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Justification, stability and relevance in incomplete argumentation frameworks

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Abstract. We explore the computational complexity of justification, stability and relevance in incomplete argumentation frameworks (IAFs). IAFs are abstract argumentation frameworks that encode qualitative uncertainty by distinguishing between certain and uncertain arguments and attacks. These IAFs can be completed by deciding for each uncertain argument or attack whether it is present or absent. Such a completion is an abstract argumentation framework, for which it can be decided which arguments are acceptable under a given semantics. The *justification* status of an argument in a completion then expresses whether the argument is accepted (IN), not accepted because it is attacked by an accepted argument (OUT) or neither (UNDEC). For a given IAF and certain argument, the justification status of that argument need not be the same in all completions. This is the issue of *stability*, where an argument is stable if its justification status is the same in all completions. For arguments that are not stable in an IAF, the *relevance* problem is of interest: which uncertain arguments or attacks should be investigated for the argument to become stable? In this paper, we define justification, stability and relevance for IAFs and provide a complexity analysis for these problems under grounded, complete, preferred and stable semantics.

Keywords: Incomplete argumentation frameworks, stability, relevance, complexity

1. Introduction

A central concept in computational argumentation is that of argumentation frameworks (AFs), in which arguments and the attack relation between them are represented as a directed graph where nodes correspond to arguments and edges display the attack relation [10]. One of the assumptions in such Dung-style argumentation frameworks is that all arguments and attacks are known. However, in practice, argumentation is a dynamic process in which not all arguments and attacks may be known in advance. For example, not all the evidence on which arguments are based might already have been observed, making the presence of a specific argument uncertain [16]. AFs are not able to represent qualitative uncertainty on the existence of specific arguments and attacks. For this reason, incomplete argumentation frameworks (IAFs) have been designed [3,4,7,15]. IAFs are an extension to AFs in which not only certain arguments are specified, but also arguments and attacks for which it is uncertain whether they are present. By deciding for every uncertain argument and attack whether it is present or absent, it is possible to "complete" an IAF, turning it into an AF. Thus, an IAF represents a set of possible AFs, its *completions*.

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Since every completion of an IAF is an AF, standard Dung semantics can be applied to determine the extensions of any completion [10]. Based on these extensions, one can determine the *justification* status of every argument in every completion of the IAF. The justification status is defined in terms of labels like IN (expressing that the argument is in some, or all, extensions and therefore should be accepted), OUT (expressing that the argument is attacked by an IN argument and therefore should be rejected) and UNDEC (for all other arguments, which are undecided) [6]. Note that the notion of justification status is only defined on the AFs that make up the completions of an IAF, but not on the IAF itself.

Now suppose that we are interested in the justification status of a particular argument, but are faced with an IAF containing uncertainties concerning the presence of (other) arguments and attacks. Then it would be interesting to know whether it is required to resolve these uncertainties. In a situation where the argument we are interested in has the same justification status in each completion of the IAF, there is no need for further investigation into the uncertain arguments and attacks. Then we say that the argument is stable with respect to the IAF and justification status. The detection of such *stability* has practical applications, for instance as a termination criterion for argumentative dialogue agents: in the agent architecture for inquiry proposed in [16,17], stability detection prevents the agent from asking unnecessary questions. In addition, [15] proposes an application of stability detection in negotiating agents, to recognise situations in which an agent should stop negotiating and accept its opponent's offer.

In the case that the argument of interest is not stable in the given IAF, we know that the agent should collect more information. However, not all information that is currently unknown can influence the justification status of the argument of interest. A natural question would therefore be: which uncertainties should we resolve in order to reach a point where the argument is stable? In other words: which uncertain arguments or attacks are still *relevant* for the justification status? Adding relevance to an inquiry or negotiation process ensures that the questions that are asked contribute to reaching stability.

The problem of detecting stability has been studied for structured (ASPIC$^+$) approaches to argumentation in [16,17,23]. The algorithms proposed in this line of research have been implemented in an inquiry system for the intake of online trade fraud that has been used by hundreds of users every day since its launch in September 2019 [16]. For abstract approaches to argumentation, the problem of detecting stability was introduced in [15] and the subsequent problem of detecting relevance in incomplete (abstract) argumentation frameworks was introduced in our earlier work in [18]. Given the high potential for applications of stability and relevance in inquiry and negotiation, it would be useful to have efficient algorithms for solving these problems in abstract approaches to argumentation as well. In this paper we take a first step in the development of such algorithms, by investigating in which complexity class the problems of detecting stability and relevance are situated: insights in the complexity of the problems establish the possible efficiency of any algorithm to solve them.

The contribution of this paper, which expands on [18], is the extensive study of the complexity of justification, stability and relevance in the context of IAFs. Specifically, we present precise complexity results for each of these three problems under grounded, complete, stable and preferred semantics.[1] Table 1 provides an overview of all these results. We further provide full proofs and illustrative examples for all the complexity results of justification, stability and relevance.

The paper is structured as follows. In Section 2, we provide the necessary preliminaries. In Section 3, we study the complexity of identifying the justification status of an argument. These results are used

[1]In [18] we only discussed the complexity of relevance for IN/OUT justification statuses under grounded semantics, whereas this paper is the first to also give complexity results for relevance for all justification statuses under all semantics (stable, complete, grounded, preferred).

Table 1

Overview of all complexity results related to this paper. We considered four different semantics: stable (ST), complete (CP), grounded (GR) or preferred (PR). The semantics and acceptance strategies are defined formally in Section 2.1. If a reference is specified, this complexity result is trivial from an earlier result in the literature. New results are printed bold; we refer to the corresponding proposition by "P" and the proposition number

Semantics	Acceptance strategy	Label	JUSTIFICATION	STABILITY	RELEVANCE
ST	credulous	IN/OUT	NP-c [8,9]	Π_2^p-c [3]	**Σ_2^p-c** P17
ST	credulous	UNDEC	**Trivial (no)** P4	**Trivial (no)** P10	**Trivial (no)** P18
ST	sceptical	IN/OUT	CoNP-c [9]	CoNP-c [3]	**Σ_2^p-c** P17
ST	sceptical	UNDEC	**CoNP-c** P5	**CoNP-c** P9	**Σ_2^p-c** P19
ST	sceptical-existent	IN/OUT	DP-c [12]	Π_2^p-c [3]	**Σ_2^p-c** P20
ST	sceptical-existent	UNDEC	**Trivial (no)** P4	**Trivial (no)** P10	**Trivial (no)** P18
CP	credulous	IN/OUT	NP-c [8,9]	Π_2^p-c [3]	**Σ_2^p-c** P13
CP	credulous	UNDEC	**P-c** P1	**CoNP-c** P6	**NP-c** P12
CP	sceptical	IN/OUT	P-c [10]	CoNP-c [3]	**NP-c** P12
CP	sceptical	UNDEC	**CoNP-c** P2	**CoNP-c** P7	**Σ_2^p-c** P15
GR	credulous	IN/OUT	P-c [10]	CoNP-c [3]	**NP-c** P12
GR	credulous	UNDEC	**P-c** P1	**CoNP-c** P6	**NP-c** P12
GR	sceptical	IN/OUT	P-c [10]	CoNP-c [3]	**NP-c** P12
GR	sceptical	UNDEC	**P-c** P1	**CoNP-c** P6	**NP-c** P12
PR	credulous	IN/OUT	NP-c [8,9]	Π_2^p-c [3]	**Σ_2^p-c** P13
PR	credulous	UNDEC	**Σ_2^p-c** P3	**Π_3^p-c** P8	**Σ_3^p-c** P14
PR	sceptical	IN/OUT	Π_2^p-c [11]	Π_2^p-c [3]	**Σ_3^p-c** P16
PR	sceptical	UNDEC	**CoNP-c** P2	**CoNP-c** P7	**Σ_2^p-c** P15

in Section 4 in our complexity analysis of the stability problem. We then introduce the relevance problem for IAFs in Section 5 and provide complexity results. Related work is discussed in Section 6; we conclude in Section 7.

2. Preliminaries

In this section, we recall the most important notions from abstract argumentation and the associated semantics [10], as well as incomplete argumentation frameworks and their completions [3,4,7,15]. We also provide a brief introduction to the polynomial hierarchy, which is required for our complexity study.

2.1. Argumentation frameworks and semantics

An argumentation framework (AF) $\langle \mathcal{A}, \mathcal{R} \rangle$, as introduced in [10], consists of a finite set $\mathcal{A}$ of arguments and a binary attack relation $\mathcal{R} \subseteq \mathcal{A} \times \mathcal{A}$ on them, where $(A, B) \in \mathcal{R}$ indicates that argument A attacks argument B. In this paper, we evaluate arguments using the semantics of [10].

Definition 1 (Extension-based semantics). Let $AF = \langle \mathcal{A}, \mathcal{R} \rangle$ be an AF and $S \subseteq \mathcal{A}$. Then:

- S is *conflict-free* iff for each $X, Y \in S : (X, Y) \notin \mathcal{R}$;
- $X \in \mathcal{A}$ is *acceptable with respect to* S iff for each $Y \in \mathcal{A}$ such that $(Y, X) \in \mathcal{R}$, there is a $Z \in S$ such that $(Z, Y) \in \mathcal{R}$;

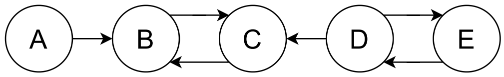

Fig. 1. An example of an argumentation framework. Arguments are depicted as circles, whereas attacks are depicted as arrows.

- S is an *admissible set* iff S is conflict free and $X \in S$ implies that X is acceptable with respect to S;
- S is a *complete extension* (CP) iff S is admissible and for each X: if $X \in \mathcal{A}$ is acceptable with respect to S then $X \in S$;
- S is a *preferred extension* (PR) iff it is a set inclusion maximal admissible set;
- S is the *grounded extension* (GR) iff it is the set inclusion minimal complete extension; and
- S is a *stable extension* (ST) iff it is complete and attacks all the arguments in $\mathcal{A} \setminus S$.

Example 1. Fig. 1 shows an example of an argumentation framework $AF = \langle \mathcal{A}, \mathcal{R} \rangle$ where $\mathcal{A} = \{A, B, C, D, E\}$ and $\mathcal{R} = \{(A, B), (B, C), (C, B), (D, C), (D, E), (E, D)\}$. The grounded extension of AF is $\{A\}$. This is also a complete extension. Additionally, there are two complete extensions that are also preferred and stable: $\{A, C, E\}$ and $\{A, D\}$.

In order to decide if an argument should be accepted w.r.t. a given AF and semantics, one can choose between different strategies. We will refer to these as acceptability strategies, following [5, Definition 1]. Dependent on the strategy, the argument is accepted iff it occurs in one and/or all extensions.

Definition 2 (Acceptability strategies)**.** Let $AF = \langle \mathcal{A}, \mathcal{R} \rangle$ be an argumentation framework and σ some semantics in $\{$GR, CP, PR, ST$\}$ and let A be some argument in $\mathcal{A}$.

- A is sceptically accepted w.r.t. σ semantics iff A belongs to *each* σ-extension of AF;
- A is credulously accepted w.r.t. σ semantics iff A belongs to *some* σ-extension of AF; and
- A is sceptically-existent accepted w.r.t. σ semantics iff AF has at least one σ-extension and A belongs to *each* σ-extension of AF.

We refer to *sceptical*, *credulous* and *sceptical-existent* as acceptability strategies.

Note that the sceptical-existent acceptability strategy only differs from the sceptical strategy for ST semantics: for GR, CP and PR semantics, each AF has at least one extension, while it is possible for an AF to have no ST extension.

Example 2. In the AF illustrated in Fig. 1, only the argument A is sceptically accepted w.r.t. CP semantics. The arguments C, D and E are all credulously accepted. Argument B is not accepted for any acceptability strategy.

2.2. *Incomplete argumentation frameworks*

Incomplete argumentation frameworks (IAFs) [3,4,15] are an extension to AFs, initially proposed as partial AFs in [7]. In an IAF, the set of arguments and attacks is split into two disjoint parts: a certain part ($\mathcal{A}$ and $\mathcal{R}$) and an uncertain part ($\mathcal{A}^?$ and $\mathcal{R}^?$). For the uncertain elements, it is not known whether they are part of the argumentation framework or not. They may be added in the future, for example because more information is acquired in an inquiry dialogue, or removed, for example because after investigation, this element turned out not to be present in the given setting.

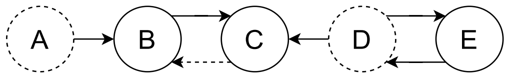

Fig. 2. An example of an incomplete argumentation framework. Certain arguments are depicted as circles with solid borders, whereas uncertain arguments are circles with dashed borders. Attacks are depicted as arrows, which have a solid line if they represent certain attacks and a dashed line if they represent uncertain attacks.

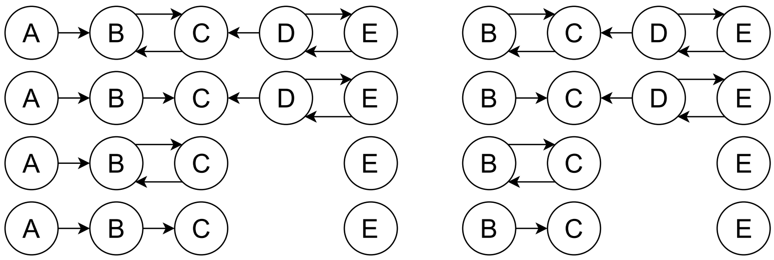

Fig. 3. The eight completions of our incomplete argumentation framework.

Definition 3 (Incomplete argumentation framework)**.** An incomplete argumentation framework is a tuple $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$, where $\mathcal{A} \cap \mathcal{A}^? = \emptyset$, $\mathcal{R} \cap \mathcal{R}^? = \emptyset$ and:

- $\mathcal{A}$ is the set of certain arguments;
- $\mathcal{A}^?$ is the set of uncertain arguments;
- $\mathcal{R} \subseteq (\mathcal{A} \cup \mathcal{A}^?) \times (\mathcal{A} \cup \mathcal{A}^?)$ is the certain attack relation; and
- $\mathcal{R}^? \subseteq (\mathcal{A} \cup \mathcal{A}^?) \times (\mathcal{A} \cup \mathcal{A}^?)$ is the uncertain attack relation.

Example 3 (IAF)**.** Fig. 2 shows an example of an incomplete argumentation framework $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$ where $\mathcal{A} = \{B, C, E\}$, $\mathcal{A}^? = \{A, D\}$, $\mathcal{R} = \{(A, B), (B, C), (D, C), (D, E), (E, D)\}$ and $\mathcal{R}^? = \{(C, B)\}$. Arguments A and D are currently absent but may be added in the future. The attack from A to B is certainly present if A is present. Similarly, the attacks (D, C), (D, E) and (E, D) are certainly present if D is present. The attack from C to B, on the other hand, is uncertain: although the arguments B and C are certainly present, the attack itself is currently absent but may still be added.

An incomplete argumentation framework can be *completed* by deciding for all uncertain arguments and attacks whether or not they are present, as defined below.

Definition 4 (Completions)**.** Given an IAF $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$, a completion is any AF $\langle \mathcal{A}', \mathcal{R}' \rangle$ that satisfies $\mathcal{A} \subseteq \mathcal{A}' \subseteq \mathcal{A} \cup \mathcal{A}^?$ and $\mathcal{R}|_{\mathcal{A}'} \subseteq \mathcal{R}' \subseteq (\mathcal{R} \cup \mathcal{R}^?)|_{\mathcal{A}'}$ where the restriction $\mathcal{R}|_{\mathcal{A}'}$ of an attack $\mathcal{R}$ to a set of arguments $\mathcal{A}'$ is defined as $\mathcal{R}|_{\mathcal{A}'} = \{(A, B) \in \mathcal{R} | A \in \mathcal{A}' \text{ and } B \in \mathcal{A}'\}$.

Example 4 (Completions)**.** The incomplete argumentation framework from our previous example has eight completions. These are illustrated in Fig. 3.

Since completions are abstract argumentation frameworks, we can use the semantics from Section 2.1 to evaluate arguments in the completions of an incomplete argumentation framework. This leads to two ways of defining acceptance for incomplete argumentation frameworks, proposed in [3, pages 6-7]: necessary and possible acceptance. Informally, some argument is necessarily accepted if it is accepted in each completion, whereas the argument is possibly accepted if this holds for some completion. The definitions below make a distinction between the different acceptability strategies.

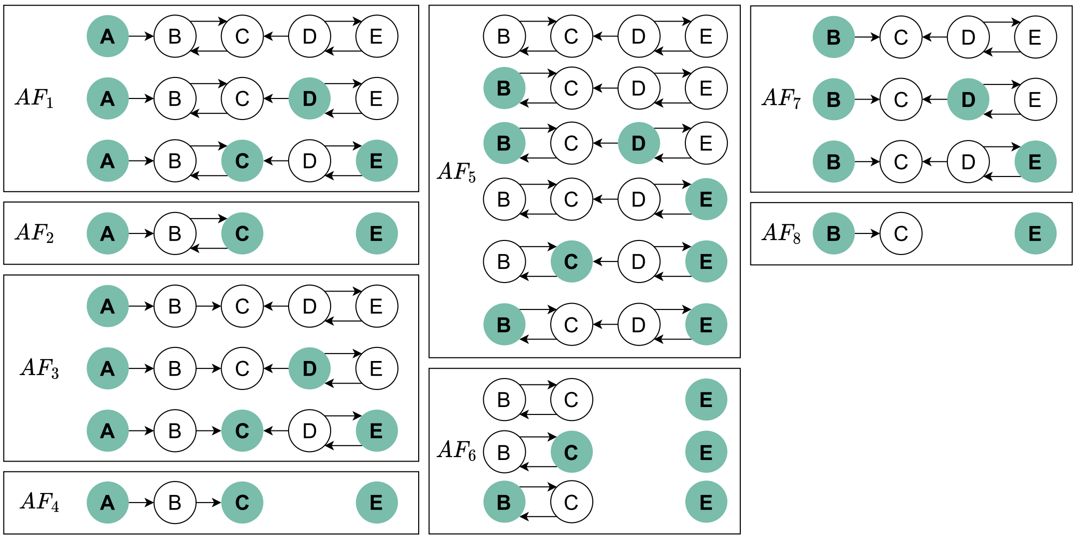

Fig. 4. Visualisation of the CP extensions of each of the eight completions of $\mathcal{I}$ from Example 3, where the completion is repeated for each CP extension and argument that are present in that extension are coloured green.

Definition 5 (Necessary acceptance)**.** Given an incomplete argumentation framework $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$, a certain argument $A \in \mathcal{A}$, some semantics σ in $\{$GR, CP, PR, ST$\}$ and some acceptability strategy $\alpha \in \{$*sceptical*, *credulous*, *sceptical-existent*$\}$, A is necessarily α-σ accepted w.r.t. $\mathcal{I}$ iff A is α-σ accepted in *each* completion of $\mathcal{I}$.

Definition 6 (Possible acceptance)**.** Given an incomplete argumentation framework $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$, a certain argument $A \in \mathcal{A}$, some semantics σ in $\{$GR, CP, PR, ST$\}$ and an acceptability strategy $\alpha \in \{$*sceptical*, *credulous*, *sceptical-existent*$\}$, A is possibly α-σ accepted w.r.t. $\mathcal{I}$ iff A is α-σ accepted in *some* completion of $\mathcal{I}$.

Example 5 (Necessary and possible acceptance)**.** Fig. 4 displays the CP extensions of each of the eight completions of $\mathcal{I}$ from Example 4, where the completion is repeated for each CP extension. None of the certain arguments in $\mathcal{I}$ is necessarily sceptically accepted w.r.t. CP semantics, because no argument is in all extensions of all completions. Note that A, although it is present in all extensions of all completions of $\mathcal{I}$ in which it occurs, is not necessarily sceptically accepted w.r.t. CP semantics: A is not a certain argument in $\mathcal{I}$. The only argument that is necessarily credulously accepted w.r.t. CP semantics is E. B is possibly sceptically accepted w.r.t. CP semantics, because the argument is in each extension of the completion AF_7 (as well as AF_8). C and E are possibly sceptically accepted w.r.t. CP semantics as well, thanks to their presence in the only extension of AF_2. Finally, all certain arguments of $\mathcal{I}$ (B, C and E) are possibly credulously accepted w.r.t. CP semantics.

The definitions of necessary and possible acceptance are particularly interesting for our work, as they are strongly related to the notion of stability, to be defined formally in Section 4. Before we do so, we first explain the polynomial hierarchy in the next section.

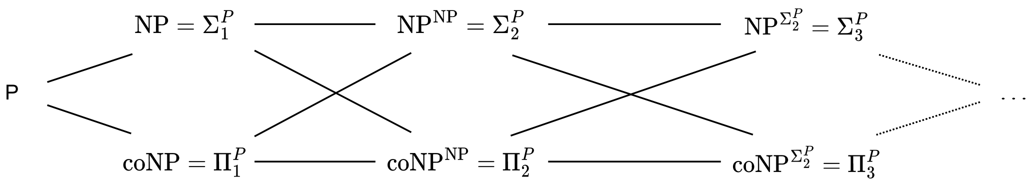

Fig. 5. The relation between complexity classes in the polynomial hierarchy. The lines between classes denote a subset relationship: all problems in the class of the left side are also contained in the class on the right side.

2.3. *The polynomial hierarchy*

In this section, we give a brief introduction to the polynomial hierarchy – for a more detailed explanation, we refer to [14]. This is required for the complexity study presented in this paper as the problems we study are situated on various levels in the polynomial hierarchy.

The polynomial hierarchy [19] is a hierarchy of complexity classes defined using oracle machines, i.e., Turing machines that are allowed to call a subroutine (oracle), deciding some fixed problem in constant time. For a class of decision problems $\mathcal{C}$ and a class $\mathcal{X}$ defined by resource bounds, $\mathcal{X}^{\mathcal{C}}$ denotes the class of problems decidable on a Turing machine with a resource bound given by $\mathcal{X}$ and an oracle for a problem in $\mathcal{C}$. For example, problems in NP are decidable with a resource bound given by NP and an oracle for a problem in P, therefore $\mathrm{NP} = \mathrm{NP}^{\mathrm{P}}$.

Based on these notions, the sets Σ_k^p and Π_k^p are defined as follows: $\Sigma_0^p = \Pi_0^p = P$, $\Sigma_{k+1}^p = \mathrm{NP}^{\Sigma_k^p}$ and $\Pi_{k+1}^p = \mathrm{CoNP}^{\Sigma_k^p}$. So problems in Σ_2^p are decidable with a resource bound given NP and an oracle for a problem in $\Sigma_1^p = \mathrm{NP}$. The polynomial hierarchy (PH) is then defined as the union of these complexity classes: $\mathrm{PH} = \bigcup_{k=0}^{\infty} \Sigma_k^p = \bigcup_{k=0}^{\infty} \Pi_k^p$. Finally, note that the definition over classes in the polynomial hierarchy imply a subset relation, illustrated in Fig. 5. By this relation, for each $i \in \mathbb{N}$, each problem in Σ_i^p is also in Σ_{i+1}^p, as well as in Π_{i+1}^p.

A problem that is Σ_2^p-complete is Σ_2-SAT [22]; in this paper we will use the CNF formulation that is also used in, e.g. [3,4]. An instance of Σ_2-SAT would be (Φ, X, Y), where Φ is an input formula in CNF over the pairwise disjoint sets of propositional variables X and Y. Then the Σ_2-SAT problem is to decide if there exists a truth value assignment τ_X to variables of X such that for each truth value assignment τ_Y: $\Phi[\tau_X, \tau_Y] = \mathrm{False}$, where $\Phi[\tau_X, \tau_Y]$ is the truth value that Φ evaluates to when applying the assignment τ_X to X and τ_Y to Y.

3. Justification status

In this section, we define the notion of justification status and study the complexity of determining this status for a given argument. The justification status [6] is a notion of acceptance for arguments in abstract argumentation frameworks that is more fine-grained than only considering the presence or absence in extensions. Given an AF $\langle \mathcal{A}, \mathcal{R} \rangle$, an argument A and a semantics σ, A's justification status can be determined by either considering all σ-extensions (sceptical and sceptical-existent) or at least one σ-extension of the AF (credulous). In this context, an argument can be IN (part of all/some σ-extensions); OUT (attacked by all/some σ-extensions), or UNDEC (otherwise).

Definition 7 (Argument justification status)**.** Let $AF = \langle \mathcal{A}, \mathcal{R} \rangle$ be an argumentation framework and σ some semantics in $\{\mathrm{GR}, \mathrm{CP}, \mathrm{PR}, \mathrm{ST}\}$. Let A be some argument in $\mathcal{A}$.

- The justification statuses for the IN label are:
  - ∗ A is σ-sceptical-IN iff A belongs to each σ-extension of AF;
  - ∗ A is σ-credulous-IN iff A belongs to some σ-extension of AF; and
  - ∗ A is σ-sceptical-existent-IN iff AF has a σ-extension and A belongs to each σ-extension of AF.
- The justification statuses for the OUT label are:
  - ∗ A is σ-sceptical-OUT iff for each σ-extension S of AF, A is attacked by some argument in S;
  - ∗ A is σ-credulous-OUT iff for some σ-extension S of AF, A is attacked by some argument in S; and
  - ∗ A is σ-sceptical-existent-OUT iff AF has a σ-extension and for each σ-extension S of AF, A is attacked by some argument in S.
- The justification statuses for the UNDEC label are:
  - ∗ A is σ-sceptical-UNDEC iff for each σ-extension S of AF, A is not in S and not attacked by any argument in S;
  - ∗ A is σ-credulous-UNDEC iff for some σ-extension S of AF, A is not in S and not attacked by any argument in S; and
  - ∗ A is σ-sceptical-existent-UNDEC iff AF has a σ-extension and for each σ-extension S of AF, A is not in S and not attacked by any argument in S.

The justification statuses that we consider in this paper are {GR, CP, PR, ST} × {sceptical, credulous, sceptical-existent} × {IN, OUT, UNDEC}. In the remainder of this paper, we refer to IN-, OUT or UNDEC-justification in case the semantics and acceptability strategy is obvious or irrelevant.

Example 6 (Argument justification status)**.** Consider the argumentation framework AF_1, as illustrated in Fig. 4. For $\sigma \in$ {GR, CP, PR, ST}, A is σ-sceptical-IN, while B is σ-sceptical-OUT. For $\sigma \in$ {CP, PR, ST}, the arguments C, D and E are σ-credulous-IN, σ-credulous-OUT as well as σ-credulous-UNDEC.

We formulate the identification of justification status as a decision problem j-JUSTIFICATION (where $j \in$ {GR, CP, PR, ST} × {sceptical, credulous, sceptical-existent} × {IN, OUT, UNDEC}):

j-JUSTIFICATION	
Given:	An argumentation framework $\langle \mathcal{A}, \mathcal{R} \rangle$, a justification status j and an argument $A \in \mathcal{A}$
Question:	Does A's justification status in $\langle \mathcal{A}, \mathcal{R} \rangle$ equal j?

The remainder of this section provides proofs for complexity results related to JUSTIFICATION problems. Whereas the complexity of IN-JUSTIFICATION has been studied before (as acceptance problems, see [8–11]), this is not the case for OUT- and UNDEC-JUSTIFICATION. This is unfortunate, as detecting OUT- and UNDEC-JUSTIFICATION has interesting applications as well: since arguments that are UNDEC are "more acceptable" than arguments that are OUT, these justification statuses provide a more fine-grained notion of acceptability than the distinction between accepted and not accepted. In addition, more insight in the complexity of detecting these justification statuses could for example be helpful in developing fast algorithms. Furthermore, we will use the complexities of OUT- and UNDEC-JUSTIFICATION later for finding the complexities of OUT- and UNDEC-STABILITY (Section 4) and OUT- and UNDEC-RELEVANCE (Section 5). We will therefore provide complexity proofs for these types of JUSTIFICATION under GR, CP, ST and PR semantics. Our strategy in proving these complexities is

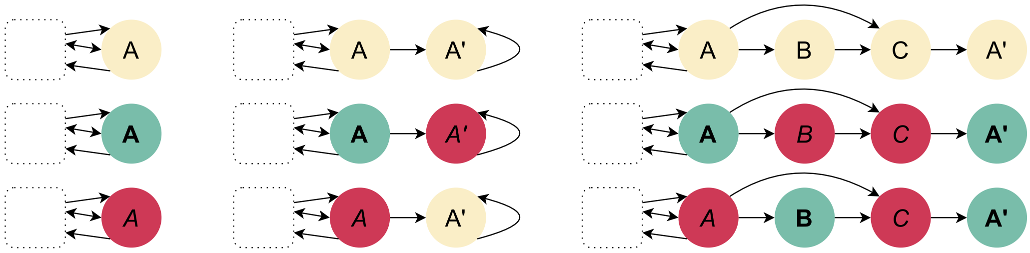

Fig. 6. Illustration of the argumentation frameworks that are used in transformations between justification problem instances used for proving Lemma 2. The figure illustrates three argumentation frameworks: $AF = \langle \mathcal{A}, \mathcal{R} \rangle$ (first column), $AF' = \langle \mathcal{A} \cup \{A'\}, \mathcal{R} \cup \{(A, A'), (A', A')\} \rangle$ (second column) and $AF'' = \langle \mathcal{A} \cup \{A', B, C\}, \mathcal{R} \cup \{(A, B), (A, C), (B, C), (C, A')\} \rangle$ (third column). The dotted, rounded rectangles represent all arguments in $\mathcal{A}$ except A. Each of the argumentation frameworks is displayed three times, corresponding to different extensions of AF: the first row, where A is coloured yellow, represents extensions that do not contain A or any attacker of A ("A is UNDEC"). The second row, where A is green and with boldface font, represents extensions containing A ("A is IN"). Finally, the third row, where A is red and with italic font, represents extensions containing some attacker of A ("A is OUT"). The colours and typesetting refer to the justification statuses of arguments, where green and boldface font stands for IN; yellow and regular font for UNDEC and red and italic font for OUT.

to relate them to the complexities of IN-JUSTIFICATION and using existing results from the complexity of acceptance problems. Short proofs will be given directly after the lemmas and propositions, whereas we provide sketches for longer proofs; the full proofs can be found in Appendix A. We start with a lemma for JUSTIFICATION that shows that IN-JUSTIFICATION and OUT-JUSTIFICATION are in the same complexity class:

Lemma 1 (Justification status IN and OUT)**.** *For any given $\sigma \in \{\text{GR}, \text{CP}, \text{PR}, \text{ST}\}$ and $c \in \{$sceptical, credulous$\}$, the complexity of σ-c-OUT-JUSTIFICATION equals the complexity of σ-c-IN-JUSTIFICATION.*

Proof. Let $AF = \langle \mathcal{A}, \mathcal{R} \rangle$ be an AF and $A \in \mathcal{A}$ an argument. Now construct $\mathcal{A}'$ as $\mathcal{A} \cup \{B\}$ (where $B \notin \mathcal{A}$) and $\mathcal{R}' = \mathcal{R} \cup \{(A, B)\}$; let $AF' = \langle \mathcal{A}', \mathcal{R}' \rangle$. Then A is σ-c-OUT in AF iff B is σ-c-IN in AF'; in addition, A is σ-c-IN in AF iff B is σ-c-OUT in AF'. □

Contrary to OUT-JUSTIFICATION, UNDEC-JUSTIFICATION is not necessarily in the same complexity class as IN-JUSTIFICATION. However, under GR, CP and PR semantics, there is another relation, as we will show in Lemma 3.[2] Before we can do so, we prove relations between credulous-IN- and sceptical-UNDEC-JUSTIFICATION, as well as between credulous-UNDEC- and sceptical-IN-JUSTIFICATION, in the following lemma. The transformations used in this lemma are illustrated in Fig. 6.

Lemma 2 (Complementary relation IN- and UNDEC-JUSTIFICATION)**.** *For any given $\sigma \in \{\text{GR}, \text{CP}, \text{PR}\}$, for each argumentation theory $AF = \langle \mathcal{A}, \mathcal{R} \rangle$ and argument $A \in \mathcal{A}$, each of the following holds:*

(1) *A is σ-credulous-IN in AF iff A' is **not** σ-sceptical-UNDEC in $\langle \mathcal{A} \cup \{A'\}, \mathcal{R} \cup \{(A, A'), (A', A')\} \rangle$;*
(2) *A is σ-sceptical-IN in AF iff A' is **not** σ-credulous-UNDEC in $\langle \mathcal{A} \cup \{A'\}, \mathcal{R} \cup \{(A, A'), (A', A')\} \rangle$;*
(3) *A is σ-credulous-UNDEC in AF iff A' is **not** σ-sceptical-IN in $\langle \mathcal{A} \cup \{A', B, C\}, \mathcal{R} \cup \{(A, B), (A, C), (B, C), (C, A')\} \rangle$; and*

[2]This relation does not exist for ST semantics as arguments cannot be ST-credulous-UNDEC. We will prove the complexity of ST-credulous-, ST-sceptical-existent- and ST-sceptical-JUSTIFICATION later in this section.

(4) *A is σ-sceptical-*UNDEC *in AF iff A' is **not** σ-credulous-*IN *in* $\langle\mathcal{A}\cup\{A', B, C\}, \mathcal{R}\cup\{(A, B), (A, C), (B, C), (C, A')\}\rangle$.

Proof sketch. We give the proof for the first item here. For full proofs of all items, we refer to Appendix A. Consider an arbitrary semantics $\sigma \in \{\text{GR}, \text{CP}, \text{PR}\}$, argumentation theory $AF = \langle\mathcal{A}, \mathcal{R}\rangle$ and argument $A \in \mathcal{A}$. Construct $AF' = \langle\mathcal{A} \cup \{A'\}, \mathcal{R} \cup \{(A, A'), (A', A')\}\rangle$; for an illustration, see the first and second columns of Fig. 6.

$\Rightarrow$ Suppose that A is σ-credulous-IN in AF: then there is some σ-extension S of AF containing A. Note that S also must be a σ-extension of AF': all arguments in $\mathcal{A}$ attacking attackers of A are still in $\mathcal{A} \cup \{A'\}$ and $S \cup \{A'\}$ is not a σ-extension as it is not conflict-free. Then there exists some σ-extension (i.e. S) of AF' in which A' is attacked by S, so A' is not σ-sceptical-UNDEC in AF'.

