\}) \oplus bd(r)$$
$$= (\big(NSpm(PPTX) \oplus MPTA_{n-1} \oplus Ass(r)\big) \oplus \big(NSpm(T_\pi) \setminus \{\sigma\}\big) \oplus \big(bd(r) \setminus Ass(r)\big)$$
$$= NSpm(PPTX) \oplus MPTA_n \oplus NSpm(T_{\pi'})$$
$$= NSpm(PPT_n) \oplus MPTA_n$$

(2) The proponent executes step (1b) at stage n by attacking an opponent's profiled partial argument $\pi = (T, h) \in OPT_{n-1}$ at an assumption $\alpha \in Ass(T)$ resulting in:
$PPT_n = PPT_{n-1} \cup \{(T', h')\}$ where $T' = [\overline{\alpha}]$ and $h' = ([\overline{\alpha}], n)$; and
$PTA_n = PTA_{n-1}$
$OPT_n = OPT_{n-1} \setminus \{\pi\}$
Let $S = Spm(T)$. It is clear $\alpha \in S$.
From the induction hypothesis, we have
$PS_{n-1} = NSpm(PPT_{n-1}) \oplus MPTA_{n-1}$;
$OS_{n-1} = \{Spm(T_p) | p \in OPT_{n-1}\}$;
Let $OS_n = OS_{n-1} \setminus \{S\}$.
$PS_n = PS_{n-1} \oplus \{\overline{\alpha}\}$
It is obvious that the sequence dd', $\langle PS_n, OS_n\rangle$ is a flatten grounded dispute derivation.
We now proceed to show that it satisfies the lemma.
It holds:

$$OS_n = OS_{n-1} \setminus \{S\}$$
$$= \big\{Spm(T_p) | p \in OPT_{n-1}\big\} \setminus \big\{Spm(T)\big\} \quad \text{(induction hypothesis)}$$
$$= \big\{Spm(T_p) | p \in OPT_n\big\};$$
$$PS_n = PS_{n-1} \oplus \{\overline{\alpha}\}$$
$$= NSpm(PPT_{n-1}) \oplus MPTA_{n-1} \cup \{\overline{\alpha}\} \quad \text{(induction hypothesis)}$$
$$= NSpm(PPT_{n-1}) \oplus \{\overline{\alpha}\} \oplus MPTA_n$$
$$= NSpm(PPT_n) \oplus MPTA_n$$

(3) The opponent expands at stage n an opponent ppa $\pi = (T, h) \in OPT_{n-1}$ at a non-final leaf node $N \in T$ labeled by a non-assumption sentence σ resulting in:
$PPT_n = PPT_{n-1}$
$PTA_n = PTA_{n-1}$
$OPT_n = OPT^0 \cup OPT^1$ where $OPT^0 = OPT_{n-1} \setminus \{\pi\}$ and
$OPT^1 = \{(T', h') | T' \in CE(T, N), h' = h.(T', n)\}$
From the induction hypothesis, we have
$PS_{n-1} = NSpm(PPT_{n-1}) \oplus MPTA_{n-1}$;
$OS_{n-1} = \{Spm(T_p) | p \in OPT_{n-1}\}$;
Let $S = Spm(T)$ and
$PS_n = PS_{n-1}$ and

$OS_n = (OS_{n-1} \setminus \{S\}) \oplus \{(S \setminus \{\sigma\}) \oplus bd(r)|hd(r) = \sigma\}$.
It is obvious that the sequence dd', $\langle PS_n, OS_n\rangle$ is a flatten grounded dispute derivation.
We now proceed to show that it satisfies the lemma.
For $T' = \exp(T, N, r)$, it holds $Spm(T') = (Spm(T) \setminus \{\sigma\}) \oplus bd(r) = (S \setminus \{\sigma\}) \oplus bd(r)$.
It holds obviously that

$$\begin{aligned} OS_n &= \big(OS_{n-1} \setminus \{S\}\big) \oplus \big\{\big(S \setminus \{\delta\}\big) \oplus bd(r)\big|hd(r) = \delta\big\} \\ &= \big\{Spm(T_p|p \in OPT^0\big\} \oplus \big\{Spm(T_p|p \in OPT^1\big\} \\ &= \big\{Spm(T_p)|p \in OPT_n\big\} \\ PS_n &= PS_{n-1} \\ &= NSpm(PPT_{n-1}) \cup MPTA_{n-1} \quad \text{(induction hypothesis)} \\ &= NSpm(PPT_n) \cup MPTA_n \end{aligned}$$