$\Leftarrow$ Suppose that A' is not σ-sceptical-UNDEC in AF'; then there exists some σ-extension S of AF' such that either $A' \in S$ or some argument attacking A' is in S. Given that A' is self-attacking, $A' \notin S$, so A' is attacked by some argument in S, which can only be A. Furthermore note that S is also a σ-extension of AF, since the arguments that are acceptable w.r.t. S in AF are exactly the same as the arguments acceptable w.r.t. S in AF'. To conclude, there exists some σ-extension (i.e. S) of AF in which $A \in S$, so A is σ-credulous-IN in AF. $\square$

Lemma 2 is used in the following lemma to show that credulous-IN-JUSTIFICATION and sceptical-UNDEC-JUSTIFICATION are in complementary complexity classes, as well as credulous-UNDEC-JUSTIFICATION and sceptical-IN-JUSTIFICATION, under GR, CP and PR semantics.

Lemma 3 (Complexities UNDEC-JUSTIFICATION)**.** *For any given $\sigma \in \{\text{GR}, \text{CP}, \text{PR}\}$:*

(1) *If the complexity of σ-credulous-*IN-JUSTIFICATION *is $\mathcal{C}$, then the complexity of σ-sceptical-*UNDEC-JUSTIFICATION *is co-$\mathcal{C}$; and*
(2) *If the complexity of σ-sceptical-*IN-JUSTIFICATION *is $\mathcal{C}$, then the complexity of σ-credulous-*UNDEC-JUSTIFICATION *is co-$\mathcal{C}$.*

Proof sketch. We give a proof sketch for the first item here and refer to the full proofs for both items to Appendix A. The first item can be proved by two reductions:

- Each instance $I_1 = (\langle\mathcal{A}, \mathcal{R}\rangle, A)$ of σ-credulous-IN-JUSTIFICATION can, in polynomial time, be converted to an instance $I_2 = (\langle\mathcal{A} \cup \{A'\}, \mathcal{R} \cup \{(A, A'), (A', A')\}\rangle, A')$ of σ-sceptical-UNDEC-JUSTIFICATION where, by Lemma 2 item 1, I_1 is a positive instance iff I_2 is a negative instance.
- Similarly, each instance $I_1 = (\langle\mathcal{A}, \mathcal{R}\rangle, A)$ of σ-sceptical-UNDEC-JUSTIFICATION can, in polynomial time, be converted to an instance $I_2 = (\langle\mathcal{A}\cup\{A', B, C\}, \mathcal{R}\cup\{(A, B), (A, C), (B, C), (C, A')\}\rangle, A')$ of σ-credulous-IN-JUSTIFICATION where, by Lemma 2 item 4, I_1 is a positive instance iff I_2 is a negative instance. $\square$

Using Lemma 3, we can now directly derive justification statuses for UNDEC-JUSTIFICATION under CP, GR and PR semantics in Propositions 1, 2 and 3.

Proposition 1. CP-*credulous*-UNDEC-JUSTIFICATION*,* GR-*credulous*-UNDEC-JUSTIFICATION *and* GR-*sceptical*-UNDEC-JUSTIFICATION *are P-complete.*

Proof. This follows directly from Lemma 3 in combination with P-completeness of CP-sceptical-IN-JUSTIFICATION, GR-sceptical-IN-JUSTIFICATION and GR-credulous-IN-JUSTIFICATION [10] and the fact that P = CoP. □

Proposition 2. CP-*sceptical*-UNDEC-JUSTIFICATION *and* PR-*sceptical*-UNDEC-JUSTIFICATION *are CoNP-complete.*

Proof. This follows directly from Lemma 3 and the fact that CP-credulous-IN-JUSTIFICATION and PR-credulous-IN-JUSTIFICATION are NP-complete [8,9]. □

Proposition 3. PR-*credulous*-UNDEC-JUSTIFICATION *is* Σ_2^p*-complete.*

Proof. This follows directly from Lemma 3 and the fact that PR-sceptical-IN-JUSTIFICATION is Π_2^p-complete [11]. □

For ST semantics, the situation is different, since for each ST extension S, each argument is either in S or attacked by some argument in S. In the remainder of this section, we give the complexity results of UNDEC-JUSTIFICATION for ST semantics.

Proposition 4. ST-*credulous*-UNDEC-JUSTIFICATION *and* ST-*sceptical-existent*-UNDEC-JUSTIFICATION *are trivial.*

Proof. In each ST extension S, each argument is either in S or attacked by some argument in S. Consequently, each instance of ST-credulous-UNDEC-JUSTIFICATION or ST-sceptical-existent-UNDEC-JUSTIFICATION is False. □

In addition, an argument can only be ST-sceptical-UNDEC in a given argumentation framework *AF* if *AF* does not have any ST extension.

Proposition 5. ST-*sceptical*-UNDEC-JUSTIFICATION *is CoNP-complete.*

Proof. For each AF $\langle \mathcal{A}, \mathcal{R} \rangle$ and argument $A \in \mathcal{A}$, A is ST-sceptical-UNDEC in $\langle \mathcal{A}, \mathcal{R} \rangle$ iff no stable extension exists for $\langle \mathcal{A}, \mathcal{R} \rangle$. The problem of deciding if a given AF has a stable extension is NP-complete [9], so the complementary problem of deciding if an AF has no stable extension is CoNP-complete. □

At this point, we have studied the complexity of the JUSTIFICATION problem for GR, CP, PR and ST semantics, for sceptical and credulous (and sceptical-existent of ST semantics) acceptance and labels IN, OUT and UNDEC. These results are summarised in Table 2. In the following sections, we will see that these definitions and complexity results can be used for defining and studying the complexity of the stability and relevance problems.

4. Stability

In this section, we will formally define stability and study its complexity. Stability can be seen as a dynamic variant of justification, defined on incomplete argumentation frameworks: whereas the notion of

Table 2

Overview of all complexity results related to justification. If a reference is specified, this complexity result is trivial from an earlier result in the literature. New results are printed bold; we refer to the corresponding proposition by "P" and the proposition number. Full proofs for each of the propositions are presented in the appendix. The complexities for IN- and OUT-JUSTIFICATION are the same; this follows from Lemma 1

Justification status j	Complexity j-JUSTIFICATION	Justification status j	Complexity j-JUSTIFICATION
ST-credulous-IN/OUT	NP-c [8,9]	GR-credulous-IN/OUT	P-c [10]
ST-credulous-UNDEC	**Trivial (no)** P4	GR-credulous-UNDEC	**P-c** P1
ST-sceptical-IN/OUT	CoNP-c [9]	GR-sceptical-IN/OUT	P-c [10]
ST-sceptical-UNDEC	**CoNP-c** P5	GR-sceptical-UNDEC	**P-c** P1
ST-sceptical-existent-IN/OUT	DP-c [12]		
ST-sceptical-existent-UNDEC	**Trivial (no)** P4		
CP-credulous-IN/OUT	NP-c [8,9]	PR-credulous-IN/OUT	NP-c [8,9]
CP-credulous-UNDEC	**P-c** P1	PR-credulous-UNDEC	$\mathbf{\Sigma_2^p}$**-c** P3
CP-sceptical-IN/OUT	P-c [10]	PR-sceptical-IN/OUT	Π_2^p-c [11]
CP-sceptical-UNDEC	**CoNP-c** P2	PR-sceptical-UNDEC	**CoNP-c** P2

justification only takes certain arguments and attacks into account, the notion of stability also considers arguments and attacks for which their presence is still uncertain. Whereas justification status is defined on arguments in an abstract argumentation framework, stability status is defined on certain arguments in an incomplete argumentation framework. Informally, a certain argument is *stable* if its justification is the same in all completions of the IAF. In the definition below, we define j-stability based on j-justification, where j can be any justification status considered in the previous section: $j \in \{\text{GR}, \text{CP}, \text{PR}, \text{ST}\} \times \{\text{sceptical}, \text{credulous}, \text{sceptical-existent}\} \times \{\text{IN}, \text{OUT}, \text{UNDEC}\}$.

Definition 8 (Stability on IAFs)**.** Given an IAF $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$, a certain argument $A \in \mathcal{A}$ and some justification status j, A is *stable-j* w.r.t. $\mathcal{I}$ iff A is j in each completion of $\mathcal{I}$.

Example 7 (Stability)**.** We reconsider the incomplete argumentation framework $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$ from Example 3. The arguments in $\mathcal{A}$ are B, C and E. Figure 7 illustrates the complete IN/OUT/UNDEC-labellings of each of the completions ($AF_1, \ldots, AF_8$) of $\mathcal{I}$. For each of these eight AFs, each complete extension is represented by colouring the argument nodes: nodes corresponding to arguments in the extension are coloured green and with boldface font; arguments attacked by an argument in the extension are coloured red and with italic font; all other arguments are yellow and with regular font. Note that for each completion of $\mathcal{I}$, there is at least one complete extension containing E. In other words: E is stable-CP-credulous-IN. Similarly, for each completion of $\mathcal{I}$, there is at least one preferred and stable extension containing E, so E is stable-PR-credulous-IN and stable-ST-credulous-IN as well. Under grounded semantics, E is not stable-IN, since there are completions (such as AF_1) for which E is not in the grounded extension.

For each $\sigma \in \{\text{GR}, \text{CP}, \text{PR}, \text{ST}\}$, there is no argument that is stable-σ-sceptical-IN, -OUT or -UNDEC. In practice, this means that a sceptical reasoner interested in one of the arguments in $\mathcal{A}$ would require more information.

Finally recall that stability is not defined for A and D, since they are in $\mathcal{A}^?$ rather than $\mathcal{A}$. So although the argument A is in each GR, CP, PR and ST extension in each completion in which the argument

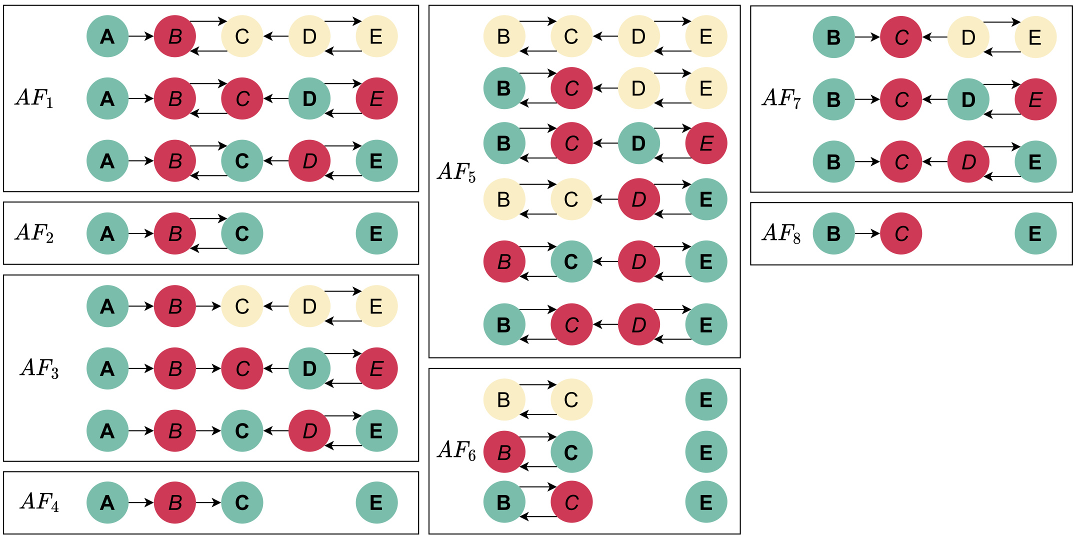

Fig. 7. Visualisation of the complete IN/OUT/UNDEC-labellings of each of the eight unique completions of $\mathcal{I}$, where the argumentation framework is repeated for each CP extension. Arguments that are in that extension are coloured green and with boldface font; argument attacked by some argument in that extension are red and with italic font; and all other arguments are yellow and with regular font.

exists, it is not stable-GR/CP/PR/ST-sceptical/credulous-IN because there are completions that do not contain A.[3]

Note that the notion of stability is strongly related to the notion of necessary acceptance, defined in Section 2. In fact, for any semantics σ, certain arguments are stable-σ-sceptical-IN if and only if they are necessarily sceptically accepted (i.e. in each extension of each completion); similarly, certain arguments are stable-σ-credulous-IN if and only if they are necessarily credulously accepted (i.e. in some extension of each completion). However, stability provides a more fine-grained notion of (non-)acceptance in IAFs than necessary acceptance. Using (OUT-)stability, it is, for example, also possible to express that, in any completion, some certain argument is attacked by an argument (not necessarily the same) in a σ-extension of that completion.

In the remainder of this section, we present our results on the complexity of the task of identifying a stability status. The following formulates this task as a decision problem.

j-STABILITY	
Given:	An incomplete argumentation framework $\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$, a justification status j and an argument $A \in \mathcal{A}$
Question:	Does A's stability status w.r.t. $\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$ equal stable-j?

[3]One may also be interested in an alternative notion of stability for uncertain arguments, which we call existent-stability here: an uncertain argument $A \in \mathcal{A}^?$ is j-existent-stable w.r.t. some IAF $\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$ if it has justification status j in all completions $\langle \mathcal{A}', \mathcal{R}' \rangle$ such that $A \in \mathcal{A}'$. Note that the set of completions $\langle \mathcal{A}', \mathcal{R}' \rangle$ of $\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$ such that $A \in \mathcal{A}'$ equals the set of completions of $\langle \mathcal{A} \cup \{A\}, \mathcal{A}^? \setminus \{A\}, \mathcal{R}, \mathcal{R}^? \rangle$. Therefore, $A \in \mathcal{A}^?$ is j-existent-stable w.r.t. some $\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$ iff A is j-stable w.r.t. $\langle \mathcal{A} \cup \{A\}, \mathcal{A}^? \setminus \{A\}, \mathcal{R}, \mathcal{R}^? \rangle$. Analogously, $B \in \mathcal{A}$ is j-stable w.r.t. some $\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$ iff B is j-existent-stable w.r.t. $\langle \mathcal{A} \setminus \{B\}, \mathcal{A}^? \cup \{B\}, \mathcal{R}, \mathcal{R}^? \rangle$. This implies that the problem of deciding j-existent-stability is in the same complexity class as the problem of deciding j-stability.

Next, we give complexity results for variants of the STABILITY problem. We start with relating IN-STABILITY to necessary (sceptical and credulous) acceptance, as defined in Definition 5 (which was adapted from [3]).

Lemma 4. *For any given* $\sigma \in \{\text{GR}, \text{CP}, \text{PR}, \text{ST}\}$ *and* $c \in \{\textit{sceptical}, \textit{credulous}\}$*, the complexity of* σ-c-IN-STABILITY *equals the complexity of necessary c acceptance w.r.t. semantics* σ*.*

Proof. This is trivial from Definition 8 of stability and Definition 5 of necessary acceptance. □

We continue with relating IN- and OUT-STABILITY. Similar to what we did for IN- and OUT-JUSTIFICATION in Lemma 1, we show that the complexity of IN-STABILITY is the same as the complexity of OUT-STABILITY.

Lemma 5. *For any given* $\sigma \in \{\text{GR}, \text{CP}, \text{PR}, \text{ST}\}$ *and* $c \in \{\textit{sceptical}, \textit{credulous}\}$*, the complexity of* σ-c-OUT-STABILITY *equals the complexity of* σ-c-IN-STABILITY.

Proof sketch. We prove this by a reduction from σ-c-OUT-STABILITY to σ-c-IN-STABILITY and by a reduction in the other direction. Below, we give proof sketch; the full proof can be found in Appendix B.

- For each instance $(\mathcal{I}, A)$ of the σ-c-OUT-STABILITY problem where $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$ and $A \in \mathcal{A}$, one can construct $\mathcal{I}^* = \langle \mathcal{A} \cup \{B\}, \mathcal{A}^?, \mathcal{R} \cup \{(A, B)\}, \mathcal{R}^? \rangle$, where $B \notin \mathcal{A} \cup \mathcal{A}^?$. Then $(\mathcal{I}, A)$ is a positive instance of σ-c-OUT-STABILITY iff $(\mathcal{I}^*, B)$ is a positive instance of σ-c-IN-STABILITY.
- Similarly, each instance $(\mathcal{I}, A)$ of the σ-c-IN-STABILITY problem, where $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$ and $A \in \mathcal{A}$, can be transformed into $(\mathcal{I}^*, B)$ where $\mathcal{I}^* = \langle \mathcal{A} \cup \{B\}, \mathcal{A}^?, \mathcal{R} \cup \{(A, B)\}, \mathcal{R}^? \rangle$ with $B \notin \mathcal{A} \cup \mathcal{A}^?$. The instance $(\mathcal{I}, A)$ is positive for σ-c-IN-STABILITY iff $(\mathcal{I}^*, B)$ is a positive instance for σ-c-OUT-STABILITY.

From these two reductions, it follows that σ-c-IN-STABILITY and σ-c-OUT-STABILITY have the same complexity. □

Using Lemma 5, we can show that for any given $\sigma \in \{\text{GR}, \text{CP}, \text{PR}, \text{ST}\}$ and $c \in \{\text{sceptical}, \text{credulous}\}$, we can derive the complexity of σ-c-OUT-STABILITY directly from the complexity of σ-c-IN-STABILITY. In the following lemma, we relate UNDEC-STABILITY for specific semantics with possible sceptical and credulous acceptance (Definition 6). Similar to Lemma 2, we prove this for GR, CP and PR semantics but not for ST semantics: the relation does not hold for ST semantics because arguments cannot be ST-credulous-UNDEC.

Lemma 6 (Complexities UNDEC-STABILITY)**.** *For any given* $\sigma \in \{\text{GR}, \text{CP}, \text{PR}\}$*:*

(1) *If possible credulous acceptance w.r.t.* σ *semantics is in the complexity class* $\mathcal{C}$*, then* σ*-sceptical-*UNDEC-STABILITY *is in the complexity class co-*$\mathcal{C}$*; and*
(2) *If possible sceptical acceptance w.r.t.* σ *semantics is in the complexity class* $\mathcal{C}$*, then* σ*-credulous-*UNDEC-STABILITY *is in the complexity class co-*$\mathcal{C}$*.*

Proof sketch. We only give a proof sketch for the first item here; full proofs of both items are in Appendix B. This proof consists of two reductions: we show that possible credulous acceptance w.r.t. σ semantics reduces to the complementary problem of σ-sceptical-UNDEC-STABILITY and the other way around. The proof strategy is similar to the proof of Lemma 2, illustrated in Fig. 6.

- Let $I_1 = (\mathcal{I}, A)$ where $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$ and $A \in \mathcal{A}$. Construct $I_2 = (\mathcal{I}^*, A')$ where $\mathcal{I}^* = \langle \mathcal{A} \cup \{A'\}, \mathcal{A}^?, \mathcal{R} \cup \{(A, A'), (A', A')\}, \mathcal{R}^? \rangle$ and $A' \notin \mathcal{A} \cup \mathcal{A}^?$. Then I_1 is a positive instance of possible credulous acceptance w.r.t. σ semantics iff I_2 is a negative instance of σ-sceptical-UNDEC-STABILITY.
- Let $I_1 = (\mathcal{I}, A)$ where $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$ and $A \in \mathcal{A}$. Now let $I_2 = (\mathcal{I}^*, A')$ where $\mathcal{I}^* = \langle \mathcal{A} \cup \{A', B, C\}, \mathcal{A}^?, \mathcal{R} \cup \{(A, B), (A, C), (B, C), (C, A')\}, \mathcal{R}^? \rangle$ and none of A', B and C is in $\mathcal{A} \cup \mathcal{A}^?$. I_1 is a positive instance of σ-sceptical-UNDEC-STABILITY iff I_2 is a negative instance of possible credulous acceptance w.r.t. σ semantics. □

The results of Lemma 6 can be used to derive the complexity classes of UNDEC-STABILITY based on existing results for possible acceptance from [3] for CP, GR and PR semantics.

Proposition 6. GR-*sceptical*-UNDEC-STABILITY, GR-*credulous*-UNDEC-STABILITY *and* CP-*credulous*-UNDEC-STABILITY *are CoNP-complete.*

Proof. This follows directly from Lemma 6 and the fact that possible sceptical acceptance under GR and CP semantics and possible credulous acceptance under GR semantics are NP-complete [3]. □

Proposition 7. CP-*sceptical*-UNDEC-STABILITY *and* PR-*sceptical*-UNDEC-STABILITY *are CoNP-complete.*

Proof. This follows directly from Lemma 6 and the fact that possible credulous acceptance under CP and PR semantics are NP-complete [3]. □

Proposition 8. PR-*credulous*-UNDEC-STABILITY *is Π_3^p-complete.*

Proof. This follows directly from Lemma 6 and the fact that possible sceptical acceptance under PR semantics is Σ_3^p-complete [3]. □

Finally, we turn to ST semantics. Our strategy for proving these complexities is similar to our approach for the ST-UNDEC-JUSTIFICATION complexity proofs in the previous section.

Proposition 9. ST-*sceptical*-UNDEC-STABILITY *is CoNP-complete.*

Proof. The problem is in CoNP, as a negative instance $(\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle, A)$ can be verified in polynomial time given a certificate (AF', S) such that AF' is a completion of $\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$, $A \in \mathcal{A}$ and S is a ST extension of AF'. If S is a ST extension then each argument in $\mathcal{A}$, including A, is either in S or attacked by S; therefore A cannot be stable-ST-sceptical-UNDEC w.r.t. $\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$.

For hardness, we can reduce from the CoNP-complete problem ST-sceptical-UNDEC-JUSTIFICATION: any instance $I_1 = (\langle \mathcal{A}, \mathcal{R} \rangle, A)$ of ST-sceptical-UNDEC-JUSTIFICATION can be solved by solving ST-sceptical-UNDEC-STABILITY for $I_2 = (\langle \mathcal{A}, \emptyset, \mathcal{R}, \emptyset \rangle, A)$. Given that $\langle \mathcal{A}, \mathcal{R} \rangle$ is the only completion of $\langle \mathcal{A}, \emptyset, \mathcal{R}, \emptyset \rangle$, I_1 is positive iff I_2 is positive. □

Proposition 10. ST-*credulous*-UNDEC-STABILITY *and* ST-*sceptical-existent*-UNDEC-STABILITY *are trivial.*

Table 3

Overview of all complexity results related to stability. If a reference is specified, this complexity result is trivial from an earlier result in the literature. New results are printed bold; we refer to the corresponding proposition by "P" and the proposition number. Full proofs for each of the propositions are presented in Appendix B. The complexities for IN- and OUT-STABILITY are the same; this follows from Lemma 5

Justification status j	Complexity j-STABILITY	Justification status j	Complexity j-STABILITY
ST-credulous-IN/OUT	Π_2^p-c [3]	GR-credulous-IN/OUT	**CoNP-c** P6
ST-credulous-UNDEC	**Trivial (no)** P10	GR-credulous-UNDEC	CoNP-c [3]
ST-sceptical-IN/OUT	CoNP-c [3]	GR-sceptical-IN/OUT	**CoNP-c** P6
ST-sceptical-UNDEC	**CoNP-c** P9	GR-sceptical-UNDEC	**CoNP-c** P6
ST-sceptical-existent-IN/OUT	Π_2^p-c [3]		
ST-sceptical-existent-UNDEC	**Trivial (no)** P10		
CP-credulous-IN/OUT	Π_2^p-c [3]	PR-credulous-IN/OUT	Π_2^p-c [3]
CP-credulous-UNDEC	**CoNP-c** P6	PR-credulous-UNDEC	$\boldsymbol{\Pi_3^p}$**-c** P8
CP-sceptical-IN/OUT	CoNP-c [3]	PR-sceptical-IN/OUT	Π_2^p-c [3]
CP-sceptical-UNDEC	**CoNP-c** P7	PR-sceptical-UNDEC	**CoNP-c** P7

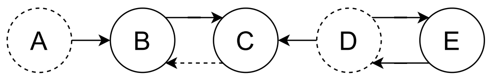

Fig. 8. Illustration of the IAF $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$ from Example 3.

Proof. For each argumentation framework $AF = \langle \mathcal{A}, \mathcal{R} \rangle$ such that a ST extension S exists, each argument in $\mathcal{A}$ is either in S or attacked by an argument in S. Therefore, A cannot be ST-credulous-UNDEC or ST-sceptical-existent-UNDEC in AF. This applies for each AF, including all completions of each possible IAF, so each instance of ST-credulous-UNDEC-STABILITY and ST-sceptical-existent-UNDEC-STABILITY must be negative. □

To conclude this section, we have studied the complexity of the STABILITY problem for GR, CP, PR and ST semantics, for sceptical and credulous (and sceptical-existent for ST semantics) acceptance and labels IN, OUT and UNDEC. For an overview of the results, we refer to Table 3. In the following section, we consider the problem of identifying relevance.

5. Relevance

For IAFs in which a given argument is not stable, a natural follow-up question would be: which uncertainties should be resolved in order to reach a point where the argument is stable? These uncertainties are *relevant* to investigate in the given IAF. In this section, we will define the problem of relevance and study its complexity. First, we give some intuition on the notion of relevance in the context of stability.

Example 8. We return to the IAF $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$ from Example 3, which is shown in Fig. 8 (which is a repetition of Fig. 2). Suppose that we want to know how we can make the certain argument C stable-GR-sceptical-IN. In order to do so, we should make sure that argument B is stable-GR-sceptical-OUT, which can only be the case if it is attacked by argument A. So in order to make sure that C is stable-GR-sceptical-IN, we need to make sure that argument A is present. In addition, argument C is attacked by the uncertain argument D. If argument D turns out to be present, then C cannot be stable-GR-sceptical-IN as there is no suitable argument to defend C from D under grounded semantics. Therefore it is relevant to make sure that argument D is absent. To conclude, the two relevant operations in this case are adding argument A and removing argument D. Note that there is still some uncertainty left in the resulting IAF: it is still unknown if the attack (C, B) should be present or absent. However, adding or removing this attack does not influence the GR-sceptical-IN-stability status of C. Therefore, these operations are not relevant.

The example showed that being certain about the existence of the attack (C, B) does not contribute anything to the stability status of C in a situation where we already know that A is present and D is absent: both with and without (C, B), C is stable-GR-sceptical-IN. In general, adding or removing an argument or attack is only relevant if there is a situation in which this argument or attack is really necessary to obtain stability. In order to define which uncertainties are relevant to be resolved for obtaining some stability status, we therefore need some notion of "partial" completions, in which only those uncertainties are resolved that are required for stability. A partial completion of an IAF $\mathcal{I}$ is an IAF $\mathcal{I}'$ such that a (possibly empty) part of the uncertain elements of $\mathcal{I}$ is resolved in $\mathcal{I}'$, while another (possibly empty) part of the uncertain elements is still uncertain.[4] Next, we formally define such partial completions.

Definition 9 (Partial completion)**.** Given an IAF $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$, a partial completion is an IAF $\mathcal{I}' = \langle \mathcal{A}', \mathcal{A}^{?\prime}, \mathcal{R}', \mathcal{R}^{?\prime} \rangle$, where:

- $\mathcal{A} \subseteq \mathcal{A}' \subseteq \mathcal{A} \cup \mathcal{A}^?$;
- $\mathcal{R}|_{(\mathcal{A}' \cup \mathcal{A}^{?\prime})} \subseteq \mathcal{R}' \subseteq (\mathcal{R} \cup \mathcal{R}^?)|_{(\mathcal{A}' \cup \mathcal{A}^{?\prime})}$;
- $\mathcal{A}^{?\prime} \subseteq \mathcal{A}^?$;
- $\mathcal{R}^{?\prime} \subseteq \mathcal{R}^?$.

Note that, since $\mathcal{I}'$ is an IAF, it must still hold that $\mathcal{A}' \cap \mathcal{A}^{?\prime} = \emptyset$; $\mathcal{R}' \cap \mathcal{R}^{?\prime} = \emptyset$; $\mathcal{R}' \subseteq (\mathcal{A}' \cup \mathcal{A}^{?\prime}) \times (\mathcal{A}' \cup \mathcal{A}^{?\prime})$ and $\mathcal{R}^{?\prime} \subseteq (\mathcal{A}' \cup \mathcal{A}^{?\prime}) \times (\mathcal{A}' \cup \mathcal{A}^{?\prime})$. We denote all possible partial completions for $\mathcal{I}$ by $\mathtt{part}(\mathcal{I})$.

In order to be able to apply the semantics of [10], which are defined on AFs, to IAFs, we define the certain projection of an IAF. This is an AF consisting of only the IAF's certain arguments and the attacks between them.

Definition 10 (Certain projection)**.** Given an IAF $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$, the certain projection $\mathtt{cert}(\mathcal{I})$ is the argumentation framework $AF = \langle \mathcal{A}, \mathcal{R}|_{\mathcal{A}} \rangle$.

Example 9 (Partial completions and certain projections)**.** Returning to the IAF $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$, given in Example 3 and illustrated in Fig. 8, the following IAFs are some (but not all) examples of partial completions in $\mathtt{part}(\mathcal{I})$:

- $\mathcal{I}_1 = \langle \mathcal{A} \cup \mathcal{A}^?, \emptyset, \mathcal{R} \cup \mathcal{R}^?, \emptyset \rangle$: all uncertain arguments and attacks have become certain.
- $\mathcal{I}_2 = \langle \mathcal{A}, \emptyset, \{(B, C)\}, \emptyset \rangle$: all uncertain arguments and attacks are removed, as well as all attacks that are not in the restriction of $\mathcal{R}$ to $\mathcal{A}$.
- $\mathcal{I}_3 = \langle \mathcal{A} \cup \{A\}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$: the argument A is moved from the uncertain to the certain part. The argument D and the attack (C, B) are still uncertain.

The partial completions $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$ are illustrated on the left side of Fig. 9. The certain projection of $\mathcal{I}_1$ is $\langle \mathcal{A} \cup \mathcal{A}^?, \mathcal{R} \cup \mathcal{R}^? \rangle$. For $\mathcal{I}_2$, the certain projection is $\langle \mathcal{A}, \{(B, C)\} \rangle$. Finally, $\mathtt{cert}(\mathcal{I}_3) = \langle \{A, B, C, E\}, \{(A, B), (B, C))\} \rangle$. These AFs are illustrated on the right side of Fig. 9.

[4]Partial completions in this paper are a corrected version of specifications in [18].

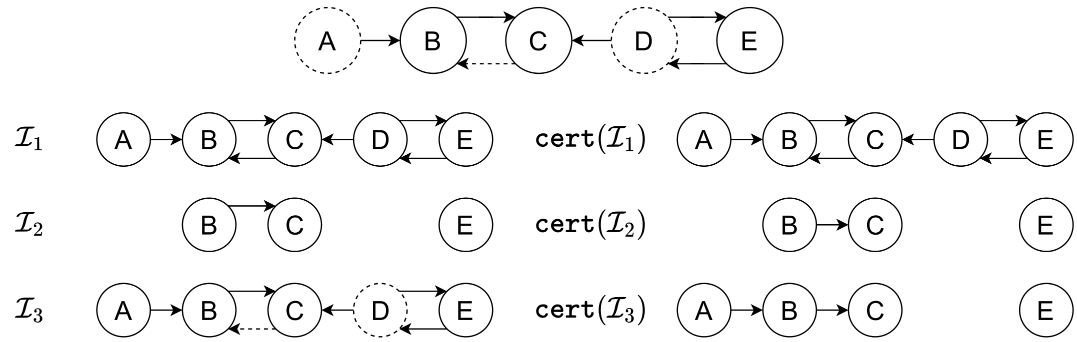

Fig. 9. Three partial completions of our incomplete argumentation framework and their certain projections.

Before proceeding to a formal definition of relevance that matches the intuitions in the example above, we define the notion of minimal stable partial completions. In this definition, j refers to the justification status: $j \in \{\text{GR}, \text{CP}, \text{PR}, \text{ST}\} \times \{\text{sceptical}, \text{credulous}, \text{sceptical-existent}\} \times \{\text{IN}, \text{OUT}, \text{UNDEC}\}$.

Definition 11 (Minimal stable-j partial completion)**.** Given an IAF $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$, a certain argument $A \in \mathcal{A}$ and a justification status j, a *minimal stable-j partial completion* for A w.r.t. $\mathcal{I}$ is a partial completion $\mathcal{I}'$ in $\mathtt{part}(\mathcal{I})$ such that A is stable-j in $\mathcal{I}'$ and there is no partial completion $\mathcal{I}''$ in $\mathtt{part}(\mathcal{I})$ such that A is stable-j in $\mathcal{I}''$, $\mathcal{I}'' \neq \mathcal{I}'$ and $\mathcal{I}' \in \mathtt{part}(\mathcal{I}'')$.

Intuitively, the minimal stable-j partial completion for A is a partial completion in which A is stable-j, while A would not be stable-j in any partial completion with more uncertain elements.

Example 10 (Minimal stable-j partial completion)**.** Recall the IAF $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$ from Example 3. Suppose that we are interested to know if argument C is stable-GR-sceptical-IN. $\mathcal{I}$ has one minimal stable-GR-sceptical-IN partial completion, which is $\mathcal{I}_4 = \langle \{A, B, C, E\}, \emptyset, \{(A, B), (B, C)\}, \mathcal{R}^? \rangle$. Given that $\mathcal{R}^?$ contains the attack (C, B) as its only uncertain element, $\mathcal{I}_4$ has three partial completions, resulting in two unique certain projections. These are $\langle \{A, B, C, E\}, \{(A, B), (B, C), (C, B)\} \rangle$ (depicted as AF_2 in Fig. 7) and $\langle \{A, B, C, E\}, \{(A, B), (B, C)\} \rangle$ (depicted as AF_4 in the same figure). Since AF_2 and AF_4 each have a grounded extension – in both cases $\{A, C, E\}$ – that contains C, C is stable-GR-sceptical-IN in $\mathcal{I}_4$. In addition, note that $\mathcal{I}_4$ is minimal in that C would not be stable-GR-sceptical-IN in partial completions with more uncertain elements:

- Suppose that the presence of argument A is unknown as in $\mathcal{I}_5 = \langle \{B, C, E\}, \{A\}, \{(A, B), (B, C)\}, \{(C, B)\} \rangle$. Then AF_6 and AF_8 would be certain projections of partial completions of $\mathcal{I}_5$. Since C is not in the grounded extension of each of these AFs, C is not stable-GR-sceptical-IN w.r.t. $\mathcal{I}_5$.
- Alternatively, suppose that the absence of argument D is yet unknown, as in $\mathcal{I}_3$, defined earlier as $\langle \{A, B, C, E\}, \{D\}, \mathcal{R}, \{(C, B)\} \rangle$. Then AF_1 and AF_3 would also be certain projections of partial completions of $\mathcal{I}_3$, where these AFs would not have C in their grounded extensions. Therefore, C is not stable-GR-sceptical-IN w.r.t. $\mathcal{I}_3$.