(4) The opponent attacks an assumption $\alpha \in PTA_{n-1}$ resulting in:
$PPT_n = PPT_{n-1}$
$PTA_n = PTA_{n-1} - \{\alpha\}$
$OPT_n = OPT_{n-1} \cup \{\pi\}$ where $\pi = ([\overline{\alpha}], ([\overline{\alpha}], n))$
From the induction hypothesis, we have
$PS_{n-1} = NSpm(PPT_{n-1}) \oplus MPTA_{n-1}$;
$OS_{n-1} = \{Spm(T_p)|p \in OPT_{n-1}\}$;
Let $PS_n = PS_{n-1} - \{\alpha\}$ and
$OS_n = OS_{n-1} \oplus \{\{\overline{\alpha}\}\}$.
It is obvious that the sequence dd', $\langle PS_n, OS_n\rangle$ is a flatten grounded dispute derivation.
We now proceed to show that it satisfies the lemma.
It holds obviously that

$$\begin{aligned} PS_n &= PS_{n-1} - \{\alpha\} \\ &= \big(NSpm(PPT_{n-1}) \oplus MPTA_{n-1}\big) - \{\alpha\} \quad \text{(induction hypothesis)} \\ &= NSpm(PPT_{n-1}) \oplus \big(MPTA_{n-1} - \{\alpha\}\big) \\ &= NSpm(PPT_n) \oplus MPTA_n \\ OS_n &= OS_{n-1} \oplus \big\{\{\overline{\alpha}\}\big\} \\ &= \big\{Spm(T_p)|p \in OPT_{n-1}\big\} \oplus \big\{\{\overline{\alpha}\}\big\} \quad \text{(induction hypothesis)} \\ &= \big\{Spm(T_p)|p \in OPT_{n-1}\big\} \oplus \big\{Spm(T_\pi)\big\} \\ &= \big\{Spm(T_p)|p \in OPT_n\big\} \qquad \square \end{aligned}$$

A.5. Appendix: Proof of Lemma 20

Lemma. *Let fdd* $= \langle PS_0, OS_0\rangle, \ldots, \langle PS_n, OS_n\rangle$ *be a flatten grounded dispute derivation for* δ.

There is a grounded dispute derivation dd $= \langle PPT_0, PTA_0, OPT_0\rangle, \ldots, \langle PPT_n, PTA_n, OPT_n\rangle$ *for* δ *such that for all* $0 \leqslant i \leqslant n$*, the following properties hold:*

- $PS_i = NSpm(PPT_i) \oplus MPTA_i$*; and*
- $OS_i = \{Spm(T_p) | p \in OPT_i\}$*;*

where $\mathcal{M}(dd) = MPTA_0, \ldots, MPTA_n$.

Proof. By induction on n.

The lemma holds obviously for $n = 0$.

Suppose that the lemma holds for $n - 1$. We show that it also holds for n.

Let $dd' = \langle PPT_0, PTA_0, OPT_0\rangle, \ldots, \langle PPT_{n-1}, PTA_{n-1}, OPT_{n-1}\rangle$ be a grounded dispute derivation for δ and $\mathcal{M}(dd') = MPTA_0, \ldots, MPTA_{n-1}$. such that for all $0 \leqslant i \leqslant n-1$, the following properties hold:

- $PS_i = NSpm(PPT_i) \oplus MPTA_i$; and
- $OS_i = \{Spm(T_p) | p \in OPT_i\}$;

We construct a triple $\langle PPT_n, PTA_n, OPT_n\rangle$ such that $dd = dd', \langle PPT_n, PTA_n, OPT_n\rangle$ is a grounded dispute derivation and $PS_n = NSpm(PPT_n) \oplus MPTA_n$ and $OS_n = \{Spm(T_p) | p \in OPT_n\}$ where $MPTA_n$ is defined from $MPTA_{n-1}$ as in Definition 20.