This shows that $\mathcal{I}_4$ is a minimal stable-GR-sceptical-IN partial completion for C w.r.t. $\mathcal{I}$. Note that, although in this case there is a *single* minimal stable-GR-sceptical-IN partial completion for C w.r.t. $\mathcal{I}$, in general there can be multiple minimal stable partial completions.

For example, there are three minimal stable-CP-credulous-UNDEC partial completions for C w.r.t. $\mathcal{I}$:

(1) $\langle \mathcal{A} \cup \{A, D\}, \emptyset, \mathcal{R}, \{(C, B)\} \rangle$ (having AF_1 and AF_3 as certain projections of partial completions);

Fig. 10. Partial completion $\mathcal{I}_4$, which is the only minimal stable-GR-sceptical-IN partial completion for C w.r.t. $\mathcal{I}$ where $\mathcal{I} = \langle\{B, C, E\}, \{A, E\}, \mathcal{R}, \mathcal{R}^?\rangle$ and $\mathcal{I}_4 = \langle\{A, B, C, E\}, \emptyset, \{(A, B), (B, C)\}, \mathcal{R}^?\rangle$.

(2) $\langle\mathcal{A} \cup \{D\}, \{A\}, \mathcal{R} \cup \{(C, B)\}, \emptyset\rangle$ (for AF_1 and AF_5); and
(3) $\langle\mathcal{A}, \{D\}, \mathcal{R} \setminus \{(A, B)\} \cup \{(C, B)\}, \emptyset\rangle$ (for AF_5 and AF_6).

Using the notion of minimal stable-j partial completions, we can now define j-RELEVANCE.

Definition 12 (j-RELEVANCE)**.** Given an IAF $\mathcal{I} = \langle\mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^?\rangle$, an argument $A \in \mathcal{A}$, an uncertain argument or attack $U \in \mathcal{A}^? \cup \mathcal{R}^?$ and a justification status j,

- *Addition of* U is j*-relevant* for A w.r.t. $\mathcal{I}$ iff there is a minimal stable-j partial completion $\mathcal{I}' = \langle\mathcal{A}', \mathcal{A}^{?\prime}, \mathcal{R}', \mathcal{R}^{?\prime}\rangle$ for A w.r.t. $\mathcal{I}$ such that $U \in \mathcal{A}' \cup \mathcal{R}'$; and
- *Removal of* U is j*-relevant* for A w.r.t. $\mathcal{I}$ iff there is a minimal stable-j partial completion $\mathcal{I}' = \langle\mathcal{A}', \mathcal{A}^{?\prime}, \mathcal{R}', \mathcal{R}^{?\prime}\rangle$ for A w.r.t. $\mathcal{I}$ such that $U \notin \mathcal{A}' \cup \mathcal{A}^{?\prime} \cup \mathcal{R}' \cup \mathcal{R}^{?\prime}$.

In other words, addition of an uncertain element U is j-relevant if a minimal stable-j partial completion can be reached by moving U from the uncertain to the certain part of the IAF $\mathcal{I}$; and removal of U is j-relevant if completely removing U from $\mathcal{I}$, possibly in combination with other actions, leads to a minimal stable-j partial completion.

Example 11 (j-RELEVANCE)**.** To illustrate j-relevance, we build on the minimal stable-j partial completions from Example 10. Recall that $\mathcal{I}_4$, illustrated in Fig. 10, is the only minimal stable-GR-sceptical-IN partial completion for C w.r.t. $\mathcal{I}$ where $\mathcal{I} = \langle\{B, C, E\}, \{A, D\}, \mathcal{R}, \mathcal{R}^?\rangle$ and $\mathcal{I}_4 = \langle\{A, B, C, E\}, \emptyset, \{(A, B), (B, C)\}, \mathcal{R}^?\rangle$. Given that A was an uncertain argument in $\mathcal{I}$ and is a certain argument in (the minimal stable-GR-sceptical-IN partial completion) $\mathcal{I}_4$, addition of A is GR-sceptical-IN-relevant for C w.r.t. $\mathcal{I}$. Furthermore, as D was an uncertain argument in $\mathcal{I}$ and is no longer present in $\mathcal{I}_4$, the removal of D is GR-sceptical-IN-relevant for C w.r.t. $\mathcal{I}$.

Considering the justification status CP-credulous-UNDEC, there are three minimal stable-CP-credulous-UNDEC partial completions for C (see Example 10). The CP-credulous-UNDEC-relevant operations are:

- Addition of A;
- Addition of D;
- Addition of (C, B); and
- Removal of A.

Note that this example shows the possibility that both the addition and removal of some uncertain argument or attack are relevant. This example also demonstrates that performing a relevant action does not necessarily lead to a stable situation, but may be just the first step in becoming stable. For instance, the addition of A to $\mathcal{I}$ does not yet result in an IAF that is stable-CP-credulous-UNDEC as this IAF still has at least one completion (to be precise, the completions AF_2 and AF_4 in Fig. 7) in which C is not CP-credulous-UNDEC. In order to become stable, an additional relevant action (in this case, the addition of D) is required.

Like for justification and stability status, we formulate the identification of j-RELEVANCE as a decision problem:

j-RELEVANCE of action **a**	
Given:	An incomplete argumentation framework $\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$, a justification status j, an action $\mathbf{a} \in \{addition, removal\}$, an argument $A \in \mathcal{A}$ and an uncertain argument or attack $U \in \mathcal{A}^? \cup \mathcal{R}^?$.
Question:	Is **a** of U j-relevant for A w.r.t. $\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$?

In the remainder of this section, we study the complexity of relevance for complete, grounded, preferred and stable semantics. This work builds on initial research in [18], where we proved the complexity of IN-RELEVANCE under grounded semantics. In order to prove complexity of j-RELEVANCE for each justification status j, we first give proofs for the upper bounds, that is: the membership in a given complexity class, in Section 5.1. Subsequently, we provide lower bounds, that is: hardness results, for RELEVANCE under grounded, complete and preferred semantics in Section 5.2. Finally, we prove the remaining hardness results for stable semantics in Section 5.3.

5.1. Upper bounds

First, we will prove a general upper bound on j-RELEVANCE. In order to do so, we first prove Lemma 7. This lemma shows that the relevance of adding an uncertain argument can be validated by checking the justification status of the certain projections of two particular future partial completions.

Lemma 7. *Given an IAF $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$, a certain argument $A \in \mathcal{A}$ and a justification status j:*

(1) *For each $U \in \mathcal{A}^?$, addition of U is j-relevant for A w.r.t. $\mathcal{I}$ iff there exists some $\mathcal{I}' = \langle \mathcal{A}', \{U\}, \mathcal{R}', \emptyset \rangle \in \mathtt{part}(\mathcal{I})$ such that A is **not** j in the certain projection of $\mathcal{I}'$, while A is j in the certain projection of $\langle \mathcal{A}' \cup \{U\}, \emptyset, \mathcal{R}', \emptyset \rangle$.*
(2) *For each $U \in \mathcal{R}^?$, addition of U is j-relevant for A w.r.t. $\mathcal{I}$ iff there exists some $\mathcal{I}' = \langle \mathcal{A}', \emptyset, \mathcal{R}', \{U\} \rangle \in \mathtt{part}(\mathcal{I})$ such that A is **not** j in the certain projection of $\mathcal{I}'$, while A is j in the certain projection of $\langle \mathcal{A}', \emptyset, \mathcal{R}' \cup \{U\}, \emptyset \rangle$.*
(3) *For each $U \in \mathcal{A}^?$, removal of U is j-relevant for A w.r.t. $\mathcal{I}$ iff there exists some $\mathcal{I}' = \langle \mathcal{A}', \{U\}, \mathcal{R}', \emptyset \rangle \in \mathtt{part}(\mathcal{I})$ such that A is j in the certain projection of $\mathcal{I}'$, while A is **not** j in the certain projection of $\langle \mathcal{A}' \cup \{U\}, \emptyset, \mathcal{R}', \emptyset \rangle$.*
(4) *For each $U \in \mathcal{R}^?$, removal of U is j-relevant for A w.r.t. $\mathcal{I}$ iff there exists some $\mathcal{I}' = \langle \mathcal{A}', \{U\}, \mathcal{R}', \emptyset \rangle \in \mathtt{part}(\mathcal{I})$ such that A is j in the certain projection of $\mathcal{I}'$, while A is **not** j in the certain projection of $\langle \mathcal{A}', \emptyset, \mathcal{R}' \cup \{U\}, \emptyset \rangle$.*

Proof sketch. Let $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$ be an incomplete argumentation framework, $A \in \mathcal{A}$ a certain argument and j a justification status. We prove both directions of the first item here. For proofs for the other items, we refer to Appendix C.

$\Rightarrow$ See Fig. 11 for an illustration of the steps in this proof.

(1) Suppose that addition of $U \in A^?$ is j-relevant for A w.r.t. $\mathcal{I}$.
(2) Then by Definition 12 there is a minimal stable-j partial completion $\mathcal{I}^* = \langle \mathcal{A}^*, \mathcal{A}^{?*}, \mathcal{R}^*, \mathcal{R}^{?*} \rangle$ for A w.r.t. $\mathcal{I}$ such that $U \in \mathcal{A}^*$.

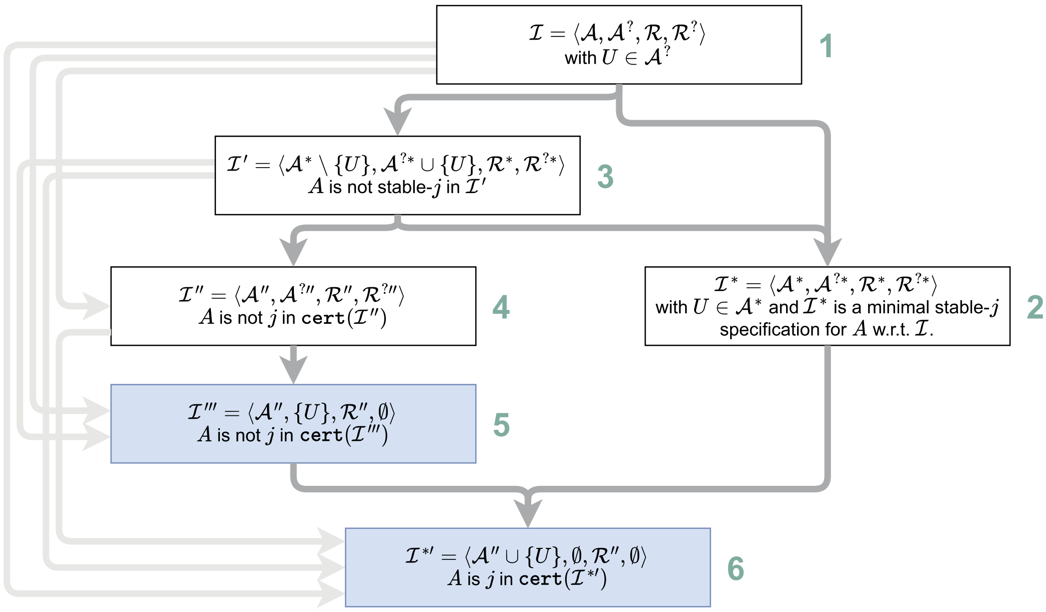

Fig. 11. Illustration of the steps in the proof of Lemma 7 item 1 (from left to right). The IAFs used in the proof are depicted as rectangles; grey arrows between these rectangles represent partial completions – note that not all partial completions of all IAFs are shown in the figure, but only those that we refer to in the proof. Rectangles corresponding to $\mathcal{I}'''$ and $\mathcal{I}^{*\prime}$ are coloured blue, as these are the partial completions of $\mathcal{I}$ for which Lemma 7 shows some properties.

(3) Now construct the IAF $\mathcal{I}'$ from $\mathcal{I}^*$ by moving U from the certain to the uncertain part: $\mathcal{I}' = \langle \mathcal{A}^* \setminus \{U\}, \mathcal{A}^{?*} \cup \{U\}, \mathcal{R}^*, \mathcal{R}^{?*} \rangle$.
(4) Given that $\mathcal{I}^*$ was minimal and $\mathcal{I}^* \in \mathtt{part}(\mathcal{I}')$, A cannot be stable-j w.r.t. $\mathcal{I}'$. So there must be some $\mathcal{I}'' = \langle \mathcal{A}'', \mathcal{A}^{?''}, \mathcal{R}'', \mathcal{R}^{?''} \rangle$ in $\mathtt{part}(\mathcal{I}')$ such that A's justification status in the certain projection of $\mathcal{I}''$ is not j – note that this means that U is not in $\mathcal{A}''$ (since A was stable-j in $\mathcal{I}^*$).
(5) Then A's justification status in the certain projection of $\mathcal{I}''' = \langle \mathcal{A}'', \{U\}, \mathcal{R}'', \emptyset \rangle$ is not j (because this is the same as the certain projection of $\mathcal{I}''$, i.e. $\langle \mathcal{A}'', \mathcal{R}''|_{\mathcal{A}''} \rangle$).
(6) Next, construct $\mathcal{I}^{*\prime} = \langle \mathcal{A}'' \cup \{U\}, \emptyset, \mathcal{R}'', \emptyset \rangle$ from $\mathcal{I}'''$ by moving U from the uncertain part to the certain part. Since $\mathcal{I}^{*\prime}$ is in $\mathtt{part}(\mathcal{I}^*)$ and A is stable-j in $\mathcal{I}^*$, A must be j in the certain projection of $\mathcal{I}^{*\prime}$.

$\Leftarrow$ Suppose that there exists some $\mathcal{I}' = \langle \mathcal{A}', \{U\}, \mathcal{R}', \emptyset \rangle \in \mathtt{part}(\mathcal{I})$ such that A is not j in $\mathtt{cert}(\mathcal{I}')$ and A is j in $\mathtt{cert}(\langle \mathcal{A}' \cup \{U\}, \emptyset, \mathcal{R}', \emptyset \rangle)$. Given that $\langle \mathcal{A}' \cup \{U\}, \emptyset, \mathcal{R}', \emptyset \rangle$ has only one completion (i.e., its certain projection), A must be stable-j w.r.t. $\langle \mathcal{A}' \cup \{U\}, \emptyset, \mathcal{R}', \emptyset \rangle$. Consequently, there must be some minimal stable-j partial completion $\mathcal{I}'' = \langle \mathcal{A}'', \mathcal{A}^{?''}, \mathcal{R}'', \mathcal{R}^{?''} \rangle$ for A w.r.t. $\mathcal{I}$ such that $\langle \mathcal{A}' \cup \{U\}, \emptyset, \mathcal{R}', \emptyset \rangle \in \mathtt{part}(\mathcal{I}'')$. Note that $U \in \mathcal{A}''$: otherwise $\langle \mathcal{A}', \emptyset, \mathcal{R}', \emptyset \rangle$ would also be in $\mathtt{part}(\mathcal{I}'')$, which contradicts the assumption that A is not j in $\mathtt{cert}(\langle \mathcal{A}', \{U\}, \mathcal{R}', \emptyset \rangle)$. To conclude, addition of U is j-relevant for A w.r.t. $\mathcal{I}$. □

In the following proposition, we use the results from Lemma 7 to prove a general upper bound on the complexity of j-RELEVANCE.

Proposition 11 (Upper bound j-RELEVANCE)**.** *Given an IAF $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$, a certain argument $A \in \mathcal{A}$, an uncertain argument or attack $U \in \mathcal{A}^? \cup \mathcal{R}^?$ and a justification status j, if the problem of deciding j's justification status in a given completion of $\mathcal{I}$ is in the complexity class $\mathcal{C}$, then the problem of deciding if addition or removal of U is j-relevant for A w.r.t. $\mathcal{I}$ is in the complexity class $NP^{\mathcal{C}}$.*

Proof. In order to validate that a given $U \in \mathcal{A}^? \cup \mathcal{R}^?$ is j-relevant for a given $A \in \mathcal{A}$, a suitable polynomial-sized certificate would be some $\mathcal{I}' = \langle \mathcal{A}', \mathcal{A}^{?\prime}, \mathcal{R}', \mathcal{R}^{?\prime} \rangle$ as specified in Lemma 7 (so $\mathcal{A}^{?\prime} \cup \mathcal{R}^{?\prime} = \{U\}$). The following procedure can be used to validate that U is j-relevant for A w.r.t. $\mathcal{I}$:

(1) Check in polynomial time if $\mathcal{I}' \in \mathtt{part}(\mathcal{I})$ and store the result in r_1;
(2) Call the $\mathcal{C}$ oracle for finding justification status j to check if A is j w.r.t. $\langle \mathcal{A}', \mathcal{R}'|_{\mathcal{A}'} \rangle$ and store the result in r_2;
(3) Let $AF' = \langle \mathcal{A}' \cup \{U\}, \mathcal{R}'|_{\mathcal{A}' \cup \{U\}} \rangle$ if $U \in A^?$ and $AF' = \langle \mathcal{A}', (\mathcal{R}' \cup \{U\})|_{\mathcal{A}'} \rangle$ otherwise. Then call the $\mathcal{C}$ oracle to check if A is j w.r.t. AF' and store the result in r_3.

Then by Lemma 7, addition of U is j-relevant for A w.r.t. $\mathcal{I}$ iff $r_1 \wedge \neg r_2 \wedge r_3$. Removal of U is j-relevant for A w.r.t. $\mathcal{I}$ iff $r_1 \wedge r_2 \wedge \neg r_3$. Checking that $r_1 \wedge \neg r_2 \wedge r_3$ (for addition) or $\mathcal{I}$ iff $r_1 \wedge r_2 \wedge \neg r_3$ (for removal) can be done in polynomial time. To conclude, the problem of deciding if addition or removal of U is j-relevant for A w.r.t. $\mathcal{I}$ is in $NP^{\mathcal{C}}$. $\square$

Proposition 11 gives a general upper bound that can be exploited to obtain an upper bound for j-RELEVANCE for all justification statuses for which we know the complexity of j-JUSTIFICATION. For example, given that ST-credulous-IN-JUSTIFICATION is in NP, both the addition and the removal variants of ST-credulous-IN-RELEVANCE must be in NP^{NP}, so in Σ_2^p. For justification statuses for which the JUSTIFICATION problem is in P, like GR-credulous-IN, the RELEVANCE problem is in NP. If the JUSTIFICATION problem is on the second level of the polynomial hierarchy, like PR-sceptical-OUT-JUSTIFICATION, then the RELEVANCE variant is in Σ_3^p. Having proved the upper bounds for all variants of the RELEVANCE problem, we turn to the lower bounds in the following two sections.

5.2. Lower bounds for grounded, complete and preferred semantics

In order to prove lower bounds for RELEVANCE under GR, CP and PR semantics, we give reductions from the complementary problem of STABILITY, to which we will refer as the INSTABILITY problem.[5] More formally, for every IAF $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$, justification status j and certain argument $A \in \mathcal{A}$, the instance $(\mathcal{I}, A)$ is negative for j-STABILITY iff it is positive for j-INSTABILITY. In the following lemma, we provide some relations between INSTABILITY and RELEVANCE that will turn out to be useful for reductions from INSTABILITY to RELEVANCE for specific justification statuses.

Lemma 8 (Reduction INSTABILITY to RELEVANCE)**.** *Given an incomplete argumentation framework $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$, a certain argument $A \in \mathcal{A}$, semantics $\sigma \in \{\text{GR}, \text{CP}, \text{PR}\}$ and $c \in \{sceptical, credulous\}$:*

(1) *Construct $\mathcal{I}'$ and $\mathcal{I}''$ as follows (see Fig. 12), where A', A'', U and U' are not in $\mathcal{A} \cup \mathcal{A}^?$:*
   - *$\mathcal{A}' = \mathcal{A} \cup \{A', A''\}$;*
   - *$\mathcal{R}' = \mathcal{R} \cup \{(A, A'), (A', A''), (U, A')\}$;*

[5] ST semantics will be covered in Section 5.3.

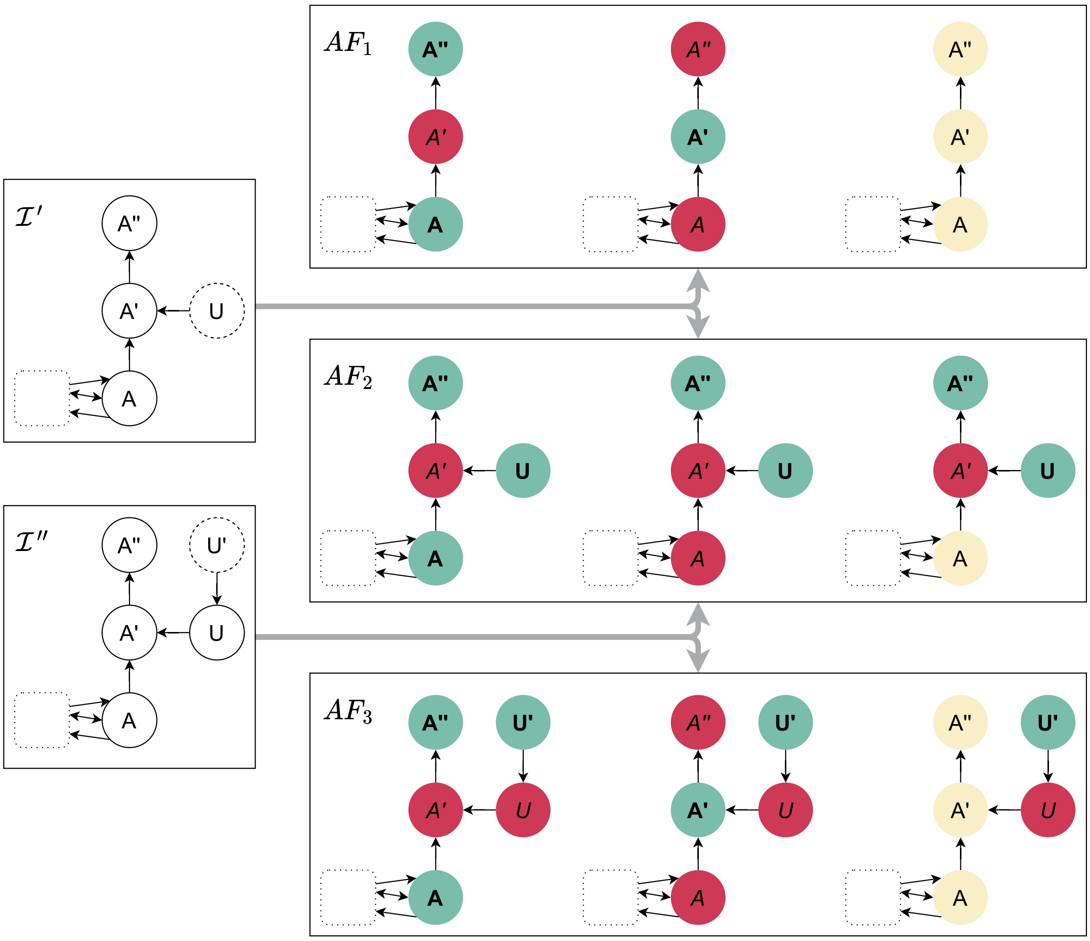

Fig. 12. Illustration of the IAFs that are used to show Lemma 8 item 1. The IAFs given on the left are $\mathcal{I}'$ (upper) and $\mathcal{I}''$ (lower). The rounded rectangle with dotted borders represents the original IAF $\mathcal{I}$ (without A and in- and outgoing attacks). The grey arrows point to certain projections AF_1, AF_2 and AF_3 of partial completions. For each of these AFs, the possible justification statuses are colour-coded: green arguments with boldface font are IN, yellow arguments are UNDEC and red arguments with italic font are OUT. Note that, for a given justification status of A, there is only one possible justification status for each of the additional arguments in $\{A', A'', U, U'\}$.

- $\mathcal{I}' = \langle \mathcal{A}', \mathcal{A}^? \cup \{U\}, \mathcal{R}', \mathcal{R}^? \rangle$; *and*
- $\mathcal{I}'' = \langle \mathcal{A}' \cup \{U\}, \mathcal{A}^? \cup \{U'\}, \mathcal{R}' \cup \{(U', U)\}, \mathcal{R}^? \rangle$.

The following three items are equivalent:

(a) *A is **not** stable-σ-c-*IN *w.r.t. $\mathcal{I}$; and*
(b) *addition of U is σ-c-*IN*-relevant for A'' w.r.t. $\mathcal{I}'$; and*
(c) *removal of U' is σ-c-*IN*-relevant for A'' w.r.t. $\mathcal{I}''$.*

(2) *Let $\mathcal{I}' = \langle \mathcal{A}, \mathcal{A}^? \cup \{U\}, \mathcal{R} \cup \{(U, A)\}, \mathcal{R}^? \rangle$ and $\mathcal{I}'' = \langle \mathcal{A} \cup \{U\}, \mathcal{A}^? \cup \{U'\}, \mathcal{R} \cup \{(U, A), (U', U)\}, \mathcal{R}^? \rangle$ where U and U' are not in $\mathcal{A} \cup \mathcal{A}^?$. The following three items are equivalent:*

(a) *A is **not** stable-σ-c-*OUT *w.r.t. $\mathcal{I}$; and*
(b) *addition of U is σ-c-*OUT*-relevant for A w.r.t. $\mathcal{I}'$; and*

(c) *removal of* U' *is* σ-c-OUT-*relevant for* A *w.r.t.* $\mathcal{I}''$.

(3) *Construct* $\mathcal{I}'$ *and* $\mathcal{I}''$ *as follows, where* A', A'', U *and* U' *are not in* $\mathcal{A} \cup \mathcal{A}^?$:

- $\mathcal{A}' = \mathcal{A} \cup \{A', A''\}$;
- $\mathcal{R}' = \mathcal{R} \cup \{(A, A'), (A', A''), (U, A'), (U, A''), (U, U)\}$;
- $\mathcal{I}' = \langle \mathcal{A}', \mathcal{A}^? \cup \{U\}, \mathcal{R}', \mathcal{R}^? \rangle$; *and*
- $\mathcal{I}'' = \langle \mathcal{A}' \cup \{U\}, \mathcal{A}^? \cup \{U'\}, \mathcal{R}' \cup \{(U', U)\}, \mathcal{R}^? \rangle$.

The following three items are equivalent:

(a) A *is* ***not*** *stable-*σ-c-UNDEC *w.r.t.* $\mathcal{I}$; *and*
(b) *addition of* U *is* σ-c-UNDEC-*relevant for* A'' *w.r.t.* $\mathcal{I}'$; *and*
(c) *removal of* U' *is* σ-c-UNDEC-*relevant for* A'' *w.r.t.* $\mathcal{I}''$.

Proof sketch. We only prove the first item here, related to IN justification statuses. For the other items, see Appendix C. Let $\mathcal{I}' = \langle \mathcal{A} \cup \{A', A''\}, \mathcal{A}^? \cup \{U\}, \mathcal{R} \cup \{(A, A'), (A', A''), (U, A')\}, \mathcal{R}^? \rangle$ and let $\mathcal{I}'' = \langle \mathcal{A} \cup \{A', A'', U\}, \mathcal{A}^? \cup \{U'\}, \mathcal{R} \cup \{(A, A'), (A', A''), (U, A'), (U', U)\}, \mathcal{R}^? \rangle$ (see Fig. 12).

(a) $\Rightarrow$ **(b) and (c)** If A is not stable-σ-c-IN w.r.t. $\mathcal{I}$ (**a**) then by Definition 8 of stability there is some completion $AF^* = \langle \mathcal{A}^*, \mathcal{R}^* \rangle$ of $\mathcal{I}$ in which A is not σ-c-IN. Next, we construct three argumentation frameworks based on AF, containing the argument A'', and discuss its status.

- First, construct $AF_1 = \langle \mathcal{A}^* \cup \{A', A''\}, \mathcal{R}^* \cup \{(A, A'), (A', A'')\} \rangle$. Given that A'' is attacked by A', which is only attacked by A in AF_1, A'' cannot be σ-c-IN in AF_1.
- Next, construct $AF_2 = \langle \mathcal{A}^* \cup \{A', A'', U\}, \mathcal{R}^* \cup \{(A, A'), (A', A''), (U, A')\} \rangle$. A'' is σ-c-IN in AF_2, since the unattacked argument U attacks the only attacker of A'' (i.e. A').
- Let $AF_3 = \langle \mathcal{A}^* \cup \{A', A'', U, U'\}, \mathcal{R}^* \cup \{(A, A'), (A', A''), (U, A'), (U', U)\} \rangle$. Given that A is not σ-c-IN in AF^*, A cannot be σ-c-IN in AF_3 either. Since the argument A'' in AF_3 is attacked by A', which is only attacked by A, A'' cannot be σ-c-IN in AF_3.

Now item **(b)** (addition of U is σ-c-IN-relevant for A'' w.r.t. $\mathcal{I}'$) follows from Lemma 7, the fact that the IAF $\langle \mathcal{A}^* \cup \{A', A''\}, \{U\}, \mathcal{R}^* \cup \{(A, A'), (A', A''), (U, A')\}, \emptyset \rangle$ is in $\mathtt{part}(\mathcal{I}')$ and the status of A'' in AF_1 and AF_2.
Similarly, item **(c)** (removal of U' is σ-c-IN-relevant for A'' w.r.t. $\mathcal{I}''$) follows from Lemma 7, the fact that the IAF $\langle \mathcal{A}^* \cup \{A', A'', U\}, \{U'\}, \mathcal{R}^* \cup \{(A, A'), (A', A''), (U, A'), (U', U)\}, \emptyset \rangle$ is in $\mathtt{part}(\mathcal{I}'')$ and the status of A'' in AF_2 and AF_3.

(b) $\Rightarrow$ **(a)** If addition of U is σ-c-IN-relevant for A'' w.r.t. $\mathcal{I}'$ then there is some $\mathcal{I}^{*\prime}$ in $\mathtt{part}(\mathcal{I}')$ such that A'' is **not** σ-c-IN in $\mathtt{cert}(\mathcal{I}^{*\prime}) = \langle \mathcal{A}^{*\prime}, \mathcal{R}^{*\prime} \rangle$. Then A cannot be σ-c-IN in $\mathtt{cert}(\mathcal{I}^{*\prime})$ either (since A attacks A', which is the only attacker of A''). Now construct $AF' = \langle \mathcal{A}', \mathcal{R}' \rangle$ where $\mathcal{A}' = \mathcal{A}^{*\prime} \setminus \{A', A'', U\}$ and $\mathcal{R}' = \mathcal{R}^{*\prime} \setminus \{(A, A'), (A', A''), (U, A')\}$. Since A is not σ-c-IN in $\mathtt{cert}(\mathcal{I}^{*\prime})$, it cannot be σ-c-IN in AF'. Given that AF' is a completion of $\mathcal{I}$, A cannot be stable-σ-c-IN w.r.t. $\mathcal{I}$.

(c) $\Rightarrow$ **(a)** If removal of U' is σ-c-IN-relevant for A'' w.r.t. $\mathcal{I}''$ then by Lemma 7, there is some $\mathcal{I}^{*\prime\prime}$ in $\mathtt{part}(\mathcal{I}'')$ such that A'' is **not** σ-c-IN in the certain projection of $\mathcal{I}^{*\prime\prime}$ – without loss of generality, let this certain projection be $AF^{*\prime\prime} = \langle \mathcal{A}^{*\prime\prime}, \mathcal{R}^{*\prime\prime} \rangle$. Now construct $AF' = \langle \mathcal{A}', \mathcal{R}' \rangle$ where $\mathcal{A}' = \mathcal{A}^{*\prime\prime} \setminus \{A', A'', U, U'\}$ and $\mathcal{R}' = \mathcal{R}^{*\prime\prime} \setminus \{(A, A'), (A', A''), (U, A'), (U', U)\}$; since A'' is not σ-c-IN in $AF^{*\prime}$, it cannot be that A is σ-c-IN in AF'. Given that AF' is a completion of $\mathcal{I}$, A cannot be stable-σ-c-IN w.r.t. $\mathcal{I}$. $\square$

Using Lemma 8 and the complexity results for STABILITY from Section 4, we obtain lower bounds for both the addition and removal variants of RELEVANCE under complete, grounded and preferred semantics. For some variants of the RELEVANCE problem, combining the upper bounds identified in Section 5.1 with these lower bounds already yields tight complexity results. We present these results in Propositions 12, 13 and 14. The "easiest" of these problems are NP-complete, as we show in Proposition 12.

Proposition 12. *The following problems are NP-complete:*

(1) CP-*credulous*-UNDEC-RELEVANCE*;*
(2) CP-*sceptical*-IN-RELEVANCE*;*
(3) CP-*sceptical*-OUT-RELEVANCE*;*
(4) GR-*credulous*-IN-RELEVANCE*;*
(5) GR-*credulous*-OUT-RELEVANCE*;*
(6) GR-*credulous*-UNDEC-RELEVANCE*;*
(7) GR-*sceptical*-IN-RELEVANCE*;*
(8) GR-*sceptical*-OUT-RELEVANCE*; and*
(9) GR-*sceptical*-UNDEC-RELEVANCE.

Proof sketch. For each of these problems, membership in NP follows from membership in P of the corresponding JUSTIFICATION problems [10], listed in Table 1, and Proposition 11 (since $\mathrm{NP} = \mathrm{NP}^{\mathrm{P}}$).

For NP-hardness, we can give a reduction from the complementary of the corresponding STABILITY problem, using Lemma 8. Here we only consider item 1; the other items are proved in Appendix C. By Proposition 6, CP-credulous-UNDEC-STABILITY is CoNP-complete, which means that the complementary problem CP-credulous-UNDEC-INSTABILITY is NP-complete. By Lemma 8 item 3, each instance $I = \langle \mathcal{I}, A \rangle$ of CP-credulous-UNDEC-INSTABILITY can be transformed into an instance I' such that I is a positive instance of CP-credulous-UNDEC-INSTABILITY iff I' is a positive instance of CP-credulous-UNDEC-RELEVANCE. □

For each of the four justification statuses j for which we discuss RELEVANCE in Proposition 13, the j-STABILITY problem is Π_2^p-complete, hence j-INSTABILITY must be Σ_2^p-complete and j-RELEVANCE (both addition and removal) must be Σ_2^p-hard. In combination with the upper bound identified in Section 5.1, this implies that the corresponding RELEVANCE problems are Σ_2^p-complete.

Proposition 13. *The following problems are* Σ_2^p*-complete:*

(1) CP-*credulous*-IN-RELEVANCE*;*
(2) CP-*credulous*-OUT-RELEVANCE*;*
(3) PR-*credulous*-IN-RELEVANCE*; and*
(4) PR-*credulous*-OUT-RELEVANCE.

Proof. For each of these problems, membership in Σ_2^p follows from NP-completeness of the corresponding JUSTIFICATION problems [8,9] in combination with Proposition 11. For Σ_2^p-hardness, we can give a reduction from the corresponding INSTABILITY problem, using Lemma 8.

(1) Since CP-credulous-IN-STABILITY is Π_2^p-complete (by Lemma 4 and [3, Theorem 24]), the complementary problem (CP-credulous-IN-INSTABILITY) is Σ_2^p-complete. By Lemma 8 item 1, each instance $I = \langle \mathcal{I}, A \rangle$ of CP-credulous-IN-INSTABILITY can be transformed into an instance

$I' = \langle \mathcal{I}', A'', U \rangle$ (for addition) or $I' = \langle \mathcal{I}'', A'', U' \rangle$ (for removal) such that I is a positive instance of CP-credulous-IN-INSTABILITY iff I' is a positive instance of CP-credulous-IN-RELEVANCE.

(2) CP-credulous-OUT-STABILITY is Π_2^p-complete (by Lemma 4 and [3, Theorem 24] in combination with Lemma 5), which means that CP-credulous-IN-INSTABILITY is Σ_2^p-complete. By Lemma 8 item 2, each instance $I = \langle \mathcal{I}, A \rangle$ of CP-credulous-OUT-INSTABILITY can be transformed into an instance $I' = \langle \mathcal{I}', A, U \rangle$ (for addition) or $I' = \langle \mathcal{I}'', A, U' \rangle$ (for removal) such that I is a positive instance of CP-credulous-OUT-INSTABILITY iff I' is a positive instance of CP-credulous-OUT-RELEVANCE.

(3) Since PR-credulous-IN-STABILITY is Π_2^p-complete (by Lemma 4 and [3, Theorem 24]), the complementary problem (PR-credulous-IN-INSTABILITY) is Σ_2^p-complete. By Lemma 8 item 1, each instance $I = \langle \mathcal{I}, A \rangle$ of PR-credulous-IN-INSTABILITY can be transformed into an instance $I' = \langle \mathcal{I}', A'', U \rangle$ (for addition) or $I' = \langle \mathcal{I}'', A'', U' \rangle$ (for removal) such that I is a positive instance of PR-credulous-IN-INSTABILITY iff I' is a positive instance of PR-credulous-IN-RELEVANCE.

(4) PR-credulous-OUT-STABILITY is Π_2^p-complete (by Lemma 4 and [3, Theorem 24] in combination with Lemma 5), which means that PR-credulous-IN-INSTABILITY is Σ_2^p-complete. By Lemma 8 item 2, each instance $I = \langle \mathcal{I}, A \rangle$ of PR-credulous-OUT-INSTABILITY can be transformed into an instance $I' = \langle \mathcal{I}', A, U \rangle$ (for addition) or $I' = \langle \mathcal{I}'', A, U' \rangle$ (for removal) such that I is a positive instance of PR-credulous-OUT-INSTABILITY iff I' is a positive instance of PR-credulous-OUT-RELEVANCE. □

Similarly, in Proposition 14 we show that PR-credulous-UNDEC-RELEVANCE must be Σ_3^p-complete, as PR-credulous-UNDEC-INSTABILITY is Σ_3^p-hard and PR-credulous-UNDEC-JUSTIFICATION is on the second level of the polynomial hierarchy.

Proposition 14. PR-*credulous*-UNDEC-RELEVANCE *is* Σ_3^p*-complete.*

Proof. Membership in Σ_3^p follows from Σ_2^p-completeness of the corresponding JUSTIFICATION problem (Proposition 3) in combination with Proposition 11.

For Σ_3^p-hardness, we reduce from PR-credulous-UNDEC-INSTABILITY, which is Σ_3^p-complete since the co-problem PR-credulous-UNDEC-STABILITY is Π_3^p-complete (Proposition 8). By Lemma 8 item 3, each instance $I = \langle \mathcal{I}, A \rangle$ of PR-credulous-UNDEC-INSTABILITY can be transformed into an instance $I' = \langle \mathcal{I}', A'', U \rangle$ (for addition) or $I' = \langle \mathcal{I}'', A'', U' \rangle$ (for removal) such that I is a positive instance of PR-credulous-UNDEC-INSTABILITY iff I' is a positive instance of PR-credulous-UNDEC-RELEVANCE. □

For some other variants of the RELEVANCE problem, the strategy used in Propositions 12, 13 and 14 does not yield tight complexity results. For instance, for CP-sceptical-UNDEC-RELEVANCE, using this strategy we could only derive that the problem is in Σ_2^p (because CP-sceptical-UNDEC-JUSTIFICATION is CoNP-complete by Proposition 2) and that it is NP-hard (because CP-sceptical-UNDEC-STABILITY is CoNP-complete by Proposition 7). For these variants, we use another approach, based on an alternative reduction: for the **sceptical IN- and OUT**-RELEVANCE variants, we reduce from **credulous UNDEC**-INSTABILITY; for the **sceptical UNDEC**-RELEVANCE variants, we reduce from **credulous IN**-INSTABILITY. We will use these reductions to prove the remaining complexities for RELEVANCE under complete, grounded and preferred semantics in Propositions 15 and 16. In order to use these reductions, we need the following lemma:

Lemma 9 (Reduction co-inverted-STABILITY to RELEVANCE)**.** *Given an incomplete argumentation framework $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$, a certain argument $A \in \mathcal{A}$, semantics $\sigma \in \{\text{GR}, \text{CP}, \text{PR}\}$:*

(1) *Construct $\mathcal{I}'$ and $\mathcal{I}''$ as follows, where A', U and U' are fresh arguments not in $\mathcal{A} \cup \mathcal{A}^?$:*

- $\mathcal{A}' = \mathcal{A} \cup \{A'\}$;
- $\mathcal{R}' = \mathcal{R} \cup \{(A, A'), (A', A'), (U, A')\}$;
- $\mathcal{I}' = \langle \mathcal{A}', \mathcal{A}^? \cup \{U\}, \mathcal{R}', \mathcal{R}^? \rangle$; *and*
- $\mathcal{I}'' = \langle \mathcal{A}' \cup \{U\}, \mathcal{A}^? \cup \{U'\}, \mathcal{R}' \cup \{(U', U)\}, \mathcal{R}^? \rangle$.

The following items are equivalent:

(a) *A is **not** stable-σ-**credulous**-**IN** w.r.t. $\mathcal{I}$; and*
(b) *addition of U' is σ-**sceptical**-**UNDEC**-relevant for A' w.r.t. $\mathcal{I}''$; and*
(c) *removal of U is σ-**sceptical**-**UNDEC**-relevant for A' w.r.t. $\mathcal{I}'$.*

(2) *Construct $\mathcal{I}'$ and $\mathcal{I}''$ as follows, where A_1, A_2, A_3, A_4, U and U' are fresh arguments not in $\mathcal{A} \cup \mathcal{A}^?$:*

- $\mathcal{A}' = \mathcal{A} \cup \{A_1, A_2, A_3, A_4\}$;
- $\mathcal{R}' = \mathcal{R} \cup \{(A, A_1), (A, A_2), (A_1, A_2), (A_2, A_3), (A_3, A_4), (U, U), (U, A_3)\}$;
- $\mathcal{I}' = \langle \mathcal{A}', \mathcal{A}^? \cup \{U\}, \mathcal{R}', \mathcal{R}^? \rangle$; *and*
- $\mathcal{I}'' = \langle \mathcal{A}' \cup \{U\}, \mathcal{A}^? \cup \{U'\}, \mathcal{R}' \cup \{(U', U)\}, \mathcal{R}^? \rangle$.

The following items are equivalent:

(a) *A is **not** stable-σ-**credulous**-**UNDEC** w.r.t. $\mathcal{I}$; and*
(b) *addition of U' is σ-**sceptical**-**IN**-relevant for A_3 w.r.t. $\mathcal{I}''$; and*
(c) *removal of U is σ-**sceptical**-**IN**-relevant for A_3 w.r.t. $\mathcal{I}'$; and*
(d) *addition of U' is σ-**sceptical**-**OUT**-relevant for A_4 w.r.t. $\mathcal{I}''$; and*
(e) *removal of U is σ-**sceptical**-**OUT**-relevant for A_4 w.r.t. $\mathcal{I}'$.*

Proof idea. The proof for this lemma, given in Appendix C, is structured in the same way as the proof for Lemma 8. □

In the following two propositions, we will use Lemma 9 to prove the remaining complexities for RELEVANCE under CP and PR semantics. First, we prove that CP-sceptical-UNDEC-RELEVANCE and PR-sceptical-UNDEC-RELEVANCE are Σ_2^p-complete.

Proposition 15. *CP-sceptical-UNDEC-RELEVANCE and PR-sceptical-UNDEC-RELEVANCE are Σ_2^p-complete.*

Proof. Membership in Σ_2^p follows from CoNP-completeness of the corresponding JUSTIFICATION problems (Proposition 2) in combination with Proposition 11.

For Σ_2^p-hardness and $\sigma \in \{\text{CP}, \text{PR}\}$, we reduce from σ-credulous-IN-INSTABILITY, which is Σ_2^p-complete since the co-problem σ-credulous-IN-STABILITY is Π_2^p-complete (Lemma 4 and the results for necessary credulous acceptance from [3, Theorem 24]). By Lemma 9 item 1, each instance $I = \langle \mathcal{I}, A \rangle$ of σ-credulous-IN-INSTABILITY can be transformed into an instance $I' = \langle \mathcal{I}', A'', U \rangle$ (for addition) or $I' = \langle \mathcal{I}'', A'', U' \rangle$ (for removal) such that I is a positive instance of σ-credulous-UNDEC-INSTABILITY iff I' is a positive instance of σ-sceptical-UNDEC-RELEVANCE. □

In Proposition 16 we use Lemma 9 to show that PR-sceptical-IN- and -OUT-RELEVANCE are Σ_3^p-hard. Together with upper bounds from Section 5.1, this implies that PR-sceptical-IN- and -OUT-RELEVANCE are Σ_3^p-complete.

Proposition 16. PR-*sceptical*-IN-RELEVANCE *and* PR-*sceptical*-OUT-RELEVANCE *are* Σ_3^p*-complete.*

Proof. Membership in Σ_3^p follows from Σ_2^p-completeness of the corresponding JUSTIFICATION problems ([11] and Lemma 1) in combination with Proposition 11.

For Σ_3^p-hardness, we reduce from PR-credulous-UNDEC-INSTABILITY, which is Σ_3^p-complete since the co-problem PR-credulous-UNDEC-STABILITY is Π_3^p-complete (by Proposition 8 and Lemma 5). By Lemma 9 item 2, each instance $I = \langle \mathcal{I}, A \rangle$ of PR-credulous-UNDEC-INSTABILITY can be transformed into:

(1) an instance $I' = \langle \mathcal{I}'', A''', U' \rangle$ such that I is a positive instance of PR-credulous-UNDEC-INSTABILITY iff I' is a positive instance of PR-sceptical-IN-RELEVANCE (for addition); or
(2) an instance $I' = \langle \mathcal{I}', A''', U \rangle$ such that I is a positive instance of PR-credulous-UNDEC-INSTABILITY iff I' is a positive instance of PR-sceptical-IN-RELEVANCE (for removal); or
(3) an instance $I' = \langle \mathcal{I}'', A'', U' \rangle$ such that I is a positive instance of PR-credulous-UNDEC-INSTABILITY iff I' is a positive instance of PR-sceptical-OUT-RELEVANCE (for addition); or
(4) an instance $I' = \langle \mathcal{I}', A'', U \rangle$ such that I is a positive instance of PR-credulous-UNDEC-INSTABILITY iff I' is a positive instance of PR-sceptical-OUT-RELEVANCE (for removal). $\square$

At this point, we have proven all complexity results for RELEVANCE under complete, grounded and preferred semantics as summarised in Table 4.

5.3. *Lower bounds for stable semantics*

In this section, we will consider the RELEVANCE problems under ST semantics. We will prove that the IN and OUT variants of these problems are Σ_2^p-complete in Proposition 17. For the hardness-proof in this proposition, we require two transformations from Σ_2-SAT instances into IAFs. Recall from Section 2.3 that the Σ_2-SAT problem is to decide if there is some truth value assignment τ_X to variables of X such that for each truth value assignment τ_Y: $\Phi[\tau_X, \tau_Y] = \text{False}$ where Φ is a formula in CNF. The transformations are given in the following definition:

Definition 13 (Transformations)**.** Let (ϕ, X, Y) be an instance of Σ_2-SAT and let $\Phi = \bigwedge_i c_i$ and $c_i = \bigvee_j \alpha_j$ for each clause c_i in Φ, where α_j are the literals over $X \cup Y$ that occur in clause c_i. We define two transformations for this instance. Let:

- $\mathcal{A} = \{x_i, \overline{x_i} | x_i \in X\} \cup \{y_i, \overline{y_i} | y_i \in Y\} \cup \{\overline{c_i} | c_i \in \phi\} \cup \{\phi', \overline{\phi}, \phi\}$;
- $\mathcal{A}^? = \{g_i | x_i \in X\}$;
- $\mathcal{R} = \{(\overline{x_i}, x_i), (g_i, \overline{x_i}) | x_i \in X\} \cup \{(\overline{y_i}, y_i), (y_i, \overline{y_i}) | y_i \in Y\} \cup \{(x_k, \overline{c_i}) | x_k \in c_i\} \cup \{(\overline{x_k}, \overline{c_i}) | \neg x_k \in c_i\} \cup \{(y_k, \overline{c_i}) | y_k \in c_i\} \cup \{(\overline{y_k}, \overline{c_i}) | \neg y_k \in c_i\} \cup \{(\overline{c_i}, \phi') | c_i \in \Phi\} \cup \{(\phi', \overline{\phi}), (\chi, \overline{\phi}), (\overline{\phi}, \phi)\}$.

A first transformation into an IAF can be constructed as: $T_1(\phi, X, Y) = \langle \mathcal{A}, \mathcal{A}^? \cup \{\chi\}, \mathcal{R}, \emptyset \rangle$, where χ is a fresh uncertain argument not in $\mathcal{A} \cup \mathcal{A}^?$. A second transformation is $T_2(\phi, X, Y) = \langle \mathcal{A} \cup \{\chi\}, \mathcal{A}^? \cup \{\overline{\chi}\}, \mathcal{R} \cup \{(\overline{\chi}, \chi)\}, \emptyset \rangle$ where χ and $\overline{\chi}$ are not in $\mathcal{A} \cup \mathcal{A}^?$.

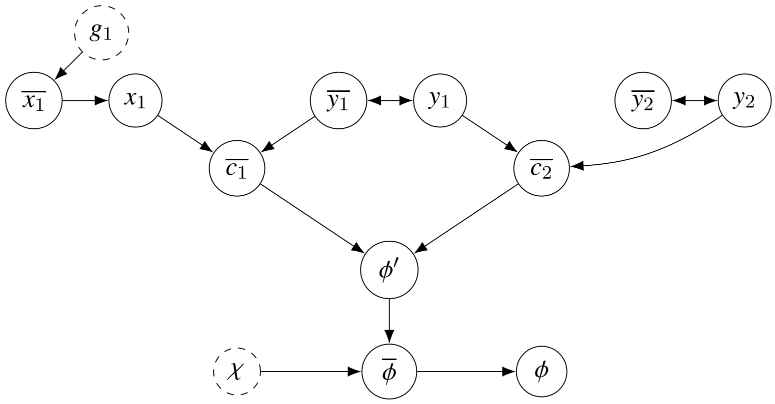

Fig. 13. Visualisation of the IAF created for the clauses $c_1 = x_1 \vee \neg y_1$ and $c_2 = y_1 \vee y_2$ using transformation T_1 of Definition 13.

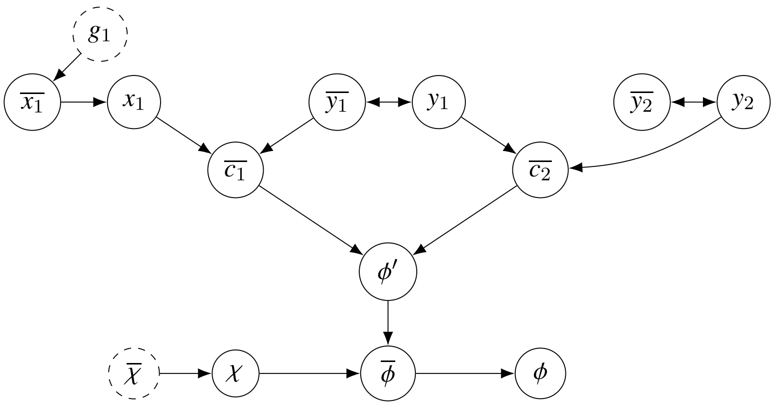

Fig. 14. Visualisation of the IAF created for the clauses $c_1 = x_1 \vee \neg y_1$ and $c_2 = y_1 \vee y_2$ using transformation T_2 of Definition 13.

An example of transformation T_1 is illustrated in Fig. 13 for the instance (Φ, X, Y) where the formula $\Phi = (x_1 \vee \neg y_1) \wedge (y_1 \vee y_2)$, $X = \{x_1\}$ and $Y = \{y_1, y_2\}$. Figure 14 shows transformation T_2 for the same Σ_2-SAT instance (Φ, X, Y).

In the following lemma, we use the transformations T_1 and T_2 to identify equivalences between instances of Σ_2-SAT and RELEVANCE.

Lemma 10. *Let (ϕ, X, Y) be an instance of Σ_2-SAT and let $\phi = \bigwedge_i c_i$ and $c_i = \bigvee_j \alpha_j$ for each clause c_i in ϕ, where α_j are the literals over $X \cup Y$ that occur in clause c_i. Now let $\mathcal{I}_1 = T_1(\phi, X, Y)$ and let $\mathcal{I}_2 = T_2(\phi, X, Y)$, using the transformations T_1 and T_2 specified in Definition 13. The following items are equivalent:*

(1) *(ϕ, X, Y) is a positive instance of Σ_2-SAT;*
(2) *Removal of χ is* ST*-sceptical-*IN*-relevant for $\overline{\phi}$ w.r.t. $\mathcal{I}_1$;*
(3) *Addition of χ is* ST*-credulous-*IN*-relevant for ϕ w.r.t. $\mathcal{I}_1$;*
(4) *Addition of $\overline{\chi}$ is* ST*-sceptical-*IN*-relevant for $\overline{\phi}$ w.r.t. $\mathcal{I}_2$;*
(5) *Removal of $\overline{\chi}$ is* ST*-credulous-*IN*-relevant for ϕ w.r.t. $\mathcal{I}_2$;*

(6) *Removal of* χ *is* ST-*sceptical*-OUT-*relevant for* ϕ *w.r.t.* $\mathcal{I}_1$*;*
(7) *Addition of* χ *is* ST-*credulous*-OUT-*relevant for* $\overline{\phi}$ *w.r.t.* $\mathcal{I}_1$*;*
(8) *Addition of* $\overline{\chi}$ *is* ST-*sceptical*-OUT-*relevant for* ϕ *w.r.t.* $\mathcal{I}_2$*; and*
(9) *Removal of* $\overline{\chi}$ *is* ST-*credulous*-OUT-*relevant for* $\overline{\phi}$ *w.r.t.* $\mathcal{I}_2$*.*

Proof sketch. We introduce an auxiliary statement, for which we prove that it equals all of the items above:

(0) There is some $\mathcal{I}^* \in \mathtt{part}(\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \emptyset\rangle)$ such that $\overline{\phi}$ is ST-sceptical-IN in its certain projection AF^* (where $\mathcal{A}$, $\mathcal{A}^?$ and $\mathcal{R}$ are chosen as in Definition 13).

Using this additional item, we prove these items separately. For brevity, we only show the equivalence between items 0 and 1 and 0 and 2 here; for full proofs for all equivalences, we refer to Appendix C.

(0) ⇒ (1) Suppose that there is some $\mathcal{I}^* \in \mathtt{part}(\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \emptyset\rangle)$ such that $\overline{\phi}$ is ST-sceptical-IN in the certain projection $AF^* = \langle \mathcal{A}^*, \mathcal{R}^*\rangle$ of $\mathcal{I}^*$. Let τ_X be an assignment to variables in X such that it assigns True to all $x_i \in X$ such that $g_i \in \mathcal{A}^*$ and False otherwise. Let τ_Y be an arbitrary assignment to all variables in Y. Given that $\overline{\phi}$ is ST-sceptical-IN in AF^*, for each ST extension S of AF^*, at least one of the arguments $\overline{c_i}$ must have been in S, so there is at least one clause in the formula Φ for which none of the variables was assigned True by τ_X and τ_Y. Since we chose τ_Y arbitrarily, we have that (ϕ, X, Y) is a positive instance of Σ_2-SAT.

(1) ⇒ (0) Let (ϕ, X, Y) be a positive instance of Σ_2-SAT. Then there is some assignment τ_X to all variables of X such that for each assignment τ_Y to the variables of Y, $\Phi[\tau_X, \tau_Y]$ is False. Let $G = \{g_i | x_i \in X$ and x_i is assigned True by $\tau_X\}$. Construct $\mathcal{I}^* = \langle \mathcal{A} \cup G, \emptyset, \mathcal{R}|_{\mathcal{A}\cup G}, \emptyset\rangle$ and let AF^* be its certain projection. Note that $\mathcal{I}^* \in \mathtt{part}(\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \emptyset\rangle)$. Given that all arguments in G are unattacked, each ST extension of AF^* contains all arguments in G. Furthermore, for each argument $x \in X$, each ST extension of AF^* contains either x (if x is assigned True by τ_X) or $\overline{y}$ (if x is assigned False by τ_Y). Additionally, for each argument $y \in Y$, each ST extension of AF^* contains either y or $\overline{y}$. Given that for each assignment τ_Y to the variables of Y, $\phi[\tau_X, \tau_Y]$ is False, it must be that for each ST extension S of this AF, at least one of the clause arguments $\overline{c_i}$ is in S, so ϕ' is attacked by an argument in S; therefore $\overline{\phi} \in S$. Thus, $\overline{\phi}$ is ST-sceptical-IN in AF^*.

(0) ⇒ (2) Suppose that there is some $\mathcal{I}^* = \langle \mathcal{A}^*, \mathcal{A}^{?*}, \mathcal{R}|_{\mathcal{A}^*\cup\mathcal{A}^{?*}}, \emptyset\rangle$ in $\mathtt{part}(\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \emptyset\rangle)$ such that $\overline{\phi}$ is ST-sceptical-IN in its certain projection AF^*. This implies that for $\mathcal{I}' = \langle \mathcal{A}^*, \{\chi\}, \mathcal{R}|_{\mathcal{A}^*\cup\{\chi\}}, \emptyset\rangle$ it also holds that $\overline{\phi}$ is ST-sceptical-IN in its certain projection $AF' = AF^* = \langle \mathcal{A}^*, \mathcal{R}|_{\mathcal{A}^*}\rangle$. Note that $\mathcal{I}' \in \mathtt{part}(\mathcal{I}_1)$. Now consider $\mathcal{I}'' = \langle \mathcal{A}^* \cup \{\chi\}, \emptyset, \mathcal{R}|_{\mathcal{A}^*\cup\{\chi\}}, \emptyset\rangle$ and its certain projection AF''. In AF'', the argument $\overline{\phi}$ is attacked by the unattacked argument χ, so $\overline{\phi}$ is not ST-sceptical-IN in AF''. Then by Lemma 7 item 3, removal of χ is ST-sceptical-IN-relevant for $\overline{\phi}$ w.r.t. $\mathcal{I}_1$.

(2) ⇒ (0) If removal of χ is ST-sceptical-IN-relevant for $\overline{\phi}$ w.r.t. $\mathcal{I}_1$ then by Lemma 7 item 3 there is some $\mathcal{I}' = \langle \mathcal{A}^*, \emptyset, \mathcal{R}|_{\mathcal{A}^*}, \emptyset\rangle$ in $\mathcal{I}_1$ (where $\chi \notin \mathcal{A}^*$, so $\mathcal{I}' \in \mathtt{part}(\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \emptyset\rangle)$) such that $\overline{\phi}$ is ST-sceptical-IN in $\mathtt{cert}(\mathcal{I}')$. □

The equivalences proven in Lemma 10 are exploited in Proposition 17 to show Σ_2^p-completeness of some of the RELEVANCE instances under ST semantics.

Proposition 17. *The following problems are* Σ_2^p*-complete:*

- ST-*credulous*-IN-RELEVANCE*;*

- ST-*sceptical*-IN-RELEVANCE*;*
- ST-*credulous*-OUT-RELEVANCE*; and*
- ST-*sceptical*-OUT-RELEVANCE.

Proof. From Proposition 11 and [8,9] it follows that these problems are in Σ_2^p.

For the hardness proofs, we use Lemma 10.

(1) ST-credulous-IN-RELEVANCE is Σ_2^p-hard because Σ_2-SAT can be reduced to this problem:

- For the *addition* variant, convert a given Σ_2-SAT instance (ϕ, X, Y) to the IAF $\mathcal{I}_1 = T_1(\phi, X, Y)$ and check that addition of χ is ST-credulous-IN-relevant for ϕ w.r.t. $\mathcal{I}_1$; this is the case iff (ϕ, X, Y) is a positive Σ_2-SAT instance (by the equality between item 1 and item 3 of Lemma 10).
- For the *removal* variant, convert a given Σ_2-SAT instance (ϕ, X, Y) to the IAF $\mathcal{I}_2 = T_2(\phi, X, Y)$ and check that removal of $\overline{\chi}$ is ST-credulous-IN-relevant for ϕ w.r.t. $\mathcal{I}_2$; this is the case iff (ϕ, X, Y) is a positive Σ_2-SAT instance (by the equality between item 1 and item 5 of Lemma 10).

(2) ST-sceptical-IN-RELEVANCE is Σ_2^p-hard because Σ_2-SAT can be reduced to this problem: for *addition*, convert a given Σ_2-SAT instance (ϕ, X, Y) to the IAF $\mathcal{I}_2 = T_2(\phi, X, Y)$ and check that addition of $\overline{\chi}$ is ST-sceptical-IN-relevant for $\overline{\phi}$ w.r.t. $\mathcal{I}_2$; this is the case iff (ϕ, X, Y) is a positive Σ_2-SAT instance (by the equality between item 1 and item 4 of Lemma 10). For *removal*, the proof is similar but we use the equality between item 1 and item 2.

(3) ST-credulous-OUT-RELEVANCE is Σ_2^p-hard because Σ_2-SAT can be reduced to this problem: for *addition*, convert a given Σ_2-SAT instance (ϕ, X, Y) to the IAF $\mathcal{I}_1 = T_1(\phi, X, Y)$ and check that addition of χ is ST-credulous-OUT-relevant for $\overline{\phi}$ w.r.t. $\mathcal{I}_1$; this is the case iff (ϕ, X, Y) is a positive Σ_2-SAT instance (by the equality between item 1 and item 7 of Lemma 10). For *removal*, the proof is similar but we use the equality between item 1 and item 9.

(4) ST-sceptical-OUT-RELEVANCE is Σ_2^p-hard because Σ_2-SAT can be reduced to this problem: for *addition*, convert a given Σ_2-SAT instance (ϕ, X, Y) to the IAF $\mathcal{I}_2 = T_2(\phi, X, Y)$ and check that addition of $\overline{\chi}$ is ST-credulous-IN-relevant for ϕ w.r.t. $\mathcal{I}_2$; this is the case iff (ϕ, X, Y) is a positive Σ_2-SAT instance (by the equality between item 1 and item 8 of Lemma 10). For *removal*, the proof is similar but we use the equality between item 1 and item 6. □

The problems ST-credulous-UNDEC-RELEVANCE and ST-sceptical-existent-UNDEC-RELEVANCE are easy, as we show in Proposition 18, because these problems only have negative instances.

Proposition 18. ST-*credulous*-UNDEC-RELEVANCE *and* ST-*sceptical-existent*-UNDEC-RELEVANCE *are trivial.*

Proof. For each argumentation framework $AF = \langle \mathcal{A}, \mathcal{R} \rangle$ such that a ST extension S exists, each argument in $\mathcal{A}$ is either in S or attacked by an argument in S. Therefore, A cannot be ST-credulous-UNDEC or ST-sceptical-existent-UNDEC in AF. This applies for each AF, including all certain projections of all partial completions of each possible IAF, so each instance of ST-credulous-UNDEC-RELEVANCE and ST-sceptical-existent-UNDEC-RELEVANCE must be negative. □

Next, we consider the ST-sceptical-UNDEC-RELEVANCE problem.

Proposition 19. ST-*sceptical*-UNDEC-RELEVANCE *is* Σ_2^p*-complete.*

Proof sketch. First, we will show that ST-sceptical-UNDEC-RELEVANCE is in Σ_2^p. By Proposition 5, ST-sceptical-UNDEC-JUSTIFICATION is CoNP-complete. By Proposition 11, this implies that ST-sceptical-UNDEC-RELEVANCE is in Σ_2^p.

For the hardness proof, we use an existing result on the problem of necessary nonempty existence under ST semantics from [21], which is defined as follows: given an IAF $\mathcal{I}$, does each completion AF' of $\mathcal{I}$ have a nonempty ST extension? It is shown in [21, Theorem 21] that this problem is Π_2^p-hard.

Let $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$ be an arbitrary instance of the necessary nonempty existence problem under ST semantics. If $\mathcal{A} = \emptyset$ then $\mathcal{I}$ is a negative instance, because there is a completion $AF' = \langle \mathcal{A}', \mathcal{R}' \rangle$ of $\mathcal{I}$ where $\mathcal{A}' = \emptyset$, which means that AF' has no nonempty ST extension. Alternatively, assume that $\mathcal{A}$ contains at least one argument and let A be an arbitrary argument in $\mathcal{A}$. Then we transform $\mathcal{I}$ into an instance $(\mathcal{I}', A, U)$ of the argument removal variant of the ST-sceptical-UNDEC-RELEVANCE problem, where:

- U is a fresh uncertain argument, not occurring in $\mathcal{A} \cup \mathcal{A}^?$; and
- $\mathcal{I}' = \langle \mathcal{A}, \mathcal{A}^? \cup \{U\}, \mathcal{R} \cup \{(U, B) | B \in \mathcal{A} \cup \mathcal{A}^?\}, \mathcal{R}^? \rangle$.

Then $\mathcal{I}$ is a *positive* instance of necessary nonempty existence under ST semantics iff $(\mathcal{I}', A, U)$ is a *negative* instance of ST-sceptical-UNDEC-RELEVANCE – we prove this in Appendix C. This appendix also contains proofs for the argument addition, attack addition and attack removal variants of the ST-sceptical-UNDEC-RELEVANCE problem.

Given that the necessary nonempty existence problem under ST semantics is Π_2^p-hard, the complementary problem of ST-sceptical-UNDEC-RELEVANCE must be Σ_2^p-hard. Together with the membership result from the beginning of this proof, this implies that ST-sceptical-UNDEC-RELEVANCE is Σ_2^p-complete. $\square$

The final variants of the RELEVANCE problem are ST-sceptical-existent-IN-RELEVANCE and ST-sceptical-existent-OUT-RELEVANCE. These are on the second level of the polynomial hierarchy, as we prove in Proposition 20.

Proposition 20. ST-*sceptical-existent*-IN-RELEVANCE *and* ST-*sceptical-existent*-OUT-RELEVANCE *are* Σ_2^p*-complete.*

Proof sketch. Membership in Σ_2^p directly follows from the complexity of ST-sceptical-existent-IN- and -OUT-JUSTIFICATION and Proposition 11, in the following way: ST-sceptical-existent-IN-JUSTIFICATION is DP-complete by [12, page 92]. By Lemma 1, ST-sceptical-existent-OUT-JUSTIFICATION is DP-complete as well. Then by Proposition 11 the problems of ST-sceptical-existent-IN-RELEVANCE and ST-sceptical-existent-OUT-RELEVANCE are in $NP^{DP} = \Sigma_2^p$; note that $NP^{DP} \subseteq \Sigma_2^p$ as any DP query can be answered by two (adaptive) SAT queries.

In order to prove Σ_2^p-hardness, we reduce from possible sceptical-existent acceptance under ST semantics (Definition 6), which was proven to be Σ_2^p-hard in [3, Theorem 25]. Let $(\mathcal{I}, A)$ be an arbitrary instance of possible sceptical-existent acceptance under ST semantics where $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$. We transform this into an instance $(\mathcal{I}', A, U)$ of ST-sceptical-existent-IN-RELEVANCE where U is a fresh uncertain argument that is not in $\mathcal{A} \cup \mathcal{A}^?$ and $\mathcal{I}' = \langle \mathcal{A}, \mathcal{A}^? \cup \{U\}, \mathcal{R} \cup \{(U, U)\}, \mathcal{R}^? \rangle$. Then $(\mathcal{I}, A)$ is a positive instance of possible sceptical-existent acceptance under ST semantics iff $(\mathcal{I}', A, U)$ is a positive instance of ST-sceptical-existent-IN-RELEVANCE; this is proven in Appendix C.