There are four cases:

(1) The proponent selects a non-assumption $\sigma \in PS_{n-1}$, a rule r with $hd(r) = \sigma$ and,
$PS_n = (PS_{n-1} \setminus \{\sigma\}) \oplus bd(r)$
$OS_n = OS_{n-1}$.
Since $PS_{n-1} = NSpm(PPT_{n-1}) \oplus MPTA_{n-1}$ (from the induction hypothesis), it follows there exists $\pi = (T, h) \in PPT_{n-1}$ such that $\sigma \in NSpm(T)$.
Let N be a leaf-node in T labeled by σ and $T' = \exp(T, N, r)$.
Let $PPT_n = (PPT_{n-1} \setminus \{\pi\}) \cup \{(T', h')\}$ where $h' = h.(T', n)$;
$PTA_n = PTA_{n-1} \cup Ass(r)$
$OPT_n = OPT_{n-1}$.
The sequence $dd, \langle PPT_n, PTA_n, OPT_n\rangle$ is hence a grounded dispute derivation. We now proceed to show that it satisfies the lemma.
It holds directly that $OS_n = OS_{n-1} = \{Spm(T_p) | p \in OPT_{n-1}\} = \{Spm(T_p) | p \in OPT_n\}$;
Let $PPTX = PPT_{n-1} \setminus \{\pi\}$.
It is easy to see that $NSpm(T') = (NSpm(T) \setminus \{\sigma\}) \oplus NAss(r)$. Therefore

$$\begin{aligned}
PS_n &= \big(PS_{n-1} \setminus \{\sigma\}\big) \oplus bd(r) \\
&= \big(\big(NSpm(PPT_{n-1}) \oplus MPTA_{n-1}\big) \setminus \{\sigma\}\big) \oplus bd(r) \quad \text{(from induction hypothesis)} \\
&= \big(\big(NSpm(PPTX) \oplus NSpm(T) \oplus MPTA_{n-1}\big) \setminus \{\sigma\}\big) \oplus \big(Ass(r) \oplus NAss(r)\big) \\
&= NSpm(PPTX) \oplus \big(\big(NSpm(T) \setminus \{\sigma\}\big) \oplus NAss(r)\big) \oplus MPTA_{n-1} \oplus Ass(r) \\
&= NSpm(PPTX) \oplus NSpm\big(T'\big) \oplus MPTA_{n-1} \oplus Ass(r) \\
&= NSpm(PPT_n) \oplus MPTA_n.
\end{aligned}$$

(2) The proponent selects $S \in OS_{n-1}$ and an assumption $\alpha \in S$ and,
$PS_n = PS_{n-1} \oplus \{\overline{\alpha}\}$
$OS_n = OS_{n-1} \setminus \{S\}$

From $OS_{n-1} = \{Spm(T_p) | p \in OPT_{n-1}\}$ (induction hypothesis), there is $\pi = (T, h) \in OPT_{n-1}$ such that $Spm(T) = S$. Hence $\alpha \in Ass(T)$.
Let $PPT_n = PPT_{n-1} \cup \{(T', h')\}$ where $T' = [\overline{\alpha}]$ and $h' = ([\overline{\alpha}], n)$; and
$PTA_n = PTA_{n-1}$
$OPT_n = OPT_{n-1} \setminus \{\pi\}$
The sequences $dd, \langle PPT_n, PTA_n, OPT_n\rangle$ is hence a grounded dispute derivation. We now proceed to show that it satisfies the lemma.
It holds:

$$\begin{aligned} OS_n &= OS_{n-1} \setminus \{S\} \\ &= \big\{Spm(T_p) | p \in OPT_{n-1}\big\} \setminus \big\{Spm(T\big\} \quad \text{(induction hypothesis)} \\ &= \big\{Spm(T_p) | p \in OPT_n\big\}. \\ PS_n &= PS_{n-1} \oplus \{\overline{\alpha}\} \\ &= NSpm(PPT_{n-1}) \oplus MPTA_{n-1} \oplus \{\overline{\alpha}\} \quad \text{(induction hypothesis)} \\ &= NSpm(PPT_n) \oplus MPTA_n \end{aligned}$$