Table 4

Overview of all complexity results related to relevance. We refer to the corresponding proposition by "P" and the proposition number. Full proofs for each of the propositions are presented in the appendix

Justification status j	Complexity j-RELEVANCE	Justification status j	Complexity j-RELEVANCE
ST-credulous-IN/OUT	$\mathbf{\Sigma_2^p}$**-c** P17	GR-credulous-IN/OUT	**NP-c** P12
ST-credulous-UNDEC	**Trivial (no)** P18	GR-credulous-UNDEC	**NP-c** P12
ST-sceptical-IN/OUT	$\mathbf{\Sigma_2^p}$**-c** P17	GR-sceptical-IN/OUT	**NP-c** P12
ST-sceptical-UNDEC	$\mathbf{\Sigma_2^p}$**-c** P19	GR-sceptical-UNDEC	**NP-c** P12
ST-sceptical-existent-IN/OUT	$\mathbf{\Sigma_2^p}$**-c** P20		
ST-sceptical-existent-UNDEC	**Trivial (no)** P18		
CP-credulous-IN/OUT	$\mathbf{\Sigma_2^p}$**-c** P13	PR-credulous-IN/OUT	$\mathbf{\Sigma_2^p}$**-c** P13
CP-credulous-UNDEC	**NP-c** P12	PR-credulous-UNDEC	$\mathbf{\Sigma_3^p}$**-c** P14
CP-sceptical-IN/OUT	**NP-c** P12	PR-sceptical-IN/OUT	$\mathbf{\Sigma_3^p}$**-c** P16
CP-sceptical-UNDEC	$\mathbf{\Sigma_2^p}$**-c** P15	PR-sceptical-UNDEC	$\mathbf{\Sigma_2^p}$**-c** P15

For ST-sceptical-existent-OUT-RELEVANCE, we transform an arbitrary instance of possible sceptical-existent acceptance under ST semantics $(\mathcal{I}, A)$ to an instance $(\mathcal{I}'', A, U)$ of ST-sceptical-existent-OUT-RELEVANCE, where U is a fresh uncertain argument that is not in $\mathcal{A} \cup \mathcal{A}^?$, B is a fresh argument that is not in $\mathcal{A} \cup \mathcal{A}^?$ and $\mathcal{I}'' = \langle \mathcal{A} \cup \{B\}, \mathcal{A}^? \cup \{U\}, \mathcal{R} \cup \{(B, A), (U, U)\}, \mathcal{R}^? \rangle$. Then $(\mathcal{I}, A)$ is a positive instance of possible sceptical-existent acceptance under ST semantics iff $(\mathcal{I}'', A, U)$ is a positive instance of ST-sceptical-existent-OUT-RELEVANCE. $\square$

This proposition completes our study on complexity for j-RELEVANCE for all justification statuses j within the scope of this paper. These results, proven in Propositions 12–20, can be found in Table 4.

6. Related work

The computational complexity of various problems defined on argumentation frameworks is well-studied; see [14] for an overview. Most studies only identify accepted arguments and do not distinguish other justification statuses. Notable exceptions are [13] and [2], but neither of these works give complexity results for separate statuses, as we do. In [13], the complexity of justification is studied for the eight justification statuses proposed in [25]. These are related to, but not the same as the justification statuses that we study: in their work, every subset of {IN, UNDEC, OUT} is a justification status, which should be interpreted as "there is at least one extension where the argument has this label". Accordingly, the justification status {IN} w.r.t. some semantics σ from [25] corresponds to our σ-sceptical-IN justification status; their {OUT} corresponds to our σ-sceptical-OUT and their {UNDEC} corresponds to our σ-sceptical-UNDEC justification status. There is however no direct mapping between the other statuses. As an additional difference, the authors of [13] give an aggregated result for all statuses, whereas we prove the complexity for each justification status separately. [2] consider the same three justification statuses as we do, but give complexity results for an aggregation of these statuses: they introduce the notions of determinism, totality and functionality and provide complexity results for determining these for a given argument. They define an argument as deterministic if has the same label (IN, OUT or UNDEC) in all extensions. An argument is total if it is IN or OUT in all extensions; arguments are functional if they are both deterministic and total.

Complexity studies on problems defined on IAFs emerged more recently. For example, variants of the verification problem on IAFs are studied in [4]. The problems of stability and relevance differ from the verification problem as they are defined on arguments rather than sets of arguments. More related is [3]: the authors study potential and necessary credulous and sceptical acceptance in IAFs, where necessary sceptical acceptance of a given argument A, for example, means that in each partial completion's certain projection, each extension (under a given semantics) contains A. The notions of necessary credulous and sceptical acceptance are very similar to specific stability problems: in fact, we used results regarding their complexity for proving the complexity of stable-IN statuses. Finally, the notion of stability, which was originally defined on structured argumentation frameworks in [23], is transposed to the context of IAFs in [15] and preliminary complexity results for stability under four semantics are provided. In our work, we define a more fine-grained notion of stability and provide more precise complexity characterisations.

Our notion of relevance has not been introduced or studied before the early version of this work in [18]. Relevance is related to the notion of influenced sets in e.g. [1], which intuitively are sets of arguments whose justification status may change after an update. However, this notion is less strict than relevance: there are situations in which some argument A would be in the influenced set of adding an uncertain attack (B, C), while addition of (B, C) is not relevant for A. Other work related to relevance is [20] on the notion of independence in abstract argumentation. Building on the graph-theoretical criterion of d-separation, the authors introduce independence between argument sets, where the evaluation of one set of arguments can be independent of the evaluation of another set of arguments if the status of a third set of arguments is already known. This seems to be related to our notion of relevance, which also can be seen as a kind of dependence, but in contrast to (ir)relevance, their notion of independence is conditional. Finally, our notion of relevance is related to work on repairing abstract argumentation frameworks [24]. An AF can be repaired if it is possible to remove (a subset-minimal set of) arguments such that some argument becomes accepted. It is therefore related to our notion of IN-relevance, but a difference is that relevance is defined on incomplete argumentation frameworks rather than normal AFs, and therefore puts a constraint on the arguments that can be removed.

7. Conclusion

We have studied the complexity of detecting stability and relevance in incomplete argumentation frameworks. First, we redefined stability [15–17,23] on IAFs. Our definition is a more fine-grained notion than the existing definition on IAFs [15], since it distinguishes between IN, OUT and UNDEC justification statuses. This distinction is appropriate in, for example, applications in inquiry [17,23], where a dialogue discussing a given argument should be terminated if more information cannot change the argument's (exact) justification status.

As second main contribution of this paper, we performed a complexity analysis for the relevance problem on incomplete argumentation frameworks. Relevance was introduced before for incomplete argumentation frameworks in an early version of this work in [18], but that paper did not contain a full complexity analysis for all relevance statuses considered in this paper. In contrast to the earlier version, we provide complexity results for complete, preferred and stable semantics and study not only IN-, but also OUT- and UNDEC-relevance. Returning to the application in inquiry [16,17], the identification of relevant elements can be used to select the next question, reaching a stable situation in an efficient way.

It is unlikely that the stability and relevance problem itself can be solved efficiently for all inputs: our complexity analysis revealed that the nontrivial variants of the relevance and stability problems have a complexity ranging from the first to the third level of the polynomial hierarchy (cf. Table 1). Interestingly, even within the same semantics, there are differences in the complexity of UNDEC-STABILITY problems and the corresponding IN-STABILITY problems – we consider this to be an additional reason to study a fine-grained notion of stability and relevance.

In order to apply these theoretical concepts in practice, we plan to develop algorithms for evaluating or estimating stability and relevance in future work. In addition, we will study stability and relevance in structured argumentation frameworks, such as a dynamic version of ASPIC+, for various semantics.

Appendix A. Proofs justification status

Lemma 2 (Complementary relation IN- and UNDEC-JUSTIFICATION)**.** *For any given* $\sigma \in \{\text{GR}, \text{CP}, \text{PR}\}$*, for each argumentation theory* $AF = \langle \mathcal{A}, \mathcal{R} \rangle$ *and argument* $A \in \mathcal{A}$*, each of the following holds:*

(1) *A is σ-credulous-*IN *in AF iff A' is* ***not*** *σ-sceptical-*UNDEC *in* $\langle \mathcal{A} \cup \{A'\}, \mathcal{R} \cup \{(A, A'), (A', A')\}\rangle$*;*

(2) *A is σ-sceptical-*IN *in AF iff A' is* ***not*** *σ-credulous-*UNDEC *in* $\langle \mathcal{A} \cup \{A'\}, \mathcal{R} \cup \{(A, A'), (A', A')\}\rangle$*;*

(3) *A is σ-credulous-*UNDEC *in AF iff A' is* ***not*** *σ-sceptical-*IN *in* $\langle \mathcal{A} \cup \{A', B, C\}, \mathcal{R} \cup \{(A, B), (A, C), (B, C), (C, A')\}\rangle$*; and*

(4) *A is σ-sceptical-*UNDEC *in AF iff A' is* ***not*** *σ-credulous-*IN *in* $\langle \mathcal{A} \cup \{A', B, C\}, \mathcal{R} \cup \{(A, B), (A, C), (B, C), (C, A')\}\rangle$*.*

Proof. Consider an arbitrary semantics $\sigma \in \{\text{GR}, \text{CP}, \text{PR}\}$, argumentation theory $AF = \langle \mathcal{A}, \mathcal{R} \rangle$ and argument $A \in \mathcal{A}$. We prove the four items separately.

(1) Construct $AF' = \langle \mathcal{A} \cup \{A'\}, \mathcal{R} \cup \{(A, A'), (A', A')\}\rangle$; for an illustration, see the first and second columns of Fig. 6.

- $\Rightarrow$ Suppose that A is σ-credulous-IN in AF: then there is some σ-extension S of AF containing A. Note that S also must be a σ-extension of AF': all arguments in $\mathcal{A}$ attacking attackers of A are still in $\mathcal{A} \cup \{A'\}$ and $S \cup \{A'\}$ is not a σ-extension as it is not conflict-free. Then there exists some σ-extension (i.e. S) of AF' in which A' is attacked by S, so A' is not σ-sceptical-UNDEC in AF'.
- $\Leftarrow$ Suppose that A' is not σ-sceptical-UNDEC in AF'; then there exists some σ-extension S of AF' such that either $A' \in S$ or some argument attacking A' is in S. Given that A' is self-attacking, $A' \notin S$, so A' is attacked by some argument in S, which can only be A. Furthermore note that S is also a σ-extension of AF, since the arguments that are acceptable w.r.t. S in AF are exactly the same as the arguments acceptable w.r.t. S in AF'. To conclude, there exists some σ-extension (i.e. S) of AF in which $A \in S$, so A is σ-credulous-IN in AF.

(2) Construct $AF' = \langle \mathcal{A} \cup \{A'\}, \mathcal{R} \cup \{(A, A'), (A', A')\}\rangle$.

- $\Rightarrow$ Suppose that A is σ-sceptical-IN in AF: A is in each σ-extension of AF. Then A is also in each σ-extension of AF', so for each σ-extension S of AF', A' is attacked by S. Accordingly, A' is not σ-credulous-UNDEC in AF'.
- $\Leftarrow$ If A' is not σ-credulous-UNDEC in AF' then each σ-extension of AF' contains an argument attacking A', which can only be A. Given that each σ-extension of AF is also a σ-extension of AF', A must be in each σ-extension of AF, and therefore is σ-sceptical-IN w.r.t. AF.

(3) Construct $AF' = \langle \mathcal{A} \cup \{A', B, C\}, \mathcal{R} \cup \{(A, B), (A, C), (B, C), (C, A')\}\rangle$; for an illustration, see the first and third columns of Fig. 6.

$\Rightarrow$ Suppose that A is σ-credulous-UNDEC in AF: there is some σ-extension S of AF such that neither A, nor any attacker of A is in S. Given that all arguments in S are acceptable w.r.t. S and all attackers of A in AF' are attackers of A in AF, there must be some σ-extension S' of AF' such that $S \subseteq S'$. Having that neither A, nor any of its attackers is in S', B is not in S' or attacked by any argument in S'. Then, C is not attacked by any argument in S', so A' cannot be in S'. This implies that $S' = S$ is a σ-extension of AF' not containing A', so A' is not σ-sceptical-IN in AF'.

$\Leftarrow$ If A' is not σ-sceptical-IN in AF' then some σ-extension S of AF' does not contain A'. This implies that the arguments A and B are not in S either: otherwise, they would defend A' against C. So S contains exactly those arguments in $\mathcal{A}$ that are acceptable w.r.t. S, which implies that S is a σ extension of AF as well. Finally note that S cannot attack any argument in $\mathcal{A}$ attacking A: otherwise, such an argument would defend B against A and then B would be in S. Given that A is not in S, nor attacked by any argument in S, while S is a σ extension of AF, we derive that A is σ-credulous-UNDEC in AF.

(4) Construct $AF' = \langle \mathcal{A} \cup \{A', B, C\}, \mathcal{R} \cup \{(A, B), (A, C), (B, C), (C, A')\}\rangle$.

$\Rightarrow$ If A is σ-sceptical-UNDEC in AF then no σ-extension of AF contains A or any attacker of A. This implies that no σ-extension of AF' would contain A or any of its attackers. Since each σ-extension of AF' is complete, it could not contain B or C, and therefore not A'. This implies that A' is not σ-credulous-IN in AF'.

$\Leftarrow$ If A' is not σ-credulous-IN in AF' then no σ-extension of AF' contains A'. Because of the completeness criterion of σ semantics, no σ-extension of AF' would contain A, B or any argument in $\mathcal{A}$ attacking A. Therefore for each σ-extension S of AF', all arguments in S are in $\mathcal{A} \setminus \{A\}$ and do not attack A. Then no σ-extension of AF can contain A or any attacker of A either. This implies that A is σ-sceptical-UNDEC in AF. $\square$

Lemma 3 (Complexities UNDEC-JUSTIFICATION)**.** *For any given $\sigma \in \{\text{GR}, \text{CP}, \text{PR}\}$:*

(1) *If the complexity of σ-credulous-*IN-JUSTIFICATION *is $\mathcal{C}$, then the complexity of σ-sceptical-*UNDEC-JUSTIFICATION *is co-$\mathcal{C}$; and*
(2) *If the complexity of σ-sceptical-*IN-JUSTIFICATION *is $\mathcal{C}$, then the complexity of σ-credulous-*UNDEC-JUSTIFICATION *is co-$\mathcal{C}$.*

Proof. We prove these two items separately:

(1) The first item can be proved by two reductions:

- Each instance $I_1 = (\langle \mathcal{A}, \mathcal{R}\rangle, A)$ of σ-credulous-IN-JUSTIFICATION can, in polynomial time, be converted to an instance $I_2 = (\langle \mathcal{A} \cup \{A'\}, \mathcal{R} \cup \{(A, A'), (A', A')\}\rangle, A')$ of σ-sceptical-UNDEC-JUSTIFICATION where, by Lemma 2 item 1, I_1 is a positive instance iff I_2 is a negative instance. So σ-credulous-IN-JUSTIFICATION reduces to σ-sceptical-UNDEC-JUSTIFICATION.
- Similarly, each instance $I_1 = (\langle \mathcal{A}, \mathcal{R}\rangle, A)$ of σ-sceptical-UNDEC-JUSTIFICATION can, in polynomial time, be converted to an instance $I_2 = (\langle \mathcal{A} \cup \{A', B, C\}, \mathcal{R} \cup \{(A, B), (A, C), (B, C), (C, A')\}\rangle, A')$ of σ-credulous-IN-JUSTIFICATION where, by Lemma 2 item 4, I_1 is a positive instance iff I_2 is a negative instance. So σ-sceptical-UNDEC-JUSTIFICATION reduces to σ-credulous-IN-JUSTIFICATION.

(2) The second item can be proved by two other reductions:

- Each instance $I_1 = (\langle\mathcal{A}, \mathcal{R}\rangle, A)$ of σ-sceptical-IN-JUSTIFICATION can, in polynomial time, be converted to an instance $I_2 = (\langle\mathcal{A} \cup \{A'\}, \mathcal{R} \cup \{(A, A'), (A', A')\}\rangle, A')$ of σ-credulous-UNDEC-JUSTIFICATION where, by Lemma 2 item 2, I_1 is a positive instance iff I_2 is a negative instance. So σ-sceptical-IN-JUSTIFICATION reduces to σ-credulous-UNDEC-JUSTIFICATION.
- Similarly, each instance $I_1 = (\langle\mathcal{A}, \mathcal{R}\rangle, A)$ of σ-credulous-UNDEC-JUSTIFICATION can, in polynomial time, be converted to an instance $I_2 = (\langle\mathcal{A} \cup \{A', B, C\}, \mathcal{R} \cup \{(A, B), (A, C), (B, C), (C, A')\}\rangle, A')$ of σ-sceptical-IN-JUSTIFICATION where, by Lemma 2 item 3, I_1 is a positive instance iff I_2 is a negative instance. So σ-credulous-UNDEC-JUSTIFICATION reduces to σ-sceptical-IN-JUSTIFICATION. □

Appendix B. Proofs stability

Lemma 5. *For any given $\sigma \in \{\text{GR}, \text{CP}, \text{PR}, \text{ST}\}$ and $c \in \{sceptical, credulous\}$, the complexity of σ-c-OUT-STABILITY equals the complexity of σ-c-IN-STABILITY.*

Proof. We prove this by a reduction from σ-c-OUT-STABILITY to σ-c-IN-STABILITY and by a reduction in the other direction.

- First, consider an arbitrary instance $(\mathcal{I}, A)$ of the σ-c-OUT-STABILITY problem where $\mathcal{I} = \langle\mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^?\rangle$ and $A \in \mathcal{A}$. Construct $\mathcal{I}^* = \langle\mathcal{A} \cup \{B\}, \mathcal{A}^?, \mathcal{R} \cup \{(A, B)\}, \mathcal{R}^?\rangle$, where $B \notin \mathcal{A} \cup \mathcal{A}^?$.
  - ∗ If the instance $(\mathcal{I}, A)$ is positive, then A is stable-σ-c-OUT in $\mathcal{I}$. Let $AF^{*\prime} = \langle\mathcal{A}^{*\prime}, \mathcal{R}^{*\prime}\rangle$ be an arbitrary completion of $\mathcal{I}^*$. Note that $\langle\mathcal{A}^{*\prime} \setminus \{B\}, \mathcal{R}^{*\prime} \setminus \{(A, B)\}\rangle$ must be a completion of $\mathcal{I}$ where A is σ-c-OUT. Then A must also be σ-c-OUT in $AF^{*\prime}$, which implies that B is σ-c-IN in $AF^{*\prime}$. Generalising over all completions of $\mathcal{I}^*$, we derive that B is stable-σ-c-IN w.r.t. $\mathcal{I}^*$. So $(\mathcal{I}^*, B)$ is a positive instance of σ-c-IN-STABILITY.
  - ∗ Alternatively, the instance is negative: A is not stable-σ-c-OUT in $\mathcal{I}$. Then there is some completion $AF' = \langle\mathcal{A}', \mathcal{R}'\rangle$ of $\mathcal{I}$ such that A is not σ-c-OUT in AF'. Construct $AF^{*\prime} = \langle\mathcal{A}' \cup \{B\}, \mathcal{R}' \cup \{(A, B)\}\rangle$ and note that $AF^{*\prime}$ is a completion of $\mathcal{I}^*$. Since A was not σ-c-OUT in AF', it cannot be σ-c-OUT in $AF^{*\prime}$ either. As a consequence, B is not σ-c-IN in $AF^{*\prime}$, hence $(\mathcal{I}^*, B)$ is a negative instance of σ-c-IN-STABILITY.

  To conclude, σ-c-OUT-STABILITY can be reduced to σ-c-IN-STABILITY.
- Now, consider an arbitrary instance $(\mathcal{I}, A)$ of the σ-c-IN-STABILITY problem where $\mathcal{I} = \langle\mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^?\rangle$ and $A \in \mathcal{A}$. Construct $\mathcal{I}^* = \langle\mathcal{A} \cup \{B\}, \mathcal{A}^?, \mathcal{R} \cup \{(A, B)\}, \mathcal{R}^?\rangle$ with $B \notin \mathcal{A} \cup \mathcal{A}^?$.
  - ∗ If the instance $(\mathcal{I}, A)$ is positive, then A is stable-σ-c-IN in $\mathcal{I}$. Let $AF^{*\prime} = \langle\mathcal{A}^{*\prime}, \mathcal{R}^{*\prime}\rangle$ be an arbitrary completion of $\mathcal{I}^*$. Note that $\langle\mathcal{A}^{*\prime} \setminus \{B\}, \mathcal{R}^{*\prime} \setminus \{(A, B)\}\rangle$ must be a completion of $\mathcal{I}$ where A is σ-c-IN. Then A must also be σ-c-IN in $AF^{*\prime}$, which implies that B is σ-c-OUT in $AF^{*\prime}$. Generalising over all completions of $\mathcal{I}^*$, we derive that B is stable-σ-c-OUT w.r.t. $\mathcal{I}^*$. So $(\mathcal{I}^*, B)$ is a positive instance of σ-c-OUT-STABILITY.
  - ∗ Alternatively, the instance is negative: A is not stable-σ-c-IN in $\mathcal{I}$. Then there is some completion $AF' = \langle\mathcal{A}', \mathcal{R}'\rangle$ of $\mathcal{I}$ such that A is not σ-c-IN in AF'. Construct $AF^{*\prime} = \langle\mathcal{A}' \cup \{B\}, \mathcal{R}' \cup \{(A, B)\}\rangle$ and note that $AF^{*\prime}$ is a completion of $\mathcal{I}^*$. Since A was not σ-c-IN in AF', it cannot be σ-c-IN in

$AF^{*\prime}$ either. As a consequence, B is not σ-c-OUT in $AF^{*\prime}$, hence $(\mathcal{I}^*, B)$ is a negative instance of σ-c-OUT-STABILITY.

To conclude, we have also shown that σ-c-IN-STABILITY can be reduced to σ-c-OUT-STABILITY.

From these two reductions, it follows that σ-c-IN-STABILITY and σ-c-OUT-STABILITY have the same complexity. $\square$

Lemma 6 (Complexities UNDEC-STABILITY)**.** *For any given* $\sigma \in \{\text{GR}, \text{CP}, \text{PR}\}$*:*

(1) *If possible credulous acceptance w.r.t.* σ *semantics is in the complexity class* $\mathcal{C}$*, then* σ*-sceptical-*UNDEC-STABILITY *is in the complexity class co-*$\mathcal{C}$*; and*
(2) *If possible sceptical acceptance w.r.t.* σ *semantics is in the complexity class* $\mathcal{C}$*, then* σ*-credulous-*UNDEC-STABILITY *is in the complexity class co-*$\mathcal{C}$*.*

Proof. We prove the two items of this lemma separately.

(1) The proof for the first item consists of two reductions: we show that possible credulous acceptance w.r.t. σ semantics reduces to the complementary problem of σ-sceptical-UNDEC-STABILITY and the other way around.

- Let $I_1 = (\mathcal{I}, A)$ be an instance of possible credulous acceptance w.r.t. σ semantics where $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$ and $A \in \mathcal{A}$. Construct $I_2 = (\mathcal{I}^*, A')$, an instance of σ-sceptical-UNDEC-STABILITY, where $\mathcal{I}^* = \langle \mathcal{A} \cup \{A'\}, \mathcal{A}^?, \mathcal{R} \cup \{(A, A'), (A', A')\}, \mathcal{R}^? \rangle$ and $A' \notin \mathcal{A} \cup \mathcal{A}^?$. For an illustration, see the first and second columns in Fig. 15. I_1 is a positive instance iff I_2 is a negative instance, as we show next:
  - $\Rightarrow$ Suppose that I_1 is positive: $\mathcal{I}$ has some completion $\langle \mathcal{A}', \mathcal{R}' \rangle$ for which there is a σ extension S containing A. Then S is also a σ extension of $AF' = \langle \mathcal{A}' \cup \{A'\}, \mathcal{R}' \cup \{(A, A'), (A', A')\} \rangle$, so A is **not** σ-sceptical-UNDEC in AF'. Note that AF' is a completion of $\mathcal{I}^*$, so A' is **not** stable-σ-sceptical-UNDEC w.r.t. $\mathcal{I}^*$. Therefore, I_2 is a negative instance.
  - $\Leftarrow$ Suppose that I_2 is negative: there is some completion $\langle \mathcal{A} \cup \mathcal{A}^{*\prime} \cup \{A'\}, \mathcal{R} \cup \mathcal{R}^{*\prime} \cup \{(A, A'), (A', A')\} \rangle$ of $\mathcal{I}^*$ with some σ extension S such that S either contains A' or some argument attacking A'. S cannot contain A' as S must be conflict-free under σ semantics, so there must be some argument attacking A' in S, which can only be A. Since S is also a σ extension of $\langle \mathcal{A} \cup \mathcal{A}^*, \mathcal{R} \cup \mathcal{R}^* \rangle$, which is a completion of $\mathcal{I}'$, I_1 is a positive instance of possible credulous acceptance w.r.t. σ semantics.

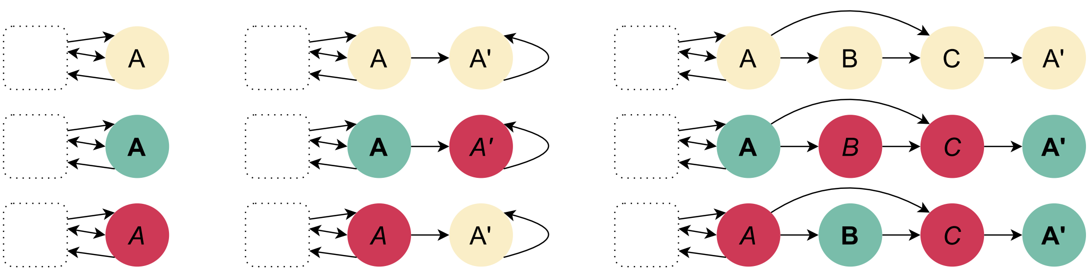

Fig. 15. Illustration of the incomplete argumentation frameworks that are used in transformations between stability problem instances that are used for proving Lemma 6. This figure is a repetition of Fig. 6 that was used for proving Lemma 2, but now the dotted, rounded rectangles represent the part of the *incomplete* argumentation framework except (attacks related to) A.

- Let $I_1 = (\mathcal{I}, A)$ be an instance of σ-sceptical-UNDEC-STABILITY where $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$ and $A \in \mathcal{A}$. Now let $I_2 = (\mathcal{I}^*, A')$ be an instance of possible credulous acceptance w.r.t. σ semantics, where $\mathcal{I}^* = \langle \mathcal{A} \cup \{A', B, C\}, \mathcal{A}^?, \mathcal{R} \cup \{(A, B), (A, C), (B, C), (C, A')\}, \mathcal{R}^? \rangle$ and none of A', B and C is in $\mathcal{A} \cup \mathcal{A}^?$. For an illustration, see the first and third columns in Fig. 15. We claim that I_1 is a positive instance iff I_2 is a negative instance:
  - $\Rightarrow$ Suppose that I_1 is positive: there is no completion of $\mathcal{I}$ such that A nor any argument attacking A is in any σ extension of that completion. Now suppose towards a contradiction that A' is possibly credulously accepted w.r.t. $\mathcal{I}^*$. In other words, there is some completion $\langle \mathcal{A}^{*\prime}, \mathcal{R}^{*\prime} \rangle$ of $\mathcal{I}^*$ which has some extension S containing A'. Given that A' is only attacked by C in $\langle \mathcal{A}^{*\prime}, \mathcal{R}^{*\prime} \rangle$ and C is attacked by A and B (and σ is complete), it must be that either A or B is in S. In case B is in S, it must be defended against A, so some argument attacking A must be in S. But that would imply that there is some completion of $\mathcal{I}$ having some σ extension S' such that either $A \in S'$ or some argument attacking A is in S', which would contradict our assumption. So A' is **not** possibly credulously accepted w.r.t. $\mathcal{I}^*$. Therefore, I_2 is a negative instance.
  - $\Leftarrow$ Suppose that I_2 is negative: no completion of $\mathcal{I}^*$ has a σ extension that contains A'. Towards a contradiction, suppose that I_1 is negative as well. Then there is some completion $\langle \mathcal{A}', \mathcal{R}' \rangle$ of $\mathcal{I}$ that has some σ extension S containing A or an argument attacking A.
    - $*$ First suppose that $A \in S$. Now consider the argumentation framework $AF^{*\prime} = \langle \mathcal{A}' \cup \{B, C, A'\}, \mathcal{R}' \cup \{(A, B), (A, C), (B, C), (C, A')\} \rangle$ and the set $S^{*\prime} = S \cup \{A'\}$. Given that S is a σ extension of $\langle \mathcal{A}', \mathcal{R}' \rangle$, it must be that $S^{*\prime}$ is a σ extension of $AF^{*\prime}$.
    - $*$ Alternatively, suppose that some argument attacking A is in S. Now consider the argumentation framework $AF^{*\prime} = \langle \mathcal{A}' \cup \{B, C, A'\}, \mathcal{R}' \cup \{(A, B), (A, C), (B, C), (C, A')\} \rangle$ and the set $S^{*\prime} = S \cup \{B, A'\}$. Given that S is a σ extension of $\langle \mathcal{A}', \mathcal{R}' \rangle$, it must be that $S^{*\prime}$ is a σ extension of $AF^{*\prime}$.

    In both cases, there is a completion of $\mathcal{I}^*$ with a σ extension that contains A'. But this contradicts our assumption that I_2 is negative, so I_1 must have been positive.

(2) The proof for the second item is similar and consists of two reductions as well.

- Let $I_1 = (\mathcal{I}, A)$ be an instance of possible sceptical acceptance w.r.t. σ semantics where $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$ and $A \in \mathcal{A}$. Let $I_2 = (\mathcal{I}^*, A')$ be an instance of σ-credulous-UNDEC-STABILITY, where $\mathcal{I}^* = \langle \mathcal{A} \cup \{A'\}, \mathcal{A}^?, \mathcal{R} \cup \{(A, A'), (A', A')\}, \mathcal{R}^? \rangle$ and $A' \notin \mathcal{A} \cup \mathcal{A}^?$. For an illustration, see the first and second columns in Fig. 15. I_1 is positive iff I_2 is negative:
  - $\Rightarrow$ If I_1 is positive then there exists some completion $\langle \mathcal{A}', \mathcal{R}' \rangle$ of $\mathcal{I}$ such that each of its σ extensions contains A. Each σ extension of $AF' = \langle \mathcal{A}' \cup \{A'\}, \mathcal{R}' \cup \{(A, A'), (A', A')\} \rangle$ is also a σ extension of $\langle \mathcal{A}', \mathcal{R}' \rangle$, so A' is **not** σ-credulous-UNDEC in AF'. Since AF' is a completion of $\mathcal{I}^*$, A' is **not** stable-σ-credulous-UNDEC w.r.t. $\mathcal{I}^*$, hence I_2 is a negative instance.
  - $\Leftarrow$ If I_2 is negative then there exists some completion $AF^* = \langle \mathcal{A} \cup \mathcal{A}^* \cup \{A'\}, \mathcal{R} \cup \mathcal{R}^* \cup \{(A, A'), (A', A')\} \rangle$ of $\mathcal{I}^*$ such that each σ extension contains some argument attacking A', which can only be A. Since each σ extension of $AF' = \langle \mathcal{A} \cup \mathcal{A}^*, \mathcal{R} \cup \mathcal{R}^* \rangle$ is also a σ extension of AF^*, A must be σ-sceptical-IN in AF'. Since AF' is a completion of $\mathcal{I}$, I_1 is a positive instance of possible sceptical acceptance w.r.t. σ semantics.

- Let $I_1 = (\mathcal{I}, A)$ be an instance of σ-sceptical-UNDEC-STABILITY where $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$ and $A \in \mathcal{A}$. Now let $I_2 = (\mathcal{I}^*, A')$ be an instance of possible credulous acceptance w.r.t. σ semantics, where $\mathcal{I}^* = \langle \mathcal{A} \cup \{A', B, C\}, \mathcal{A}^?, \mathcal{R} \cup \{(A, B), (A, C), (B, C), (C, A')\}, \mathcal{R}^? \rangle$ and none of A', B and C is in $\mathcal{A} \cup \mathcal{A}^?$. For an illustration, see the first and third columns in Fig. 15. I_1 is a positive instance iff I_2 is a negative instance:
  - $\Rightarrow$ If I_1 is positive then for each completion of $\mathcal{I}$ and for each σ extension S of that completion, A is not in S and not attacked by any argument in S. If I_2 would be positive as well, then there would be some completion $AF^{*\prime}$ of $\mathcal{I}^*$ such that A' is in some σ extension S. Then, by completeness of σ, either A or B (and therefore an argument attacking A) must have been in S. This implies that A was not σ-sceptical-UNDEC (but -IN/-OUT) in $AF^{*\prime} = \langle \mathcal{A} \cup \mathcal{A}^{*\prime} \cup \{A', B, C\}, \mathcal{R} \cup \mathcal{R}^{*\prime} \cup \{(A, B), (A, C), (B, C), (C, A')\} \rangle$. However, then $S \setminus \{A', B, C\}$ must have been a σ extension of AF' where $AF' = \langle \mathcal{A} \cup \mathcal{A}^{*\prime}, \mathcal{R} \cup \mathcal{R}^{*\prime} \rangle$, which is a completion of $\mathcal{I}$. This contradicts our earlier assumption that I_1 is positive, so I_2 must have been negative.
  - $\Leftarrow$ If I_2 is a negative instance then for each completion of $\mathcal{I}^*$ and for each σ extension S in that completion, A' was not in S. Assume towards a contradiction that I_1 was a negative instance; then there is some completion $AF' = \langle \mathcal{A}', \mathcal{R}' \rangle$ of $\mathcal{I}$ having some σ extension S such that A is either in S or attacked by any argument in S. Now construct $AF^* = \langle \mathcal{A}' \cup \{A', B, C\}, \mathcal{R}' \cup \{(A, B), (A, C), (B, C), (C, A')\} \rangle$. Given that S is a σ extension of AF', either $S \cup \{A'\}$ (if $A \in S$) or $S \cup \{B, A'\}$ (if $A \notin S$) must be a σ extension of AF^*. In any case, A' is contained in a σ extension of AF^*, while AF^* is a completion of $\mathcal{I}^*$, which contradicts the fact that I_2 is negative. We must therefore conclude that I_1 was negative. □

Appendix C. Proofs relevance

Lemma 7. *Given an IAF $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$, a certain argument $A \in \mathcal{A}$ and a justification status j:*

(1) *For each $U \in \mathcal{A}^?$, addition of U is j-relevant for A w.r.t. $\mathcal{I}$ iff there exists some $\mathcal{I}' = \langle \mathcal{A}', \{U\}, \mathcal{R}', \emptyset \rangle \in \texttt{part}(\mathcal{I})$ such that A is **not** j in the certain projection of $\mathcal{I}'$, while A is j in the certain projection of $\langle \mathcal{A}' \cup \{U\}, \emptyset, \mathcal{R}', \emptyset \rangle$.*

(2) *For each $U \in \mathcal{R}^?$, addition of U is j-relevant for A w.r.t. $\mathcal{I}$ iff there exists some $\mathcal{I}' = \langle \mathcal{A}', \emptyset, \mathcal{R}', \{U\} \rangle \in \texttt{part}(\mathcal{I})$ such that A is **not** j in the certain projection of $\mathcal{I}'$, while A is j in the certain projection of $\langle \mathcal{A}', \emptyset, \mathcal{R}' \cup \{U\}, \emptyset \rangle$.*

(3) *For each $U \in \mathcal{A}^?$, removal of U is j-relevant for A w.r.t. $\mathcal{I}$ iff there exists some $\mathcal{I}' = \langle \mathcal{A}', \{U\}, \mathcal{R}', \emptyset \rangle \in \texttt{part}(\mathcal{I})$ such that A is j in the certain projection of $\mathcal{I}'$, while A is **not** j in the certain projection of $\langle \mathcal{A}' \cup \{U\}, \emptyset, \mathcal{R}', \emptyset \rangle$.*

(4) *For each $U \in \mathcal{R}^?$, removal of U is j-relevant for A w.r.t. $\mathcal{I}$ iff there exists some $\mathcal{I}' = \langle \mathcal{A}', \{U\}, \mathcal{R}', \emptyset \rangle \in \texttt{part}(\mathcal{I})$ such that A is j in the certain projection of $\mathcal{I}'$, while A is **not** j in the certain projection of $\langle \mathcal{A}', \emptyset, \mathcal{R}' \cup \{U\}, \emptyset \rangle$.*

Proof. Let $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$ be an incomplete argumentation framework, $A \in \mathcal{A}$ a certain argument and j a justification status.

(1) We prove both directions of the first item.

⇒ See Fig. 11 for an illustration of the steps in this proof.

(a) Suppose that addition of $U \in A^?$ is j-relevant for A w.r.t. $\mathcal{I}$.
(b) Then by Definition 12 there is a minimal stable-j partial completion $\mathcal{I}^* = \langle \mathcal{A}^*, \mathcal{A}^{?*}, \mathcal{R}^*, \mathcal{R}^{?*} \rangle$ for A w.r.t. $\mathcal{I}$ such that $U \in \mathcal{A}^*$.
(c) Now construct the IAF $\mathcal{I}'$ from $\mathcal{I}^*$ by moving U from the certain to the uncertain part: $\mathcal{I}' = \langle \mathcal{A}^* \setminus \{U\}, \mathcal{A}^{?*} \cup \{U\}, \mathcal{R}^*, \mathcal{R}^{?*} \rangle$.
(d) Given that $\mathcal{I}^*$ was minimal and $\mathcal{I}^* \in \texttt{part}(\mathcal{I}')$, A cannot be stable-j w.r.t. $\mathcal{I}'$. So there must be some $\mathcal{I}'' = \langle \mathcal{A}'', \mathcal{A}^{?''}, \mathcal{R}'', \mathcal{R}^{?''} \rangle$ in $\texttt{part}(\mathcal{I}')$ such that A's justification status in the certain projection of $\mathcal{I}''$ is not j – note that this means that U is not in $\mathcal{A}''$ (since A was stable-j in $\mathcal{I}^*$).
(e) Then A's justification status in the certain projection of $\mathcal{I}''' = \langle \mathcal{A}'', \{U\}, \mathcal{R}'', \emptyset \rangle$ is not j (because this is the same as the certain projection of $\mathcal{I}''$, i.e. $\langle \mathcal{A}'', \mathcal{R}''|_{\mathcal{A}''} \rangle$).
(f) Next, construct $\mathcal{I}^{*\prime} = \langle \mathcal{A}'' \cup \{U\}, \emptyset, \mathcal{R}'', \emptyset \rangle$ from $\mathcal{I}'''$ by moving U from the uncertain part to the certain part. Since $\mathcal{I}^{*\prime}$ is in $\texttt{part}(\mathcal{I}^*)$ and A is stable-j in $\mathcal{I}^*$, A must be j in the certain projection of $\mathcal{I}^{*\prime}$.

⇐ Suppose that there exists some $\mathcal{I}' = \langle \mathcal{A}', \{U\}, \mathcal{R}', \emptyset \rangle \in \texttt{part}(\mathcal{I})$ such that A is not j in $\texttt{cert}(\mathcal{I}')$ and A is j in $\texttt{cert}(\langle \mathcal{A}' \cup \{U\}, \emptyset, \mathcal{R}', \emptyset \rangle)$. Given that $\langle \mathcal{A}' \cup \{U\}, \emptyset, \mathcal{R}', \emptyset \rangle$ has only one completion (i.e., its certain projection), A must be stable-j w.r.t. $\langle \mathcal{A}' \cup \{U\}, \emptyset, \mathcal{R}', \emptyset \rangle$. Consequently, there must be some minimal stable-j partial completion $\mathcal{I}'' = \langle \mathcal{A}'', \mathcal{A}^{?''}, \mathcal{R}'', \mathcal{R}^{?''} \rangle$ for A w.r.t. $\mathcal{I}$ such that $\langle \mathcal{A}' \cup \{U\}, \emptyset, \mathcal{R}', \emptyset \rangle \in \texttt{part}(\mathcal{I}'')$. Note that $U \in \mathcal{A}''$: otherwise $\langle \mathcal{A}', \emptyset, \mathcal{R}', \emptyset \rangle$ would also be in $\texttt{part}(\mathcal{I}'')$, which contradicts the assumption that A is not j in $\texttt{cert}(\langle \mathcal{A}', \{U\}, \mathcal{R}', \emptyset \rangle)$. To conclude, addition of U is j-relevant for A w.r.t. $\mathcal{I}$.

(2) This proof is analogous to the proof of item 1, except that U is moved between certain and uncertain attacks rather than between certain and uncertain arguments.
(3) We prove both directions of the third item.

⇒ The steps in this proof are similar to the steps in item (1).

(a) Suppose that removal of $U \in A^?$ is j-relevant for A w.r.t. $\mathcal{I}$.
(b) Then by Definition 12 there is a minimal stable-j partial completion $\mathcal{I}^* = \langle \mathcal{A}^*, \mathcal{A}^{?*}, \mathcal{R}^*, \mathcal{R}^{?*} \rangle$ for A w.r.t. $\mathcal{I}$ such that $U \notin \mathcal{A}^* \cup \mathcal{A}^{?*}$.
(c) Now construct the IAF $\mathcal{I}'$ by adding U to the uncertain arguments and adding the attacks from $\mathcal{R}$ related to U to the certain attacks: $\mathcal{I}' = \langle \mathcal{A}^*, \mathcal{A}^{?*} \cup \{U\}, \mathcal{R}^* \cup \{(X, Y) \in \mathcal{R} | U \in \{X, Y\} \text{ and } \{X, Y\} \subseteq \mathcal{A}^{?*} \cup \{U\}\}, \mathcal{R}^{?*} \rangle$.
Note that $\mathcal{I}^* \in \texttt{part}(\mathcal{I}')$.
(d) Given that $\mathcal{I}^*$ was minimal, A cannot be stable-j w.r.t. $\mathcal{I}'$, hence there must be some $\mathcal{I}'' = \langle \mathcal{A}'', \mathcal{A}^{?''}, \mathcal{R}'', \mathcal{R}^{?''} \rangle$ in $\texttt{part}(\mathcal{I}')$ such that A is not j in $\texttt{cert}(\mathcal{I}'')$.
(e) Then A's justification status in the certain projection of $\mathcal{I}''' = \langle \mathcal{A}'', \emptyset, \mathcal{R}'', \emptyset \rangle$ (which is the same as $\texttt{cert}(\mathcal{I}'')$,i.e. $\langle \mathcal{A}'', \mathcal{R}''|_{\mathcal{A}''} \rangle$) is not j either. Note that $\mathcal{A}''$ must contain U: otherwise, $\mathcal{I}'''$ would have been in $\texttt{part}(\mathcal{I}^*)$.
(f) Next, construct $\mathcal{I}^{*''} = \langle \mathcal{A}'' \setminus \{U\}, \emptyset, \mathcal{R}''|_{\mathcal{A}'' \setminus \{U\}}, \emptyset \rangle$ from $\mathcal{I}'''$ by removing U from the uncertain arguments. Since $\mathcal{I}^{*''}$ is in $\texttt{part}(\mathcal{I}^*)$ and A is stable-j in $\mathcal{I}^*$, A must be j in $\texttt{cert}(\mathcal{I}^{*\prime})$. This implies that A's justification status is j as well in the certain projection of $\mathcal{I}^{*\prime} = \langle \mathcal{A}'' \setminus \{U\}, \{U\}, \mathcal{R}'', \emptyset \rangle$, which is the same as $\texttt{cert}(\mathcal{I}^{*''})$.

$\Leftarrow$ Suppose that there exists some $\mathcal{I}' = \langle \mathcal{A}', \{U\}, \mathcal{R}', \emptyset \rangle \in \texttt{part}(\mathcal{I})$ such that A is j in $\texttt{cert}(\mathcal{I}')$ and A is not j in $\texttt{cert}(\langle \mathcal{A}' \cup \{U\}, \emptyset, \mathcal{R}', \emptyset \rangle)$. Then A is stable-j w.r.t. $\langle \mathcal{A}', \emptyset, \mathcal{R}', \emptyset \rangle$. Consequently, there must be some minimal stable-j partial completion $\mathcal{I}'' = \langle \mathcal{A}'', \mathcal{A}^{?\prime\prime}, \mathcal{R}'', \mathcal{R}^{?\prime\prime} \rangle$ for A w.r.t. $\mathcal{I}$ such that $\mathcal{I} \in \texttt{part}(\mathcal{I}'')$. Note that $U \notin \mathcal{A}'' \cup \mathcal{A}^{?\prime\prime}$: otherwise $\langle \mathcal{A}' \cup \{U\}, \emptyset, \mathcal{R}', \emptyset \rangle$ would be in $\texttt{part}(\mathcal{I}'')$, which contradicts the assumption that A is not j in $\texttt{cert}(\langle \mathcal{A}' \cup \{U\}, \emptyset, \mathcal{R}', \emptyset \rangle)$. To conclude, removal of U is j-relevant for A w.r.t. $\mathcal{I}$.

(4) This proof is analogous to the proof of item 3, except that U is moved between certain and uncertain attacks rather than between certain and uncertain arguments. $\square$

Lemma 8 (Reduction INSTABILITY to RELEVANCE)**.** *Given an incomplete argumentation framework $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$, a certain argument $A \in \mathcal{A}$, semantics $\sigma \in \{\text{GR}, \text{CP}, \text{PR}\}$ and $c \in \{sceptical, credulous\}$:*

(1) *Construct $\mathcal{I}'$ and $\mathcal{I}''$ as follows (see Fig. 12), where A', A'', U and U' are not in $\mathcal{A} \cup \mathcal{A}^?$:*

- $\mathcal{A}' = \mathcal{A} \cup \{A', A''\}$;
- $\mathcal{R}' = \mathcal{R} \cup \{(A, A'), (A', A''), (U, A')\}$;
- $\mathcal{I}' = \langle \mathcal{A}', \mathcal{A}^? \cup \{U\}, \mathcal{R}', \mathcal{R}^? \rangle$; *and*
- $\mathcal{I}'' = \langle \mathcal{A}' \cup \{U\}, \mathcal{A}^? \cup \{U'\}, \mathcal{R}' \cup \{(U', U)\}, \mathcal{R}^? \rangle$.

The following three items are equivalent:

(a) *A is **not** stable-σ-c-IN w.r.t. $\mathcal{I}$; and*
(b) *addition of U is σ-c-IN-relevant for A'' w.r.t. $\mathcal{I}'$; and*
(c) *removal of U' is σ-c-IN-relevant for A'' w.r.t. $\mathcal{I}''$.*

(2) *Let $\mathcal{I}' = \langle \mathcal{A}, \mathcal{A}^? \cup \{U\}, \mathcal{R} \cup \{(U, A)\}, \mathcal{R}^? \rangle$ and $\mathcal{I}'' = \langle \mathcal{A} \cup \{U\}, \mathcal{A}^? \cup \{U'\}, \mathcal{R} \cup \{(U, A), (U', U)\}, \mathcal{R}^? \rangle$ where U and U' are not in $\mathcal{A} \cup \mathcal{A}^?$. The following three items are equivalent:*

(a) *A is **not** stable-σ-c-OUT w.r.t. $\mathcal{I}$; and*
(b) *addition of U is σ-c-OUT-relevant for A w.r.t. $\mathcal{I}'$; and*
(c) *removal of U' is σ-c-OUT-relevant for A w.r.t. $\mathcal{I}''$.*

(3) *Construct $\mathcal{I}'$ and $\mathcal{I}''$ as follows, where A', A'', U and U' are not in $\mathcal{A} \cup \mathcal{A}^?$:*

- $\mathcal{A}' = \mathcal{A} \cup \{A', A''\}$;
- $\mathcal{R}' = \mathcal{R} \cup \{(A, A'), (A', A''), (U, A'), (U, A''), (U, U)\}$;
- $\mathcal{I}' = \langle \mathcal{A}', \mathcal{A}^? \cup \{U\}, \mathcal{R}', \mathcal{R}^? \rangle$; *and*
- $\mathcal{I}'' = \langle \mathcal{A}' \cup \{U\}, \mathcal{A}^? \cup \{U'\}, \mathcal{R}' \cup \{(U', U)\}, \mathcal{R}^? \rangle$.

The following three items are equivalent:

(a) *A is **not** stable-σ-c-UNDEC w.r.t. $\mathcal{I}$; and*
(b) *addition of U is σ-c-UNDEC-relevant for A'' w.r.t. $\mathcal{I}'$; and*
(c) *removal of U' is σ-c-UNDEC-relevant for A'' w.r.t. $\mathcal{I}''$.*

Proof. In the following, we prove all three items separately.

(1) We start by proving the first item, related to IN justification statuses. Let $\mathcal{I}' = \langle \mathcal{A} \cup \{A', A''\}, \mathcal{A}^? \cup \{U\}, \mathcal{R} \cup \{(A, A'), (A', A''), (U, A')\}, \mathcal{R}^? \rangle$ and let $\mathcal{I}'' = \langle \mathcal{A} \cup \{A', A'', U\}, \mathcal{A}^? \cup \{U'\}, \mathcal{R} \cup \{(A, A'), (A', A''), (U, A'), (U', U)\}, \mathcal{R}^? \rangle$ (see Fig. 12).

(a) ⇒ (b) and (c) Suppose that A is not stable-σ-c-IN w.r.t. $\mathcal{I}$ (**a**). Then by Definition 8 of stability there is some completion $AF^* = \langle \mathcal{A}^*, \mathcal{R}^* \rangle$ in which A is not σ-c-IN. Next, we construct three argumentation frameworks based on AF, containing the argument A'', and discuss its status.

- First, construct $AF_1 = \langle \mathcal{A}^* \cup \{A', A''\}, \mathcal{R}^* \cup \{(A, A'), (A', A'')\} \rangle$. Given that A'' is attacked by A', which is only attacked by A in AF_1, A'' cannot be σ-c-IN in AF_1.
- Next, construct $AF_2 = \langle \mathcal{A}^* \cup \{A', A'', U\}, \mathcal{R}^* \cup \{(A, A'), (A', A''), (U, A')\} \rangle$. A'' is σ-c-IN in AF_2, since the unattacked argument U attacks the only attacker of A'' (i.e. A').
- Let $AF_3 = \langle \mathcal{A}^* \cup \{A', A'', U, U'\}, \mathcal{R}^* \cup \{(A, A'), (A', A''), (U, A'), (U', U)\} \rangle$. Given that A is not σ-c-IN in AF^*, A cannot be σ-c-IN in AF_3 either. Since the argument A'' in AF_3 is attacked by A', which is only attacked by A, A'' cannot be σ-c-IN in AF_3.

Now item **(b)** (addition of U is σ-c-IN-relevant for A'' w.r.t. $\mathcal{I}'$) follows from Lemma 7, the fact that the incomplete argumentation framework $\langle \mathcal{A}^* \cup \{A', A''\}, \{U\}, \mathcal{R}^* \cup \{(A, A'), (A', A''), (U, A')\}, \emptyset \rangle$ is in $\texttt{part}(\mathcal{I}')$ and the status of A'' in AF_1 and AF_2.
Similarly, item **(c)** (removal of U' is σ-c-IN-relevant for A'' w.r.t. $\mathcal{I}''$) follows from Lemma 7, the fact that the incomplete argumentation framework $\langle \mathcal{A}^* \cup \{A', A'', U\}, \{U'\}, \mathcal{R}^* \cup \{(A, A'), (A', A''), (U, A'), (U', U)\}, \emptyset \rangle$ is in $\texttt{part}(\mathcal{I}'')$ and the status of A'' in AF_2 and AF_3.

(b) ⇒ (a) Suppose that addition of U is σ-c-IN-relevant for A'' w.r.t. $\mathcal{I}'$. Then by Lemma 7, there is some $\mathcal{I}^{*\prime}$ in $\texttt{part}(\mathcal{I}')$ such that A'' is **not** σ-c-IN in the certain projection of $\mathcal{I}^{*\prime}$ – let us call this certain projection $AF^{*\prime} = \langle \mathcal{A}^{*\prime}, \mathcal{R}^{*\prime} \rangle$. Then A cannot be σ-c-IN in $AF^{*\prime}$ either (since A attacks A', which is the only attacker of A''). Now construct $AF' = \langle \mathcal{A}', \mathcal{R}' \rangle$ where $\mathcal{A}' = \mathcal{A}^{*\prime} \setminus \{A', A'', U\}$ and $\mathcal{R}' = \mathcal{R}^{*\prime} \setminus \{(A, A'), (A', A''), (U, A')\}$. Since A is not σ-c-IN in $AF^{*\prime}$, it cannot be σ-c-IN in AF'. Given that AF' is a completion of $\mathcal{I}$, A cannot be stable-σ-c-IN w.r.t. $\mathcal{I}$.

(c) ⇒ (a) Suppose that removal of U' is σ-c-IN-relevant for A'' w.r.t. $\mathcal{I}''$. Then by Lemma 7, there is some $\mathcal{I}^{*\prime\prime}$ in $\texttt{part}(\mathcal{I}'')$ such that A'' is **not** σ-c-IN in the certain projection of $\mathcal{I}^{*\prime\prime}$ – without loss of generality, let this certain projection be $AF^{*\prime\prime} = \langle \mathcal{A}^{*\prime\prime}, \mathcal{R}^{*\prime\prime} \rangle$. Now construct $AF' = \langle \mathcal{A}', \mathcal{R}' \rangle$ where $\mathcal{A}' = \mathcal{A}^{*\prime\prime} \setminus \{A', A'', U, U'\}$ and $\mathcal{R}' = \mathcal{R}^{*\prime\prime} \setminus \{(A, A'), (A', A''), (U, A'), (U', U)\}$; since A'' is not σ-c-IN in $AF^{*\prime}$, it cannot be that A is σ-c-IN in AF'. Given that AF' is a completion of $\mathcal{I}$, A cannot be stable-σ-c-IN w.r.t. $\mathcal{I}$.

(2) We proceed by proving the second item, related to OUT justification statuses. Let $\mathcal{I}' = \langle \mathcal{A}, \mathcal{A}^? \cup \{U\}, \mathcal{R} \cup \{(U, A)\}, \mathcal{R}^? \rangle$ and let $\mathcal{I}'' = \langle \mathcal{A} \cup \{U\}, \mathcal{A}^? \cup \{U'\}, \mathcal{R} \cup \{(U, A), (U', U)\}, \mathcal{R}^? \rangle$ (see Fig. 16).

(a) ⇒ (b) and (c) Suppose that A is not stable-σ-c-OUT w.r.t. $\mathcal{I}$ (**a**). Then there is some completion $AF_1 = \langle \mathcal{A}^*, \mathcal{R}^* \rangle$ of $\mathcal{I}$ such that A is not σ-c-OUT. Next, we construct two additional argumentation frameworks based on AF_1, containing the argument A.

- First, construct $AF_2 = \langle \mathcal{A}^* \cup \{U\}, \mathcal{R}^* \cup \{(U, A)\} \rangle$. A is σ-c-OUT in AF_2, since the unattacked argument U attacks A.
- Let $AF_3 = \langle \mathcal{A}^* \cup \{U, U'\}, \mathcal{R}^* \cup \{(U, A), (U', U)\} \rangle$. Given that A is not σ-c-OUT in AF_1, A cannot be σ-c-OUT in AF_3 either.

Now item **(b)** (addition of U is σ-c-OUT-relevant for A w.r.t. $\mathcal{I}'$) follows from Lemma 7, the fact that the IAF $\langle \mathcal{A}^*, \{U\}, \mathcal{R}^* \cup \{(U, A)\}, \emptyset \rangle$ is in $\texttt{part}(\mathcal{I}')$ and the status of A in AF_1 and AF_2.

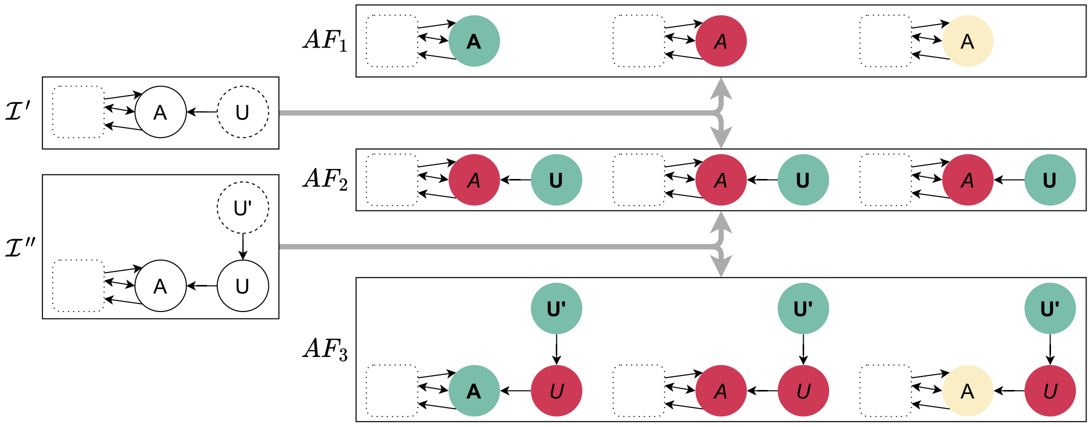

Fig. 16. Illustration of the IAF used to show Lemma 8 item 2. The IAFs given on the left are $\mathcal{I}'$ (upper) and $\mathcal{I}''$ (lower). The rounded rectangle with dotted borders represents the original IAF $\mathcal{I}$ (without A and in- and outgoing attacks). The grey arrows point to certain projections AF_1, AF_2 and AF_3 of partial completions. For each of these AFs, the possible justification statuses are colour-coded: green arguments with boldface font are IN, yellow arguments are UNDEC and red arguments with italic font are OUT. Note that, for a given justification status of A, there is only one possible justification status for each of the additional arguments in $\{U, U'\}$.

Similarly, item **(c)** (removal of U' is σ-c-OUT-relevant for A w.r.t. $\mathcal{I}''$) follows from Lemma 7, the fact that the IAF $\langle \mathcal{A}^* \cup \{U\}, \{U'\}, \mathcal{R}^* \cup \{(U, A), (U', U)\}, \emptyset\rangle$ is in $\texttt{part}(\mathcal{I}'')$ and the status of A in AF_2 and AF_3.

(b) $\Rightarrow$ **(a)** Suppose that addition of U is σ-c-OUT-relevant for A w.r.t. $\mathcal{I}'$. Then by Lemma 7, there is some $\mathcal{I}^{*\prime}$ in $\texttt{part}(\mathcal{I}')$ such that A is **not** σ-c-OUT in its certain projection $AF^{*\prime} = \langle \mathcal{A}', \mathcal{R}'\rangle$. Note that U cannot be in $\mathcal{A}'$: otherwise, A would be σ-c-OUT. Given that $AF^{*\prime}$ is a completion of $\mathcal{I}$, A cannot be stable-σ-c-OUT w.r.t. $\mathcal{I}$.

(c) $\Rightarrow$ **(a)** Suppose that removal of U' is σ-c-OUT-relevant for A w.r.t. $\mathcal{I}''$. Then by Lemma 7, there is some $\mathcal{I}^{*\prime\prime}$ in $\texttt{part}(\mathcal{I}'')$ such that A is **not** σ-c-OUT in $AF^{*\prime\prime} = \langle \mathcal{A}^{*\prime\prime}, \mathcal{R}^{*\prime\prime}\rangle$, the certain projection of $\mathcal{I}^{*\prime\prime}$. Now construct $AF' = \langle \mathcal{A}', \mathcal{R}'\rangle$ where $\mathcal{A}' = \mathcal{A}^{*\prime\prime} \setminus \{U, U'\}$ and $\mathcal{R}' = \mathcal{R}^{*\prime\prime} \setminus \{(U, A), (U', U)\}$; since A is not σ-c-OUT in $AF^{*\prime}$, it cannot be that A is σ-c-OUT in AF'. Given that AF' is a completion of $\mathcal{I}$, A cannot be stable-σ-c-OUT w.r.t. $\mathcal{I}$.

(3) Finally, we prove the third item, related to UNDEC justification statuses. Let $\mathcal{I}' = \langle \mathcal{A} \cup \{A', A''\}, \mathcal{A}^? \cup \{U\}, \mathcal{R} \cup \{(A, A'), (A', A''), (U, A'), (U, A''), (U, U)\}, \mathcal{R}^?\rangle$ and let $\mathcal{I}'' = \langle \mathcal{A} \cup \{A', A'', U\}, \mathcal{A}^? \cup \{U'\}, \mathcal{R} \cup \{(A, A'), (A', A''), (U, A'), (U, A''), (U, U), (U', U)\}, \mathcal{R}^?\rangle$ (see Fig. 17).

(a) $\Rightarrow$ **(b) and (c)** If A is not stable-σ-c-UNDEC w.r.t. $\mathcal{I}$ **(a)** then there is some completion $AF^* = \langle \mathcal{A}^*, \mathcal{R}^*\rangle$ of $\mathcal{I}$ in which A is not σ-c-UNDEC. Next, we construct three argumentation frameworks based on AF^*, containing the argument A'', and discuss its status.

- First, construct $AF_1 = \langle \mathcal{A}^* \cup \{A', A''\}, \mathcal{R}^* \cup \{(A, A'), (A', A'')\}\rangle$. Given that A'' is attacked by A', which is only attacked by A in AF^*, A'' cannot be σ-c-UNDEC in AF_1.
- Next, construct $AF_2 = \langle \mathcal{A}^* \cup \{A', A'', U\}, \mathcal{R}^* \cup \{(A, A'), (A', A''), (U, A'), (U, A''), (U, U)\}\rangle$. A'' is σ-c-UNDEC in AF_2, since the self-attacking argument U attacks each attacker of A'' (i.e. A' and U itself).
- Let $AF_3 = \langle \mathcal{A}^* \cup \{A', A'', U, U'\}, \mathcal{R}^* \cup \{(A, A'), (A', A''), (U, A'), (U, A''), (U, U), (U', U)\}\rangle$. Given that A is not σ-c-UNDEC in AF^*, A cannot be σ-c-UNDEC in AF_3 either.

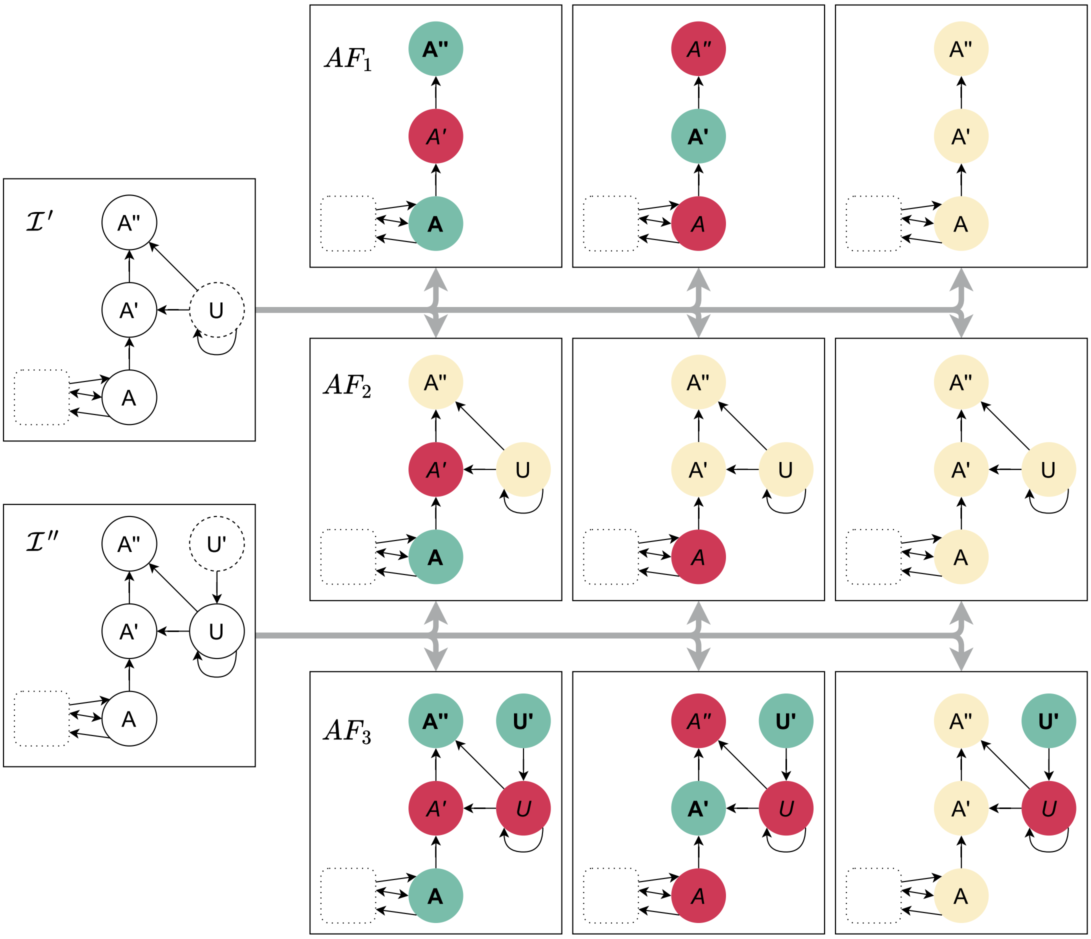

Fig. 17. Illustration of the IAF used to show Lemma 8 item 3. The IAFs given on the left are $\mathcal{I}'$ (upper) and $\mathcal{I}''$ (lower). The rounded rectangle with dotted borders represents the original IAF $\mathcal{I}$ (without A and in- and outgoing attacks). The grey arrows point to certain projections AF_1, AF_2 and AF_3 of partial completions. For each of these AFs, the possible justification statuses are colour-coded: green arguments with boldface font are IN, yellow arguments are UNDEC and red arguments with italic font are OUT. Note that, for a given justification status of A, there is only one possible justification status for each of the additional arguments in $\{A', A'', U, U'\}$.

Since the argument A'' in AF_3 is, apart from the argument U that is definitely σ-c-OUT, attacked by A', which is only attacked by A, A'' cannot be σ-c-UNDEC in AF_3.

Now item **(b)** (addition of U is σ-c-UNDEC-relevant for A'' w.r.t. $\mathcal{I}'$) follows from Lemma 7, the fact that the incomplete argumentation framework $\langle \mathcal{A}^* \cup \{A', A''\}, \{U\}, \mathcal{R}^* \cup \{(A, A'), (A', A''), (U, A'), (U, A''), (U, U)\}, \emptyset\rangle$ is in $\mathtt{part}(\mathcal{I}')$ and the status of A'' in AF_1 and AF_2.

Similarly, item **(c)** (removal of U' is σ-c-UNDEC-relevant for A'' w.r.t. $\mathcal{I}''$) follows from Lemma 7, the fact that the incomplete argumentation framework $\langle \mathcal{A}^* \cup \{A', A'', U\}, \{U'\}, \mathcal{R}^* \cup \{(A, A'), (A', A''), (U, A'), (U, A''), (U, U), (U', U)\}, \emptyset\rangle$ is in $\mathtt{part}(\mathcal{I}'')$ and the status of A'' in AF_2 and AF_3.

(b) ⇒ (a) Suppose that addition of U is σ-c-UNDEC-relevant for A'' w.r.t. $\mathcal{I}'$. Then by Lemma 7, there is some $\mathcal{I}^{*\prime}$ in $\texttt{part}(\mathcal{I}')$ such that A'' is **not** σ-c-UNDEC in the certain projection of $\mathcal{I}^{*\prime}$ – let us call this certain projection $AF^{*\prime} = \langle \mathcal{A}^{*\prime}, \mathcal{R}^{*\prime} \rangle$. Then A cannot be σ-c-UNDEC in $AF^{*\prime}$ either. Now construct $AF' = \langle \mathcal{A}', \mathcal{R}' \rangle$ where $\mathcal{A}' = \mathcal{A}^{*\prime} \setminus \{A', A'', U\}$ and $\mathcal{R}' = \mathcal{R}^{*\prime} \setminus \{(A, A'), (A', A''), (U, A'), (U, A''), (U, U)\}$. Since A is not σ-c-UNDEC in $AF^{*\prime}$, it cannot be σ-c-UNDEC in AF'. Given that AF' is a completion of $\mathcal{I}$, A cannot be stable-σ-c-UNDEC w.r.t. $\mathcal{I}$.

(c) ⇒ (a) If removal of U' is σ-c-UNDEC-relevant for A'' w.r.t. $\mathcal{I}''$ then by Lemma 7, there is some $\mathcal{I}^{*\prime\prime}$ in $\texttt{part}(\mathcal{I}'')$ such that A'' is **not** σ-c-UNDEC in the certain projection of $\mathcal{I}^{*\prime\prime}$ – let this certain projection be $AF^{*\prime\prime} = \langle \mathcal{A}^{*\prime\prime}, \mathcal{R}^{*\prime\prime} \rangle$. Now construct $AF' = \langle \mathcal{A}', \mathcal{R}' \rangle$ where $\mathcal{A}' = \mathcal{A}^{*\prime\prime} \setminus \{A', A'', U, U'\}$ and $\mathcal{R}' = \mathcal{R}^{*\prime\prime} \setminus \{(A, A'), (A', A''), (U, A'), (U, A''), (U, U), (U', U)\}$; since A'' is not σ-c-UNDEC in $AF^{*\prime}$, it cannot be that A is σ-c-UNDEC in AF'. Given that AF' is a completion of $\mathcal{I}$, A cannot be stable-σ-c-UNDEC w.r.t. $\mathcal{I}$. □

Proposition 12. *The following problems are NP-complete:*

(1) CP-*credulous*-UNDEC-RELEVANCE*;*
(2) CP-*sceptical*-IN-RELEVANCE*;*
(3) CP-*sceptical*-OUT-RELEVANCE*;*
(4) GR-*credulous*-IN-RELEVANCE*;*
(5) GR-*credulous*-OUT-RELEVANCE*;*
(6) GR-*credulous*-UNDEC-RELEVANCE*;*
(7) GR-*sceptical*-IN-RELEVANCE*;*
(8) GR-*sceptical*-OUT-RELEVANCE*; and*
(9) GR-*sceptical*-UNDEC-RELEVANCE*.*

Proof. For each of these problems, membership in NP follows from membership in P of the corresponding JUSTIFICATION problems [10], listed in Table 1, and Proposition 11 (since NP = NP^{P}).