(3) The opponent selects $S \in OS_{n-1}$ and a non-assumption $\sigma \in S$ and proceeds as follows:
$PS_n = PS_{n-1}$
$OS_n = (OS_{n-1} \setminus \{S\}) \oplus \{(S \setminus \{\sigma\}) \oplus bd(r) | hd(r) = \sigma\}$.
From the induction hypothesis, we have
$PS_{n-1} = NSpm(PPT_{n-1}) \oplus MPTA_{n-1}$; and
$OS_{n-1} = \{Spm(T_p) | p \in OPT_{n-1}\}$.
Therefore there is $\pi \in OPT_{n-1}$ such that $Spm(T_\pi) = S$. Further let N_σ be a non-final leaf node in T_π labeled by σ.
Let $PPT_n = PPT_{n-1}$
$PTA_n = PTA_{n-1}$
$OPT_n = (OPT_{n-1} \setminus \{\pi\}) \cup \{(T', h') | T' \in CE(T_\pi, N_\sigma), h' = h.(T', n)\}$
The sequences $dd, \langle PPT_n, PTA_n, OPT_n\rangle$ is hence a grounded dispute derivation. We now proceed to show that it satisfies the lemma.
It holds

$$\begin{aligned} OS_n &= (OS_{n-1} \setminus \{S\}) \oplus \{(S \setminus \{\sigma\}) \oplus bd(r) | hd(r) = \sigma\} \\ &= \{Spm(T_p) | p \in OPT_{n-1} \setminus \{\pi\}\} \oplus \{(S \setminus \{\sigma\}) \oplus bd(r) | hd(r) = \sigma\} \\ &= \{Spm(T_p) | p \in OPT_{n-1} \setminus \{\pi\}\} \oplus \{Spm(\exp(T_\pi, N_\sigma, r)) | hd(r) = \sigma\} \text{ [35]} \\ &= \{Spm(T_p) | p \in OPT_{n-1} \setminus \{\pi\}\} \oplus \{Spm(T') | T' \in CE(T_\pi, N_\sigma)\}. \\ &= \{Spm(T_p) | p \in OPT_n\}. \end{aligned}$$

Further, we have:

$$\begin{aligned} PS_n &= PS_{n-1} \\ &= NSpm(PPT_{n-1}) \oplus MPTA_{n-1} \quad \text{(induction hypothesis)} \\ &= NSpm(PPT_n) \oplus MPTA_n; \end{aligned}$$

[35] Note that $(S \setminus \{\sigma\}) \oplus bd(r) = Spm(\exp(T_\pi, N_\sigma, r))$.

(4) The opponent selects an assumption $\alpha \in PS_{n-1}$ and,
$PS_n = PS_{n-1} - \{\alpha\}$
$OS_n = OS_{n-1} \oplus \{\{\overline{\alpha}\}\}$
Let $PPT_n = PPT_{n-1}$
$PTA_n = PTA_{n-1} \setminus \{\alpha\}$
$OPT_n = OPT_{n-1} \cup \{(T, h)\}$ where $T = [\overline{\alpha}]$ and $h = ([\overline{\alpha}], n)$
The sequences dd, $\langle PPT_n, PTA_n, OPT_n\rangle$ is hence a grounded dispute derivation. We now proceed to show that it satisfies the lemma.
From the induction hypothesis, we have
$PS_{n-1} = NSpm(PPT_{n-1}) \oplus MPTA_{n-1}$;
$OS_{n-1} = \{Spm(T_p) | p \in OPT_{n-1}\}$;
It holds

$$\begin{aligned} OS_n &= OS_{n-1} \oplus \big\{\{\overline{\alpha}\}\big\} \\ &= \big\{Spm(T_p) | p \in OPT_{n-1}\big\} \oplus \big\{\{\overline{\alpha}\}\big\} \quad \text{(induction hypothesis)} \\ &= \big\{Spm(T_p) | p \in OPT_n\big\} \\ PS_n &= PS_{n-1} - \{\alpha\} \\ &= \big(NSpm(PPT_{n-1}) \oplus MPTA_{n-1}\big) - \{\alpha\} \\ &= NSpm(PPT_{n-1}) \oplus \big(MPTA_{n-1} - \{\alpha\}\big) \\ &= NSpm(PPT_n) \oplus MPTA_n \qquad \square \end{aligned}$$

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