For NP-hardness, we can give a reduction from the complementary of the corresponding STABILITY problem, using Lemma 8.

(1) By Proposition 6, CP-credulous-UNDEC-STABILITY is CoNP-complete, which means that the complementary problem CP-credulous-UNDEC-INSTABILITY is NP-complete. By Lemma 8 item 3, each instance $I = \langle \mathcal{I}, A \rangle$ of CP-credulous-UNDEC-INSTABILITY can be transformed into an instance $I' = \langle \mathcal{I}', A'', U \rangle$ (for addition) or $I' = \langle \mathcal{I}'', A'', U' \rangle$ (for removal) such that I is a positive instance of CP-credulous-UNDEC-INSTABILITY iff I' is a positive instance of CP-credulous-UNDEC-RELEVANCE.
(2) The problem CP-sceptical-IN-STABILITY is CoNP-complete [3], hence CP-sceptical-IN-INSTABILITY is NP-complete. By Lemma 8 item 1, each instance $I = \langle \mathcal{I}, A \rangle$ of CP-sceptical-IN-INSTABILITY can be transformed into an instance $I' = \langle \mathcal{I}', A'', U \rangle$ (for addition) or $I' = \langle \mathcal{I}'', A'', U' \rangle$ (for removal) such that I is a positive instance of CP-sceptical-IN-INSTABILITY iff I' is a positive instance of CP-sceptical-IN-RELEVANCE.
(3) The problem CP-sceptical-OUT-STABILITY is CoNP-complete by [3] in combination with Lemma 5, hence CP-sceptical-OUT-INSTABILITY is NP-complete. By Lemma 8 item 2, each instance $I = \langle \mathcal{I}, A \rangle$ of CP-sceptical-OUT-INSTABILITY can be transformed into an instance $I' = \langle \mathcal{I}', A, U \rangle$ (for addition) or $I' = \langle \mathcal{I}'', A, U' \rangle$ (for removal) such that I is a positive instance of CP-sceptical-OUT-INSTABILITY iff I' is a positive instance of CP-sceptical-OUT-RELEVANCE.

(4) The problem GR-credulous-IN-STABILITY is CoNP-complete [3], hence GR-credulous-IN-INSTABILITY is NP-complete. By Lemma 8 item 1, each instance $I = \langle \mathcal{I}, A \rangle$ of GR-credulous-IN-INSTABILITY can be transformed into an instance $I' = \langle \mathcal{I}', A'', U \rangle$ (for addition) or $I' = \langle \mathcal{I}'', A'', U' \rangle$ (for removal) such that I is a positive instance of GR-credulous-IN-INSTABILITY iff I' is a positive instance of GR-credulous-IN-RELEVANCE.

(5) The problem GR-credulous-OUT-STABILITY is CoNP-complete by [3] in combination with Lemma 5, hence GR-credulous-OUT-INSTABILITY is NP-complete. By Lemma 8 item 2, each instance $I = \langle \mathcal{I}, A \rangle$ of CP-sceptical-OUT-INSTABILITY can be transformed into an instance $I' = \langle \mathcal{I}', A, U \rangle$ (for addition) or $I' = \langle \mathcal{I}'', A, U' \rangle$ (for removal) such that I is a positive instance of GR-credulous-OUT-INSTABILITY iff I' is a positive instance of GR-credulous-OUT-RELEVANCE.

(6) By Proposition 6, GR-credulous-UNDEC-STABILITY is CoNP-complete, which means that the complementary problem GR-credulous-UNDEC-INSTABILITY is NP-complete. By Lemma 8 item 3, each instance $I = \langle \mathcal{I}, A \rangle$ of GR-credulous-UNDEC-INSTABILITY can be transformed into an instance $I' = \langle \mathcal{I}', A'', U \rangle$ (for addition) or $I' = \langle \mathcal{I}'', A'', U' \rangle$ (for removal) such that I is a positive instance of GR-credulous-UNDEC-INSTABILITY iff I' is a positive instance of GR-credulous-UNDEC-RELEVANCE.

(7) The problem GR-sceptical-IN-STABILITY is CoNP-complete [3], hence GR-sceptical-IN-INSTABILITY is NP-complete. By Lemma 8 item 1, each instance $I = \langle \mathcal{I}, A \rangle$ of GR-sceptical-IN-INSTABILITY can be transformed into an instance $I' = \langle \mathcal{I}', A'', U \rangle$ (for addition) or $I' = \langle \mathcal{I}'', A'', U' \rangle$ (for removal) such that I is a positive instance of GR-sceptical-IN-INSTABILITY iff I' is a positive instance of GR-sceptical-IN-RELEVANCE.

(8) The problem GR-sceptical-OUT-STABILITY is CoNP-complete by [3] in combination with Lemma 5, hence GR-sceptical-OUT-INSTABILITY is NP-complete. By Lemma 8 item 2, each instance $I = \langle \mathcal{I}, A \rangle$ of GR-sceptical-OUT-INSTABILITY can be transformed into an instance $I' = \langle \mathcal{I}', A, U \rangle$ (for addition) or $I' = \langle \mathcal{I}'', A, U' \rangle$ (for removal) such that I is a positive instance of GR-sceptical-OUT-INSTABILITY iff I' is a positive instance of GR-sceptical-OUT-RELEVANCE.

(9) By Proposition 6, GR-sceptical-UNDEC-STABILITY is CoNP-complete, which means that the complementary problem GR-sceptical-UNDEC-INSTABILITY is NP-complete. By Lemma 8 item 3, each instance $I = \langle \mathcal{I}, A \rangle$ of GR-sceptical-UNDEC-INSTABILITY can be transformed into an instance $I' = \langle \mathcal{I}', A'', U \rangle$ (for addition) or $I' = \langle \mathcal{I}'', A'', U' \rangle$ (for removal) such that I is a positive instance of GR-sceptical-UNDEC-INSTABILITY iff I' is a positive instance of GR-sceptical-UNDEC-RELEVANCE. □

Lemma 9 (Reduction co-inverted-STABILITY to RELEVANCE). *Given an incomplete argumentation framework $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$, a certain argument $A \in \mathcal{A}$, semantics $\sigma \in \{$GR, CP, PR$\}$:*

(1) *Construct $\mathcal{I}'$ and $\mathcal{I}''$ as follows, where A', U and U' are fresh arguments not in $\mathcal{A} \cup \mathcal{A}^?$:*

- $\mathcal{A}' = \mathcal{A} \cup \{A'\}$;
- $\mathcal{R}' = \mathcal{R} \cup \{(A, A'), (A', A'), (U, A')\}$;
- $\mathcal{I}' = \langle \mathcal{A}', \mathcal{A}^? \cup \{U\}, \mathcal{R}', \mathcal{R}^? \rangle$; *and*
- $\mathcal{I}'' = \langle \mathcal{A}' \cup \{U\}, \mathcal{A}^? \cup \{U'\}, \mathcal{R}' \cup \{(U', U)\}, \mathcal{R}^? \rangle$.

The following items are equivalent:

(a) *A is **not** stable-σ-**credulous**-**IN** w.r.t. $\mathcal{I}$; and*
(b) *addition of U' is σ-**sceptical**-**UNDEC**-relevant for A' w.r.t. $\mathcal{I}''$; and*

(c) *removal of U is σ-**sceptical**-*UNDEC*-relevant for A' w.r.t. $\mathcal{I}'$.*

(2) *Construct $\mathcal{I}'$ and $\mathcal{I}''$ as follows, where A_1, A_2, A_3, A_4, U and U' are fresh arguments not in $\mathcal{A} \cup \mathcal{A}^?$:*

- $\mathcal{A}' = \mathcal{A} \cup \{A_1, A_2, A_3, A_4\}$;
- $\mathcal{R}' = \mathcal{R} \cup \{(A, A_1), (A, A_2), (A_1, A_2), (A_2, A_3), (A_3, A_4), (U, U), (U, A_3)\}$;
- $\mathcal{I}' = \langle \mathcal{A}', \mathcal{A}^? \cup \{U\}, \mathcal{R}', \mathcal{R}^? \rangle$; *and*
- $\mathcal{I}'' = \langle \mathcal{A}' \cup \{U\}, \mathcal{A}^? \cup \{U'\}, \mathcal{R}' \cup \{(U', U)\}, \mathcal{R}^? \rangle$.

The following items are equivalent:

(a) *A is **not** stable-σ-**credulous**-*UNDEC* w.r.t. $\mathcal{I}$; and*
(b) *addition of U' is σ-**sceptical**-*IN*-relevant for A_3 w.r.t. $\mathcal{I}''$; and*
(c) *removal of U is σ-**sceptical**-*IN*-relevant for A_3 w.r.t. $\mathcal{I}'$; and*
(d) *addition of U' is σ-**sceptical**-*OUT*-relevant for A_4 w.r.t. $\mathcal{I}''$; and*
(e) *removal of U is σ-**sceptical**-*OUT*-relevant for A_4 w.r.t. $\mathcal{I}'$.*

Proof. We prove these two items separately.

(1) Let $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$ be an IAF, $A \in \mathcal{A}$ be a certain argument and σ some semantics in $\{$GR, CP, PR$\}$. Construct $\mathcal{I}'$ and $\mathcal{I}''$ as specified above (see also Fig. 18).

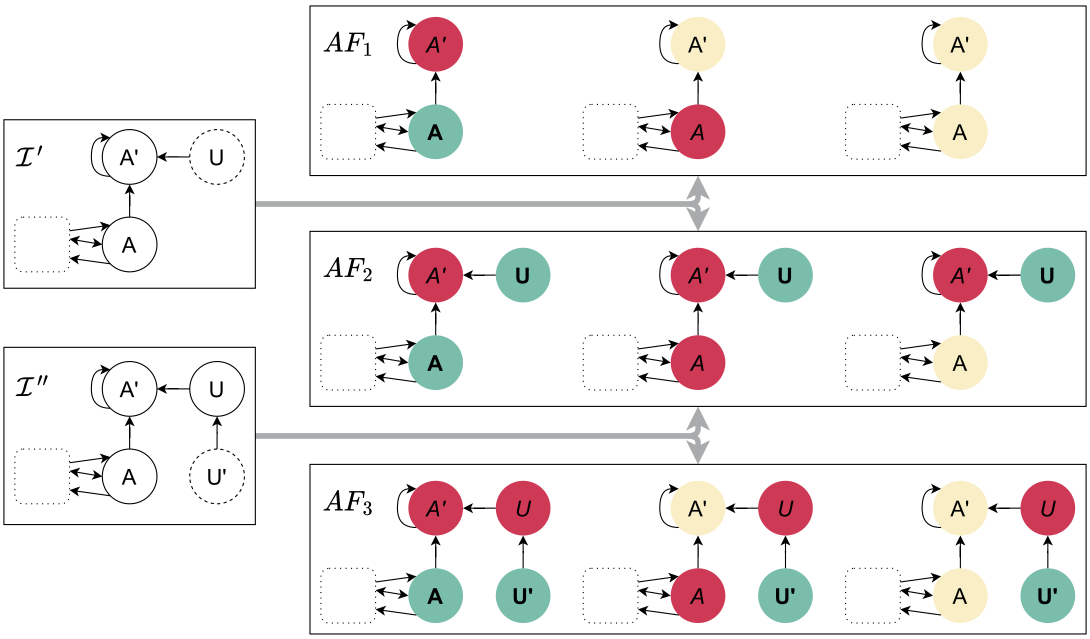

Fig. 18. Illustration of the IAF used to show Lemma 9 item 1. The IAFs given on the left are $\mathcal{I}'$ (upper) and $\mathcal{I}''$ (lower). The rounded rectangle with dotted borders represents the original IAF $\mathcal{I}$ (without A and in- and outgoing attacks). The grey arrows point to certain projections AF_1, AF_2 and AF_3 of partial completions. For each of these AFs, the possible justification statuses are colour-coded: green arguments with boldface font are IN, yellow arguments are UNDEC and red arguments with italic font are OUT. Note that, for a given justification status of A, there is only one possible justification status for each of the additional arguments in $\{A', U, U'\}$.

(a) ⇒ (b) and (c) Suppose that A is **not** stable-σ-**credulous-IN** w.r.t. $\mathcal{I}$. Then by Definition 8 of stability there is some completion $AF^* = \langle \mathcal{A}^*, \mathcal{R}^* \rangle$ of $\mathcal{I}$ such that A is not in any σ-extension. Next, construct and consider the following three AFs:

- Let $AF_1 = \langle \mathcal{A}^* \cup \{A'\}, \mathcal{R}^* \cup \{(A, A'), (A', A')\} \rangle$. Given that A is not in any σ-extension of $\langle \mathcal{A}^*, \mathcal{R}^* \rangle$ and A' is self-attacking, A' cannot be in, nor attacked by any argument in any σ-extension of AF_1. Consequently, A' is σ-**sceptical-UNDEC** in AF_1.
- Now consider $AF_2 = \langle \mathcal{A}^* \cup \{A', U\}, \mathcal{R}^* \cup \{(A, A'), (A', A'), (U, A')\} \rangle$. Since A' is attacked by the unattacked argument U, A' is attacked by an argument in some σ-extension of AF_2 and therefore **not** σ-**sceptical-UNDEC** in AF_2.
- Finally construct $AF_3 = \langle \mathcal{A}^* \cup \{A', U, U'\}, \mathcal{R}^* \cup \{(A, A'), (A', A'), (U, A'), (U', U)\} \rangle$. Note that U is attacked by the unattacked argument U', which means that U' is in each σ-extension of AF_3. Consequently, none of the arguments attacking A' (A, U and A' itself) is in any σ-extension of AF_3. In addition, A' cannot be in any σ-extension of AF_3 as it is self-attacking. This implies that A' is σ-**sceptical-UNDEC** in AF_3.

We continue proving item **(b)**. Note that the partial completion $\langle \mathcal{A}^* \cup \{A', U\}, \{U'\}, \mathcal{R}^* \cup \{(A, A'), (A', A'), (U, A'), (U', U)\}, \emptyset \rangle$ is in $\texttt{part}(\mathcal{I}'')$. Then by Lemma 7 and the status of A' in AF_2 and AF_3, addition of U' is σ-**sceptical-UNDEC**-relevant for A' w.r.t. $\mathcal{I}'$.
Item **(c)** follows from the fact that $\langle \mathcal{A}^* \cup \{A'\}, \{U\}, \mathcal{R}^* \cup \{(A, A'), (A', A'), (U, A')\}, \emptyset \rangle$ is in $\texttt{part}(\mathcal{I}')$, Lemma 7 and the status of A' in AF_1 and AF_2.

(b) ⇒ (a) Suppose that addition of U' is σ-**sceptical-UNDEC**-relevant for A' w.r.t. $\mathcal{I}''$. Then by Lemma 7, there is some $\mathcal{I}^*$ in $\texttt{part}(\mathcal{I}'')$ such that A' is σ-sceptical-UNDEC w.r.t. its certain projection $AF^* = \langle \mathcal{A}^*, \mathcal{R}^* \rangle$. This implies that A is not in any σ-extension of AF^* (otherwise, A' would be out in that extension). Now construct $AF' = \langle \mathcal{A}', \mathcal{R}' \rangle$ where $\mathcal{A}' = \mathcal{A}^* \setminus \{A', U, U'\}$ and $\mathcal{R}' = \mathcal{R}^* \setminus \{(A, A'), (A', A'), (U, A'), (U', U)\}$. Since A' is not in any σ-extension of AF^*, it cannot be in any σ-extension of AF' either (because all arguments defending A in $\mathcal{A}^*$ are also in $\mathcal{A}'$). Therefore, given that AF' is a completion of $\mathcal{I}$, A is not stable-σ-credulous-IN w.r.t. $\mathcal{I}$.

(c) ⇒ (a) Finally, suppose that removal of U is σ-**sceptical-UNDEC**-relevant for A' w.r.t. $\mathcal{I}'$. Then by Lemma 7, there is some $\mathcal{I}^*$ in $\texttt{part}(\mathcal{I}')$ such that A' is σ-sceptical-UNDEC w.r.t. its certain projection $AF^* = \langle \mathcal{A}^*, \mathcal{R}^* \rangle$. Consequently, A cannot be in any σ-extension of AF^*. Construct $AF' = \langle \mathcal{A}', \mathcal{R}' \rangle$ where $\mathcal{A}' = \mathcal{A}^* \setminus \{A', U\}$ and $\mathcal{R}' = \mathcal{R}^* \setminus \{(A, A'), (A', A'), (U, A')\}$ and note that A cannot be in any σ-extension of AF either. Given that AF' is a completion of $\mathcal{I}$, A cannot be stable-σ-credulous-IN w.r.t. $\mathcal{I}$.

(2) Let $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$ be an IAF, $A \in \mathcal{A}$ be a certain argument and σ some semantics in $\{\text{GR}, \text{CP}, \text{PR}\}$. Construct $\mathcal{I}'$ and $\mathcal{I}''$ as specified above (see also Fig. 19).

(a) ⇒ (b), (c), (d) and (e) Suppose that A is **not** stable-σ-**credulous-UNDEC** w.r.t. $\mathcal{I}$. Then there is some completion $AF^* = \langle \mathcal{A}^*, \mathcal{R}^* \rangle$ of $\mathcal{I}$ such that for each σ-extension, A is either in, or attacked by some argument in this extension. Next, construct and consider the following three AFs:

- Let $AF_1 = \langle \mathcal{A}_1, \mathcal{R}_1 \rangle$ where $\mathcal{A}_1 = \mathcal{A}^* \cup \{A_1, A_2, A_3, A_4\}$ and $\mathcal{R}_1 = \mathcal{R}^* \cup \{(A, A_1), (A, A_2), (A_1, A_2), (A_2, A_3), (A_3, A_4)\}$. Consider an arbitrary σ-extension S of AF_1 and let $S' = S \setminus \{A_1, A_2, A_3, A_4\}$. Note that S' must be a σ-extension of AF^*. Recall

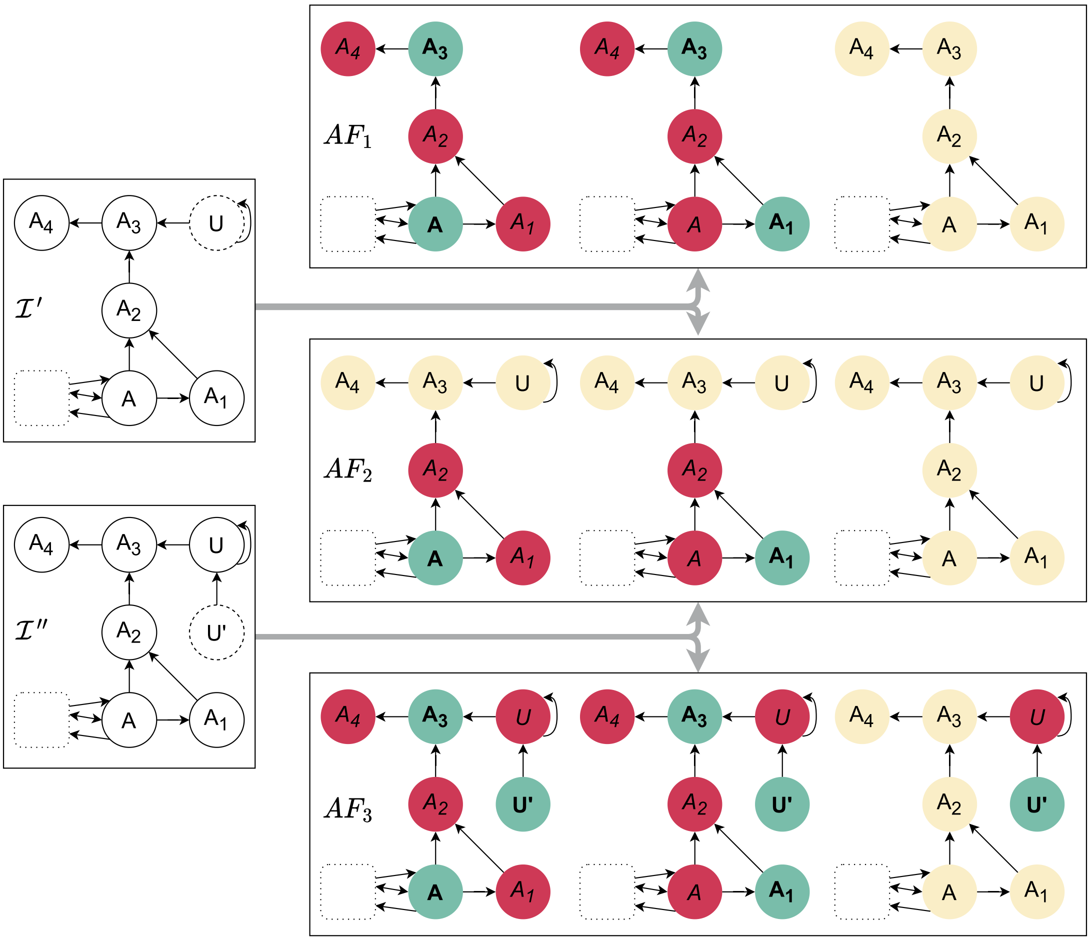

Fig. 19. Illustration of the IAF used to show Lemma 9 item 2. The IAFs given on the left are $\mathcal{I}'$ (upper) and $\mathcal{I}''$ (lower). The rounded rectangle with dotted borders represents the original IAF $\mathcal{I}$ (without A and in- and outgoing attacks). The grey arrows point to certain projections AF_1, AF_2 and AF_3 of partial completions. For each of these AFs, the possible justification statuses are colour-coded: green arguments with boldface font are IN, yellow arguments are UNDEC and red arguments with italic font are OUT. Note that, for a given justification status of A, there is only one possible justification status for each of the additional arguments in $\{A_1, A_2, A_3, A_4, U, U'\}$.

that A is either in, or attacked by an argument in S'. This implies that A is either in S or attacked by some argument in S. In both cases, A_2 is attacked by an argument in S (either A or A_1), which by completeness of σ semantics implies that $A_3 \in S$, while A_4 is attacked by an argument (A_3) in S. Thus A_3 is σ-**sceptical-IN** in AF_1 and A_4 is σ-**sceptical-OUT** in AF_1.

- Now consider $AF_2 = \langle \mathcal{A}_2, \mathcal{R}_2 \rangle$ where $\mathcal{A}_2 = \mathcal{A}^* \cup \{A_1, A_2, A_3, A_4, U\}$ and $\mathcal{R}_2 = \mathcal{R}^* \cup \{(A, A_1), (A, A_2), (A_1, A_2), (A_2, A_3), (A_3, A_4), (U, U), (U, A_3)\}$. Since AF_2 has at least one extension under σ semantics, there is some σ extension S. Given that A_3 is attacked by the self-attacking argument U, which is not attacked by any other argument, A_3 cannot be in S and A_4 is not attacked by any argument in S. This implies that A_3 is **not** σ-**sceptical-IN** in AF_2 and A_4 is **not** σ-**sceptical-OUT** in AF_2.

- Finally construct $AF_3 = \langle \mathcal{A}_3, \mathcal{R}_3 \rangle$ where $\mathcal{A}_3 = \mathcal{A}^* \cup \{A_1, A_2, A_3, A_4, U, U'\}$ and $\mathcal{R}_3 = \mathcal{R}^* \cup \{(A, A_1), (A, A_2), (A_1, A_2), (A_2, A_3), (A_3, A_4), (U, U), (U, A_3), (U', U)\}$. Consider an arbitrary σ-extension S of AF_3 and let $S' = S \setminus \{A_1, A_2, A_3, A_4, U, U'\}$. Note that S' must be a σ-extension of AF^*. Given that A is in S' or attacked by an argument in S', it must be that either A is in S and A_1 is attacked by an argument in S or A is attacked by an argument in S and A_1 is in S. In both cases, A_2 is attacked by an argument in S. Given that U' is unattacked as well, U is also attacked by an argument in S. This implies that A_3 is in S and A_4 is attacked by an argument in S. In other words, A_3 is σ-**sceptical-IN** in AF_3 and A_4 is σ-**sceptical-OUT** in AF_3.

We continue proving items **(b)** and **(d)**. Note that the partial completion $\langle \mathcal{A}_2, \{U'\}, \mathcal{R}_3, \emptyset \rangle$ is in $\texttt{part}(\mathcal{I}'')$. Then by Lemma 7 and the status of A_3 in AF_2 and AF_3, addition of U' is σ-**sceptical-IN**-relevant for A_3 w.r.t. $\mathcal{I}''$ and σ-**sceptical-OUT**-relevant for A_4 w.r.t. $\mathcal{I}''$.
Items **(c)** and **(e)** follow from the fact that $\langle \mathcal{A}_1, \{U\}, \mathcal{R}_2, \emptyset \rangle$ is in $\texttt{part}(\mathcal{I}')$, Lemma 7 and the justification statuses of A_3 and A_4 in AF_1 and AF_2.

(d) $\Rightarrow$ **(b)** If addition of U' is σ-sceptical-out-relevant for A_4 w.r.t. $\mathcal{I}''$ then by Lemma 7 there exists some $\mathcal{I}^* = \langle \mathcal{A}', \{U\}, \mathcal{R}', \emptyset \rangle \in \texttt{part}(\mathcal{I}'')$ such that A_4 is **not** σ-sceptical-out-relevant in the certain projection of $\mathcal{I}^*$, while A_4 is σ-sceptical-out-relevant in the certain projection of $\langle \mathcal{A}' \cup \{U\}, \emptyset, \mathcal{R}', \emptyset \rangle$. Given that A_4 is (only) attacked by A_3, A_3 is **not** σ-sceptical-in-relevant in the certain projection of $\mathcal{I}^*$, while A_3 is σ-sceptical-in-relevant in the certain projection of $\langle \mathcal{A}' \cup \{U\}, \emptyset, \mathcal{R}', \emptyset \rangle$. By Lemma 7, addition of U' is σ-sceptical-in-relevant for A_3 w.r.t. $\mathcal{I}''$.

(e) $\Rightarrow$ **(c)** Similarly, if removal of U is σ-sceptical-out-relevant for A_4 w.r.t. $\mathcal{I}'$ then by Lemma 7 removal of U is σ-sceptical-in-relevant for A_3 w.r.t. $\mathcal{I}'$.

(b) $\Rightarrow$ **(a)** Suppose that addition of U' is σ-**sceptical-IN**-relevant for A_3 w.r.t. $\mathcal{I}''$. Then by Lemma 7, there is some $\mathcal{I}^*$ in $\texttt{part}(\mathcal{I}'')$ such that A_3 is σ-sceptical-IN w.r.t. its certain projection $AF^* = \langle \mathcal{A}^*, \mathcal{R}^* \rangle$. Construct $AF' = \langle \mathcal{A}', \mathcal{R}' \rangle$ where $\mathcal{A}' = \mathcal{A}^* \setminus \{A_1, A_2, A_3, A_4, U, U'\}$ and $\mathcal{R}' = \mathcal{R}^* \setminus \{(A, A_1), (A, A_2), (A_1, A_2), (A_2, A_3), (A_3, A_4), (U, U), (U, A_3), (U', U)\}$. Now suppose, towards a contradiction, that A is stable-σ-credulous-UNDEC w.r.t. $\mathcal{I}$. Then there would be some σ extension S' of AF' such that $A \notin S'$ and A is not attacked by any argument in S' (since AF' is a completion of $\mathcal{I}$). Reconsidering AF^*, the set $S = S' \cup \{U'\}$ would be a σ extension of AF^*. Note that A_3 is not in S, which contradicts our assumption that A_3 is σ-sceptical-IN w.r.t. AF^*. Hence A is not stable-σ-credulous-UNDEC w.r.t. $\mathcal{I}$.

(c) $\Rightarrow$ **(a)** Finally, suppose that removal of U is σ-**sceptical-UNDEC**-relevant for A_3 w.r.t. $\mathcal{I}'$. Then by Lemma 7, there is some $\mathcal{I}^*$ in $\texttt{part}(\mathcal{I}')$ such that A_3 is σ-sceptical-UNDEC w.r.t. its certain projection $AF^* = \langle \mathcal{A}^*, \mathcal{R}^* \rangle$. Construct $AF' = \langle \mathcal{A}', \mathcal{R}' \rangle$ where $\mathcal{A}' = \mathcal{A}^* \setminus \{A_1, A_2, A_3, A_4, U\}$ and $\mathcal{R}' = \mathcal{R}^* \setminus \{(A, A_1), (A, A_2), (A_1, A_2), (A_2, A_3), (A_3, A_4), (U, U), (U, A_3)\}$. If A would be stable-σ-credulous-UNDEC w.r.t. $\mathcal{I}$, then there would be some σ extension S' of AF' such that $A \notin S'$ and A is not attacked by any argument in S' (since AF' is a completion of $\mathcal{I}$), thus $S = S' \cup \{U'\}$ would be a σ extension of AF^* not containing A_3. Since this contradicts our earlier assumption, A cannot be in any σ-extension of AF^*. □

Lemma 10. *Let (ϕ, X, Y) be an instance of Σ_2-SAT and let $\phi = \bigwedge_i c_i$ and $c_i = \bigvee_j \alpha_j$ for each clause c_i in ϕ, where α_j are the literals over $X \cup Y$ that occur in clause c_i. Now let $\mathcal{I}_1 = T_1(\phi, X, Y)$ and let*

$\mathcal{I}_2 = T_2(\phi, X, Y)$, using the transformations T_1 and T_2 specified in Definition 13. The following items are equivalent:

(1) *(ϕ, X, Y) is a positive instance of Σ_2-SAT;*
(2) *Removal of χ is* ST*-sceptical-*IN*-relevant for $\overline{\phi}$ w.r.t. $\mathcal{I}_1$;*
(3) *Addition of χ is* ST*-credulous-*IN*-relevant for ϕ w.r.t. $\mathcal{I}_1$;*
(4) *Addition of $\overline{\chi}$ is* ST*-sceptical-*IN*-relevant for $\overline{\phi}$ w.r.t. $\mathcal{I}_2$;*
(5) *Removal of $\overline{\chi}$ is* ST*-credulous-*IN*-relevant for ϕ w.r.t. $\mathcal{I}_2$;*
(6) *Removal of χ is* ST*-sceptical-*OUT*-relevant for ϕ w.r.t. $\mathcal{I}_1$;*
(7) *Addition of χ is* ST*-credulous-*OUT*-relevant for $\overline{\phi}$ w.r.t. $\mathcal{I}_1$;*
(8) *Addition of $\overline{\chi}$ is* ST*-sceptical-*OUT*-relevant for ϕ w.r.t. $\mathcal{I}_2$; and*
(9) *Removal of $\overline{\chi}$ is* ST*-credulous-*OUT*-relevant for $\overline{\phi}$ w.r.t. $\mathcal{I}_2$.*

Proof. We introduce an auxiliary statement, for which we prove that it equals all of the items above:

(0) There is some $\mathcal{I}^* \in \mathtt{part}(\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \emptyset\rangle)$ such that $\overline{\phi}$ is ST-sceptical-IN in its certain projection AF^* (where $\mathcal{A}$, $\mathcal{A}^?$ and $\mathcal{R}$ are chosen as in Definition 13).

Using this additional item, we prove these items separately.

(0) ⇒ (1) Suppose that there is some $\mathcal{I}^* \in \mathtt{part}(\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \emptyset\rangle)$ such that $\overline{\phi}$ is ST-sceptical-IN in the certain projection $AF^* = \langle \mathcal{A}^*, \mathcal{R}^*\rangle$ of $\mathcal{I}^*$. Let τ_X be an assignment to variables in X such that it assigns True to all $x_i \in X$ such that $g_i \in \mathcal{A}^*$ and False otherwise. Let τ_Y be an arbitrary assignment to all variables in Y. Given that $\overline{\phi}$ is ST-sceptical-IN in AF^*, for each ST extension S of AF^*, at least one of the arguments $\overline{c_i}$ must have been in S, so there is at least one clause in the formula Φ for which none of the variables was assigned True by τ_X and τ_Y. Since we chose τ_Y arbitrarily, we have that (ϕ, X, Y) is a positive instance of Σ_2-SAT.

(1) ⇒ (0) Let (ϕ, X, Y) be a positive instance of Σ_2-SAT. Then there is some assignment τ_X to all variables of X such that for each assignment τ_Y to the variables of Y, $\Phi[\tau_X, \tau_Y]$ is False. Let $G = \{g_i | x_i \in X \text{ and } x_i \text{ is assigned True by } \tau_X\}$. Construct $\mathcal{I}^* = \langle \mathcal{A} \cup G, \emptyset, \mathcal{R}|_{\mathcal{A}\cup G}, \emptyset\rangle$ and let AF^* be its certain projection. Note that $\mathcal{I}^* \in \mathtt{part}(\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \emptyset\rangle)$. Given that all arguments in G are unattacked, each ST extension of AF^* contains all arguments in G. Furthermore, for each argument $x \in X$, each ST extension of AF^* contains either x (if x is assigned True by τ_X) or $\overline{y}$ (if x is assigned False by τ_Y). Additionally, for each argument $y \in Y$, each ST extension of AF^* contains either y or $\overline{y}$. Given that for each assignment τ_Y to the variables of Y, $\phi[\tau_X, \tau_Y]$ is False, it must be that for each ST extension S of this AF, at least one of the clause arguments $\overline{c_i}$ is in S, so ϕ' is attacked by an argument in S; therefore $\overline{\phi} \in S$. Thus, $\overline{\phi}$ is ST-sceptical-IN in AF^*.

(0) ⇒ (2) Suppose that there is some $\mathcal{I}^* = \langle \mathcal{A}^*, \mathcal{A}^{?*}, \mathcal{R}|_{\mathcal{A}^*\cup\mathcal{A}^{?*}}, \emptyset\rangle$ in $\mathtt{part}(\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \emptyset\rangle)$ such that $\overline{\phi}$ is ST-sceptical-IN in its certain projection AF^*. This implies that for $\mathcal{I}' = \langle \mathcal{A}^*, \{\chi\}, \mathcal{R}|_{\mathcal{A}^*\cup\{\chi\}}, \emptyset\rangle$ it also holds that $\overline{\phi}$ is ST-sceptical-IN in its certain projection $AF' = AF^* = \langle \mathcal{A}^*, \mathcal{R}|_{\mathcal{A}^*}\rangle$. Note that $\mathcal{I}' \in \mathtt{part}(\mathcal{I}_1)$. Now consider $\mathcal{I}'' = \langle \mathcal{A}^* \cup \{\chi\}, \emptyset, \mathcal{R}|_{\mathcal{A}^*\cup\{\chi\}}, \emptyset\rangle$ and its certain projection AF''. In AF'', the argument $\overline{\phi}$ is attacked by the unattacked argument χ, so $\overline{\phi}$ is not ST-sceptical-IN in AF''. Then by Lemma 7 item 3, removal of χ is ST-sceptical-IN-relevant for $\overline{\phi}$ w.r.t. $\mathcal{I}_1$.

(2) ⇒ (0) If removal of χ is ST-sceptical-IN-relevant for $\overline{\phi}$ w.r.t. $\mathcal{I}_1$ then by Lemma 7 item 3 there is some $\mathcal{I}' = \langle \mathcal{A}^*, \emptyset, \mathcal{R}|_{\mathcal{A}^*}, \emptyset\rangle$ in $\mathcal{I}_1$ (where $\chi \notin \mathcal{A}^*$, so $\mathcal{I}' \in \mathtt{part}(\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \emptyset\rangle)$) such that $\overline{\phi}$ is ST-sceptical-IN in $\mathtt{cert}(\mathcal{I}')$.

(0) $\Rightarrow$ **(3)** Suppose that there is some $\mathcal{I}^* = \langle \mathcal{A}^*, \mathcal{A}^{?*}, \mathcal{R}|_{\mathcal{A}^* \cup \mathcal{A}^{?*}}, \emptyset\rangle$ in $\texttt{part}(\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \emptyset\rangle)$ such that $\overline{\phi}$ is ST-sceptical-IN in its certain projection AF^*. Then ϕ is ST-sceptical-OUT and therefore not ST-credulous-IN in AF^*. This implies that for $\mathcal{I}' = \langle \mathcal{A}^*, \{\chi\}, \mathcal{R}|_{\mathcal{A}^* \cup \{\chi\}}, \emptyset\rangle$ it also holds that ϕ is not ST-credulous-IN in its certain projection $AF' = AF^* = \langle \mathcal{A}^*, \mathcal{R}|_{\mathcal{A}^*}\rangle$. Note that $\mathcal{I}' \in \texttt{part}(\mathcal{I}_1)$. Now consider $\mathcal{I}'' = \langle \mathcal{A}^* \cup \{\chi\}, \emptyset, \mathcal{R}|_{\mathcal{A}^* \cup \{\chi\}}, \emptyset\rangle$ and its certain projection AF''. In AF'', the argument $\overline{\phi}$ is attacked by the unattacked argument χ, so $\overline{\phi}$ is not ST-sceptical-IN in AF'', so ϕ is ST-credulous-IN in AF''. Then by Lemma 7 item 1, addition of χ is ST-credulous-IN-relevant for ϕ w.r.t. $\mathcal{I}_1$.

(3) $\Rightarrow$ **(0)** If addition of χ is ST-credulous-IN-relevant for ϕ w.r.t. $\mathcal{I}_1$ then by Lemma 7 item 1 there is some $\mathcal{I}' = \langle \mathcal{A}^*, \{\chi\}, \mathcal{R}|_{\mathcal{A}^* \cup \{\chi\}}, \emptyset\rangle$ such that ϕ is not ST-credulous-IN in $\texttt{cert}(\mathcal{I}')$, so ϕ is ST-sceptical-OUT in $\texttt{cert}(\mathcal{I}')$, which implies that $\overline{\phi}$ is ST-sceptical-IN in $\texttt{cert}(\mathcal{I}')$.

(0) $\Rightarrow$ **(4)** Suppose that there is some $\mathcal{I}^* = \langle \mathcal{A}^*, \mathcal{A}^{?*}, \mathcal{R}|_{\mathcal{A}^* \cup \mathcal{A}^{?*}}, \emptyset\rangle$ in $\texttt{part}(\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \emptyset\rangle)$ such that $\overline{\phi}$ is ST-sceptical-IN in its certain projection AF^*. This implies that for $\mathcal{I}' = \langle \mathcal{A}^* \cup \{\chi, \overline{\chi}\}, \emptyset, (\mathcal{R} \cup \{(\overline{\chi}, \chi)\})|_{\mathcal{A}^* \cup \{\overline{\chi}, \chi\}}, \emptyset\rangle$ it also holds that $\overline{\phi}$ is ST-sceptical-IN in its certain projection AF'. Note that $\mathcal{I}' \in \texttt{part}(\mathcal{I}_2)$. Now consider $\mathcal{I}'' = \langle \mathcal{A}^* \cup \{\chi\}, \{\overline{\chi}\}, (\mathcal{R} \cup \{(\overline{\chi}, \chi)\})|_{\mathcal{A}^* \cup \{\overline{\chi}, \chi\}}, \emptyset\rangle$ and its certain projection AF''. In AF'', the argument $\overline{\phi}$ is attacked by the unattacked argument χ, so $\overline{\phi}$ is not ST-sceptical-IN in AF''. Then by Lemma 7 item 1, addition of $\overline{\chi}$ is ST-sceptical-IN-relevant for $\overline{\phi}$ w.r.t. $\mathcal{I}_2$.

(4) $\Rightarrow$ **(0)** If addition of $\overline{\chi}$ is ST-sceptical-IN-relevant for $\overline{\phi}$ w.r.t. $\mathcal{I}_2$ then by Lemma 7 item 1 there is some $\mathcal{I}' = \langle \mathcal{A}^* \cup \{\chi, \overline{\chi}\}, \emptyset, (\mathcal{R} \cup \{(\overline{\chi}, \chi)\})|_{\mathcal{A}^* \cup \{\overline{\chi}, \chi\}}, \emptyset\rangle$ in $\texttt{part}(\mathcal{I}_2)$ such that $\overline{\phi}$ is ST-sceptical-IN in $\texttt{cert}(\mathcal{I}')$. Then $\overline{\phi}$ would also be ST-sceptical-IN in $\texttt{cert}(\mathcal{I}'')$ where $\mathcal{I}'' = \langle \mathcal{A}^*, \emptyset, \mathcal{R}|_{\mathcal{A}^*}, \emptyset\rangle$, which is in $\texttt{part}(\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \emptyset\rangle)$.

(0) $\Rightarrow$ **(5)** Suppose that there is some $\mathcal{I}^* = \langle \mathcal{A}^*, \mathcal{A}^{?*}, \mathcal{R}|_{\mathcal{A}^* \cup \mathcal{A}^{?*}}, \emptyset\rangle$ in $\texttt{part}(\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \emptyset\rangle)$ such that $\overline{\phi}$ is ST-sceptical-IN in its certain projection AF^*; then ϕ is ST-credulous-IN in AF^*. This implies that for $\mathcal{I}' = \langle \mathcal{A}^* \cup \{\chi, \overline{\chi}\}, \emptyset, (\mathcal{R} \cup \{(\overline{\chi}, \chi)\})|_{\mathcal{A}^* \cup \{\overline{\chi}, \chi\}}, \emptyset\rangle$ it also holds that ϕ is not ST-credulous-IN in its certain projection AF'. Note that $\mathcal{I}' \in \texttt{part}(\mathcal{I}_2)$. Now consider $\mathcal{I}'' = \langle \mathcal{A}^* \cup \{\chi\}, \{\overline{\chi}\}, (\mathcal{R} \cup \{(\overline{\chi}, \chi)\})|_{\mathcal{A}^* \cup \{\overline{\chi}, \chi\}}, \emptyset\rangle$ and its certain projection AF''. In AF'', the argument $\overline{\phi}$ is attacked by the unattacked argument χ, so ϕ is ST-credulous-IN in AF''. Then by Lemma 7 item 3, removal of $\overline{\chi}$ is ST-credulous-IN-relevant for ϕ w.r.t. $\mathcal{I}_2$.

(5) $\Rightarrow$ **(0)** If removal of $\overline{\chi}$ is ST-credulous-IN-relevant for ϕ w.r.t. $\mathcal{I}_2$ then by Lemma 7 item 3 there is some $\mathcal{I}' = \langle \mathcal{A}^* \cup \{\chi, \overline{\chi}\}, \emptyset, (\mathcal{R} \cup \{(\overline{\chi}, \chi)\})|_{\mathcal{A}^* \cup \{\overline{\chi}, \chi\}}, \emptyset\rangle$ in $\texttt{part}(\mathcal{I}_2)$ such that ϕ is not ST-credulous-IN in $\texttt{cert}(\mathcal{I}')$, so ϕ is ST-sceptical-OUT and $\overline{\phi}$ is ST-sceptical-IN in $\texttt{cert}(\mathcal{I}')$. Then $\overline{\phi}$ would also be ST-sceptical-IN in $\texttt{cert}(\mathcal{I}'')$ where $\mathcal{I}'' = \langle \mathcal{A}^*, \emptyset, \mathcal{R}|_{\mathcal{A}^*}, \emptyset\rangle$, which is in $\texttt{part}(\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \emptyset\rangle)$.

(0) $\Rightarrow$ **(6)** Suppose that there is some $\mathcal{I}^* = \langle \mathcal{A}^*, \mathcal{A}^{?*}, \mathcal{R}|_{\mathcal{A}^* \cup \mathcal{A}^{?*}}, \emptyset\rangle$ in $\texttt{part}(\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \emptyset\rangle)$ such that $\overline{\phi}$ is ST-sceptical-IN in its certain projection AF^*; then ϕ is ST-sceptical-OUT in AF^*. This implies that for $\mathcal{I}' = \langle \mathcal{A}^*, \{\chi\}, \mathcal{R}|_{\mathcal{A}^* \cup \{\chi\}}, \emptyset\rangle$ it also holds that ϕ is ST-sceptical-OUT in its certain projection $AF' = AF^*$. Note that $\mathcal{I}' \in \texttt{part}(\mathcal{I}_1)$. Now consider $\mathcal{I}'' = \langle \mathcal{A}^* \cup \{\chi\}, \emptyset, \mathcal{R}|_{\mathcal{A}^* \cup \{\chi\}}, \emptyset\rangle$ and its certain projection AF''. In AF'', the argument $\overline{\phi}$ is attacked by the unattacked argument χ, so ϕ is not ST-sceptical-OUT in AF''. Then by Lemma 7 item 3, removal of χ is ST-sceptical-OUT-relevant for ϕ w.r.t. $\mathcal{I}_1$.

(6) $\Rightarrow$ **(0)** If removal of χ is ST-sceptical-OUT-relevant for ϕ w.r.t. $\mathcal{I}_1$ then by Lemma 7 item 3 there is some $\mathcal{I}' = \langle \mathcal{A}^*, \emptyset, \mathcal{R}|_{\mathcal{A}^*}, \emptyset\rangle$ in $\texttt{part}(\mathcal{I}_1)$ such that ϕ is ST-sceptical-OUT in $\texttt{cert}(\mathcal{I}')$, so $\overline{\phi}$ is ST-sceptical-IN in $\texttt{cert}(\mathcal{I}')$. Since χ is not in $\mathcal{A}^*$, $\mathcal{I}'$ is also in $\texttt{part}(\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \emptyset\rangle)$.

(0) ⇒ (7) Suppose that there is some $\mathcal{I}^* = \langle \mathcal{A}^*, \mathcal{A}^{?*}, \mathcal{R}|_{\mathcal{A}^* \cup \mathcal{A}^{?*}}, \emptyset \rangle$ in $\texttt{part}(\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \emptyset \rangle)$ such that $\overline{\phi}$ is ST-sceptical-IN in its certain projection AF^*; then $\overline{\phi}$ is not ST-credulous-OUT in AF^*. This implies that for $\mathcal{I}' = \langle \mathcal{A}^*, \{\chi\}, \mathcal{R}|_{\mathcal{A}^* \cup \{\chi\}}, \emptyset \rangle$ it also holds that $\overline{\phi}$ is not ST-credulous-OUT in its certain projection $AF' = AF^*$. Note that $\mathcal{I}' \in \texttt{part}(\mathcal{I}_1)$. Now consider $\mathcal{I}'' = \langle \mathcal{A}^* \cup \{\chi\}, \emptyset, \mathcal{R}|_{\mathcal{A}^* \cup \{\chi\}}, \emptyset \rangle$ and its certain projection AF''. In AF'', the argument $\overline{\phi}$ is attacked by the unattacked argument χ, so $\overline{\phi}$ is ST-sceptical-OUT in AF''. Then by Lemma 7 item 1, addition of χ is ST-credulous-OUT-relevant for $\overline{\phi}$ w.r.t. $\mathcal{I}_1$.

(7) ⇒ (0) If addition of χ is ST-credulous-OUT-relevant for $\overline{\phi}$ w.r.t. $\mathcal{I}_1$ then by Lemma 7 item 1 there is some $\mathcal{I}' = \langle \mathcal{A}^*, \{\chi\}, \mathcal{R}|_{\mathcal{A}^* \cup \{\chi\}}, \emptyset \rangle$ in $\texttt{part}(\mathcal{I}_1)$ such that $\overline{\phi}$ is not ST-credulous-OUT in $\texttt{cert}(\mathcal{I}')$, so $\overline{\phi}$ is ST-sceptical-IN in $\texttt{cert}(\mathcal{I}')$. Since χ is not in $\mathcal{A}^*$, $\mathcal{I}'$ is also in $\texttt{part}(\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \emptyset \rangle)$.

(0) ⇒ (8) Suppose that there is some $\mathcal{I}^* = \langle \mathcal{A}^*, \mathcal{A}^{?*}, \mathcal{R}|_{\mathcal{A}^* \cup \mathcal{A}^{?*}}, \emptyset \rangle$ in $\texttt{part}(\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \emptyset \rangle)$ such that $\overline{\phi}$ is ST-sceptical-IN in its certain projection AF^*; then ϕ is ST-sceptical-OUT in AF^*. This implies that for $\mathcal{I}' = \langle \mathcal{A}^* \cup \{\chi, \overline{\chi}\}, \emptyset, (\mathcal{R} \cup \{(\overline{\chi}, \chi)\})|_{\mathcal{A}^* \cup \{\overline{\chi}, \chi\}}, \emptyset \rangle$ it also holds that ϕ is ST-sceptical-OUT in its certain projection AF'. Note that $\mathcal{I}' \in \texttt{part}(\mathcal{I}_2)$. Now consider $\mathcal{I}'' = \langle \mathcal{A}^* \cup \{\chi\}, \{\overline{\chi}\}, (\mathcal{R} \cup \{(\overline{\chi}, \chi)\})|_{\mathcal{A}^* \cup \{\overline{\chi}, \chi\}}, \emptyset \rangle$ and its certain projection AF''. In AF'', the argument $\overline{\phi}$ is attacked by the unattacked argument χ, so ϕ is not ST-sceptical-OUT in AF''. Then by Lemma 7 item 1, addition of $\overline{\chi}$ is ST-sceptical-OUT-relevant for ϕ w.r.t. $\mathcal{I}_2$.

(8) ⇒ (0) If addition of $\overline{\chi}$ is ST-sceptical-OUT-relevant for ϕ w.r.t. $\mathcal{I}_2$ then by Lemma 7 item 1 there is some $\mathcal{I}' = \langle \mathcal{A}^* \cup \{\chi, \overline{\chi}\}, \emptyset, (\mathcal{R} \cup \{(\overline{\chi}, \chi)\})|_{\mathcal{A}^* \cup \{\overline{\chi}, \chi\}}, \emptyset \rangle$ in $\texttt{part}(\mathcal{I}_2)$ such that ϕ is ST-sceptical-OUT in $\texttt{cert}(\mathcal{I}')$, so $\overline{\phi}$ is ST-sceptical-IN in $\texttt{cert}(\mathcal{I}')$. Then $\overline{\phi}$ would also be ST-sceptical-IN in $\texttt{cert}(\mathcal{I}'')$ where $\mathcal{I}'' = \langle \mathcal{A}^*, \emptyset, \mathcal{R}|_{\mathcal{A}^*}, \emptyset \rangle$, which is in $\texttt{part}(\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \emptyset \rangle)$.

(0) ⇒ (9) Suppose that there is some $\mathcal{I}^* = \langle \mathcal{A}^*, \mathcal{A}^{?*}, \mathcal{R}|_{\mathcal{A}^* \cup \mathcal{A}^{?*}}, \emptyset \rangle$ in $\texttt{part}(\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \emptyset \rangle)$ such that $\overline{\phi}$ is ST-sceptical-IN in its certain projection AF^*; then $\overline{\phi}$ is not ST-credulous-OUT in AF^*. This implies that for $\mathcal{I}' = \langle \mathcal{A}^* \cup \{\chi, \overline{\chi}\}, \emptyset, (\mathcal{R} \cup \{(\overline{\chi}, \chi)\})|_{\mathcal{A}^* \cup \{\overline{\chi}, \chi\}}, \emptyset \rangle$ it also holds that $\overline{\phi}$ is not ST-credulous-OUT in its certain projection AF'. Note that $\mathcal{I}' \in \texttt{part}(\mathcal{I}_2)$. Now consider $\mathcal{I}'' = \langle \mathcal{A}^* \cup \{\chi\}, \{\overline{\chi}\}, (\mathcal{R} \cup \{(\overline{\chi}, \chi)\})|_{\mathcal{A}^* \cup \{\overline{\chi}, \chi\}}, \emptyset \rangle$ and its certain projection AF''. In AF'', the argument $\overline{\phi}$ is attacked by the unattacked argument χ, so $\overline{\phi}$ is ST-credulous-OUT in AF''. Then by Lemma 7 item 3, removal of $\overline{\chi}$ is ST-credulous-OUT-relevant for $\overline{\phi}$ w.r.t. $\mathcal{I}_2$.

(9) ⇒ (0) If removal of $\overline{\chi}$ is ST-credulous-OUT-relevant for $\overline{\phi}$ w.r.t. $\mathcal{I}_2$ then by Lemma 7 item 3 there is some $\mathcal{I}' = \langle \mathcal{A}^* \cup \{\chi, \overline{\chi}\}, \emptyset, (\mathcal{R} \cup \{(\overline{\chi}, \chi)\})|_{\mathcal{A}^* \cup \{\overline{\chi}, \chi\}}, \emptyset \rangle$ in $\texttt{part}(\mathcal{I}_2)$ such that $\overline{\phi}$ is not ST-credulous-OUT in $\texttt{cert}(\mathcal{I}')$, so $\overline{\phi}$ is ST-sceptical-IN in $\texttt{cert}(\mathcal{I}')$. Then $\overline{\phi}$ would also be ST-sceptical-IN in $\texttt{cert}(\mathcal{I}'')$ where $\mathcal{I}'' = \langle \mathcal{A}^*, \emptyset, \mathcal{R}|_{\mathcal{A}^*}, \emptyset \rangle$, which is in $\texttt{part}(\langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \emptyset \rangle)$. □

Proposition 19. ST-*sceptical*-UNDEC-RELEVANCE *is* Σ_2^p*-complete.*

Proof. First, we will show that ST-sceptical-UNDEC-RELEVANCE is in Σ_2^p. By Proposition 5, ST-sceptical-UNDEC-JUSTIFICATION is CoNP-complete. By Proposition 11, this implies that ST-sceptical-UNDEC-RELEVANCE is in Σ_2^p.

For the hardness proof, we use an existing result on the problem of necessary nonempty existence under ST semantics from [21], which is defined as follows: given an IAF $\mathcal{I}$, does each completion AF' of $\mathcal{I}$ have a nonempty ST extension? It is shown in [21, Theorem 21] that this problem is Π_2^p-hard.

Let $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$ be an arbitrary instance of the necessary nonempty existence problem under ST semantics. If $\mathcal{A} = \emptyset$ then $\mathcal{I}$ is a negative instance, because there is a completion $AF' = \langle \mathcal{A}', \mathcal{R}' \rangle$ of $\mathcal{I}$ where $\mathcal{A}' = \emptyset$, which means that AF' has no nonempty ST extension. Alternatively, assume that $\mathcal{A}$

contains at least one argument and let A be an arbitrary argument in $\mathcal{A}$. Then we transform $\mathcal{I}$ into an instance $(\mathcal{I}', A, U)$ of the **argument removal variant** of the ST-sceptical-UNDEC-RELEVANCE problem, where:

- U is a fresh uncertain argument, not occurring in $\mathcal{A} \cup \mathcal{A}^?$; and
- $\mathcal{I}' = \langle \mathcal{A}, \mathcal{A}^? \cup \{U\}, \mathcal{R} \cup \{(U, B) | B \in \mathcal{A} \cup \mathcal{A}^?\}, \mathcal{R}^? \rangle$.

Next, we will show that $\mathcal{I}$ is a *positive* instance of necessary nonempty existence under ST semantics iff $(\mathcal{I}', A, U)$ is a *negative* instance of ST-sceptical-UNDEC-RELEVANCE:

$\Rightarrow$ First suppose that $\mathcal{I}$ is a *positive* instance of necessary nonempty existence under ST semantics. Now let $\mathcal{I}'' = \langle \mathcal{A}'', \mathcal{A}^{?\prime\prime}, \mathcal{R}'', \mathcal{R}^{?\prime\prime} \rangle$ be an arbitrary partial completion of $\mathcal{I}'$ (which is the IAF in the transformed problem). We will prove that there is no completion of $\mathcal{I}''$ where A is ST-sceptical-UNDEC by distinguishing two options:

- If $U \in \mathcal{A}''$ then each argument other than U (including A) in each completion of $\mathcal{I}''$ is attacked by the unattacked argument U, so $\{U\}$ is a ST extension. Since A is attacked by an argument in the ST extension, it cannot be ST-sceptical-UNDEC. This holds for every completion of $\mathcal{I}''$.
- Otherwise, $U \notin \mathcal{A}''$. Then every completion of $\mathcal{I}''$ is also a completion of $\mathcal{I}$. Given that $\mathcal{I}$ is a positive instance of necessary nonempty existence under ST semantics, each completion of $\mathcal{I}$ has a (nonempty) ST extension, which must include either A or some argument attacking A. Then there is no completion of $\mathcal{I}$ in which A is ST-sceptical-UNDEC, which implies that there is no completion of $\mathcal{I}''$ either in which A is ST-sceptical-UNDEC.

Given that there is no completion of $\mathcal{I}''$ in which A is ST-sceptical-UNDEC, there can be no partial completion $\mathcal{I}'''$ of $\mathcal{I}''$ such that A is stable-ST-sceptical-UNDEC w.r.t. $\texttt{cert}(\mathcal{I}''')$. Then there is no minimal stable-ST-sceptical-UNDEC partial completion for A w.r.t. $\mathcal{I}$, which implies that the removal of U is not ST-sceptical-UNDEC relevant for A w.r.t. $\mathcal{I}''$. In other words: $(\mathcal{I}', A, U)$ is a *negative* instance of ST-sceptical-UNDEC-RELEVANCE.

$\Leftarrow$ Now suppose that $\mathcal{I}$ is a *negative* instance of necessary nonempty existence under ST semantics. Then there is some completion $AF' = \langle \mathcal{A}', \mathcal{R}' \rangle$ of $\mathcal{I}$ that has no nonempty ST extension. Recall that $\mathcal{A}'$ contains at least one argument (A), so AF' cannot have an empty ST extension (as any ST extension must contain either A or an attacker of A). This implies that AF' has no ST extension at all – which means that A must be ST-sceptical-UNDEC in AF'. Now construct $\mathcal{I}^* = \langle \mathcal{A}', \{U\}, \mathcal{R}' \cup \{(U, B) | B \in \mathcal{A} \cup \mathcal{A}^?\}, \emptyset \rangle$. Note that $\mathcal{I}^*$ is a partial completion of $\mathcal{I}'$ and that $\texttt{cert}(\mathcal{I}^*) = AF'$. So $\mathcal{I}'$ has a partial completion $\mathcal{I}^*$ such that A is ST-sceptical-UNDEC in $\texttt{cert}(\mathcal{I}^*)$.
Now consider the IAF $\mathcal{I}^{*\prime} = \langle \mathcal{A}' \cup \{U\}, \emptyset, \mathcal{R}' \cup \{(U, B) | B \in \mathcal{A} \cup \mathcal{A}^?\}, \emptyset \rangle$, which is also a partial completion of $\mathcal{I}'$. A is not ST-sceptical-UNDEC in $\texttt{cert}(\mathcal{I}^*)$, because $\texttt{cert}(\mathcal{I}^*)$ has a ST extension $\{U\}$ and U attacks A.
Given that $\mathcal{I}'$ has a partial completion $\mathcal{I}^*$ such that A is ST-sceptical-UNDEC in $\texttt{cert}(\mathcal{I}^*)$ while A is not ST-sceptical-UNDEC in $\texttt{cert}(\mathcal{I}^{*\prime})$ and thanks to specific properties of $\mathcal{I}^*$ and $\mathcal{I}^{*\prime}$ (having only U and no uncertain element), by Lemma 7 it must be that the removal of U is ST-sceptical-UNDEC-relevant w.r.t. $\mathcal{I}'$. So $(\mathcal{I}', A, U)$ is a *positive* instance of ST-sceptical-UNDEC-RELEVANCE.

Similarly, we can transform $\mathcal{I}$ into an instance $(\mathcal{I}'', A, U')$ of the **argument addition** variant of the ST-sceptical-UNDEC-RELEVANCE problem, where:

- U is a fresh argument, not occurring in $\mathcal{A} \cup \mathcal{A}^?$;

- U' is a fresh uncertain argument, not occurring in $\mathcal{A} \cup \mathcal{A}^?$; and
- $\mathcal{I}'' = \langle \mathcal{A} \cup \{U\}, \mathcal{A}^? \cup \{U'\}, \mathcal{R} \cup \{(U, B) | B \in \mathcal{A} \cup \mathcal{A}^?\} \cup \{(U', U)\}, \mathcal{R}^? \rangle$.

Then $\mathcal{I}$ is a *positive* instance of necessary nonempty existence under ST semantics iff $(\mathcal{I}'', A, U')$ is a *negative* instance of ST-sceptical-UNDEC-RELEVANCE.

For the **attack addition** variant of the ST-sceptical-UNDEC-RELEVANCE problem, the transformation from $\mathcal{I}$ into $(\mathcal{I}''', A, (U', U))$ is very similar:

- U and U' are fresh arguments, not occurring in $\mathcal{A} \cup \mathcal{A}^?$; and
- $\mathcal{I}''' = \langle \mathcal{A} \cup \{U, U'\}, \emptyset, \mathcal{R} \cup \{(U, B) | B \in \mathcal{A} \cup \mathcal{A}^?\}, \mathcal{R}^? \cup \{(U', U)\} \rangle$.

Then $\mathcal{I}$ is a *positive* instance of necessary nonempty existence under ST semantics iff $(\mathcal{I}''', A, (U', U))$ is a *negative* instance of ST-sceptical-UNDEC-RELEVANCE.

Finally, the transformation for the **attack removal** variant of the ST-sceptical-UNDEC-RELEVANCE problem from $\mathcal{I}$ into $(\mathcal{I}''', A, (U'', U'))$ is:

- U, U' and U'' are fresh arguments, not occurring in $\mathcal{A} \cup \mathcal{A}^?$; and
- $\mathcal{I}'''' = \langle \mathcal{A} \cup \{U, U', U''\}, \emptyset, \mathcal{R} \cup \{(U, B) | B \in \mathcal{A} \cup \mathcal{A}^?\} \cup \{(U', U)\}, \mathcal{R}^? \cup \{(U'', U')\} \rangle$.

Then $\mathcal{I}$ is a *positive* instance of necessary nonempty existence under ST semantics iff $(\mathcal{I}'''', A, (U'', U'))$ is a *negative* instance of ST-sceptical-UNDEC-RELEVANCE.

We have shown for each of the four variants of the relevance problem that $\mathcal{I}$ is a *positive* instance of necessary nonempty existence under ST semantics iff the transformed instance is a *negative* instance of ST-sceptical-UNDEC-RELEVANCE. Given that the necessary nonempty existence problem under ST semantics is Π_2^p-hard, the complementary problem of ST-sceptical-UNDEC-RELEVANCE must be Σ_2^p-hard. Together with the membership result from the beginning of this proof, this implies that ST-sceptical-UNDEC-RELEVANCE is Σ_2^p-complete. □

Proposition 20. ST-*sceptical-existent*-IN-RELEVANCE *and* ST-*sceptical-existent*-OUT-RELEVANCE *are* Σ_2^p*-complete.*

Proof. Membership in Σ_2^p directly follows from the complexity of ST-sceptical-existent-IN- and -OUT-JUSTIFICATION and Proposition 11, in the following way: ST-sceptical-existent-IN-JUSTIFICATION is DP-complete by [12, page 92]. By Lemma 1, ST-sceptical-existent-OUT-JUSTIFICATION is DP-complete as well. Then by Proposition 11 the problems of ST-sceptical-existent-IN-RELEVANCE and ST-sceptical-existent-OUT-RELEVANCE are in $NP^{DP} = \Sigma_2^p$; note that $NP^{DP} \subseteq \Sigma_2^p$ as any DP query can be answered by two (adaptive) SAT queries.

In order to prove Σ_2^p-hardness, we reduce from possible sceptical-existent acceptance under ST semantics (Definition 6), which was proven to be Σ_2^p-hard in [3, Theorem 25]. Let $(\mathcal{I}, A)$ be an arbitrary instance of possible sceptical-existent acceptance under ST semantics where $\mathcal{I} = \langle \mathcal{A}, \mathcal{A}^?, \mathcal{R}, \mathcal{R}^? \rangle$. We transform this into an instance $(\mathcal{I}', A, U)$ of ST-sceptical-existent-IN-RELEVANCE where U is a fresh uncertain argument that is not in $\mathcal{A} \cup \mathcal{A}^?$ and $\mathcal{I}' = \langle \mathcal{A}, \mathcal{A}^? \cup \{U\}, \mathcal{R} \cup \{(U, U)\}, \mathcal{R}^? \rangle$. Next, we will prove that $(\mathcal{I}, A)$ is a positive instance of possible sceptical-existent acceptance under ST semantics iff $(\mathcal{I}', A, U)$ is a positive instance of ST-sceptical-existent-IN-RELEVANCE.

⇒ If $(\mathcal{I}, A)$ is a positive instance of possible sceptical-existent acceptance under ST semantics then there is some completion $AF' = \langle \mathcal{A}', \mathcal{R}' \rangle$ of $\mathcal{I}$ that has a ST extension and such that A is in each ST extension of AF'.

Construct the IAF $\mathcal{I}'' = \langle \mathcal{A}', \{U\}, \mathcal{R}' \cup \{(U, U)\}, \emptyset \rangle$; note that $\texttt{cert}(\mathcal{I}'') = AF'$ and $\mathcal{I}'' \in \texttt{part}(\mathcal{I}')$. So A is ST-sceptical-existent-IN w.r.t. $\texttt{cert}(\mathcal{I}'')$.

Also construct $\mathcal{I}''' = \langle \mathcal{A}' \cup \{U\}, \emptyset, \mathcal{R}' \cup \{(U, U)\}, \emptyset \rangle$ – this is a partial completion of $\mathcal{I}'$ as well. The certain projection of $\mathcal{I}'''$ contains the self-attacking argument U, which is not attacked by any other argument. Consequently, $\texttt{cert}(\mathcal{I}''')$ cannot have a ST extension, so A cannot be ST-sceptical-existent-IN w.r.t. $\texttt{cert}(\mathcal{I}''')$.

Then by Lemma 7, the removal of U is ST-sceptical-existent-IN-relevant w.r.t. $\mathcal{I}$. In other words, $(\mathcal{I}', A, U)$ is a positive instance of ST-sceptical-existent-IN-RELEVANCE.

$\Leftarrow$ If $(\mathcal{I}, A)$ is a negative instance of possible sceptical-existent acceptance under ST semantics then there is no partial completion $\mathcal{I}''$ of $\mathcal{I}$ such that A is ST-sceptical-existent-IN in $\texttt{cert}(\mathcal{I}'')$. Next, we will show that there is no partial completion $\mathcal{I}'''$ of $\mathcal{I}'$ either for which A is ST-sceptical-existent-IN in $\texttt{cert}(\mathcal{I}''')$. Let $\mathcal{I}''' = \langle \mathcal{A}''', \mathcal{A}^{?'''}, \mathcal{R}''', \mathcal{R}^{?'''} \rangle$ be an arbitrary partial completion of $\mathcal{I}'$.

- If $U \in \mathcal{A}'''$ then $\texttt{cert}(\mathcal{I}''')$ contains the self-attacking argument U that is not attacked by any other argument. This implies that $\texttt{cert}(\mathcal{I}''')$ does not have a ST extension. Consequently, A cannot be ST-sceptical-existent-IN in $\texttt{cert}(\mathcal{I}''')$.
- Alternatively, $U \notin \mathcal{A}'''$. In that case, $\mathcal{I}'''$ is also a partial completion of $\mathcal{I}$, so A cannot be ST-sceptical-existent-IN in $\texttt{cert}(\mathcal{I}''')$.

Since $\mathcal{I}'''$ was chosen arbitrarily from $\texttt{part}(\mathcal{I}')$, there can be no partial completion of $\mathcal{I}'''$ such that A is ST-sceptical-existent-IN in $\texttt{cert}(\mathcal{I}''')$. This implies that there is no ST-sceptical-existent-IN-relevant operation w.r.t. $\mathcal{I}'$. Therefore, $(\mathcal{I}', A, U)$ is a negative instance of ST-sceptical-existent-IN-RELEVANCE.

For ST-sceptical-existent-OUT-RELEVANCE, we transform an arbitrary instance of possible sceptical-existent acceptance under ST semantics $(\mathcal{I}, A)$ to an instance $(\mathcal{I}'', A, U)$ of ST-sceptical-existent-OUT-RELEVANCE, where U is a fresh uncertain argument that is not in $\mathcal{A} \cup \mathcal{A}^?$, B is a fresh argument that is not in $\mathcal{A} \cup \mathcal{A}^?$ and $\mathcal{I}'' = \langle \mathcal{A} \cup \{B\}, \mathcal{A}^? \cup \{U\}, \mathcal{R} \cup \{(B, A), (U, U)\}, \mathcal{R}^? \rangle$. Then $(\mathcal{I}, A)$ is a positive instance of possible sceptical-existent acceptance under ST semantics iff $(\mathcal{I}'', A, U)$ is a positive instance of ST-sceptical-existent-OUT-RELEVANCE. □

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