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Argumentation frameworks with necessities and their relationship with logic programs

Farid Nouioua [a,b,*] and Sara Boutouhami [a]
[a] *Computer Science Department, University of Bordj Bou Arreridj, Algeria*
E-mails: farid.nouioua@gmail.com, boutouhaminouioua@gmail.com
[b] *LIS UMR-CNRS 7020, University of Aix-Marseille, France*
E-mail: farid.nouioua@lis-lab.fr

Abstract. This paper presents a comprehensive study of argumentation frameworks with necessities (AFNs), a bipolar extension of Dung Abstract argumentation frameworks (AFs) where the support relation captures a positive interaction between arguments having the meaning of necessity: the acceptance of an argument may require the acceptance of other argument(s). The paper discusses new main acceptability semantics for AFNs and their characterization both by a direct approach and a labelling approach. It examines the relationship between AFNs and Dung AFs and shows the gain provided by the former in terms of concision. Finally, the paper shows how to represent an AFN as a normal logic program (LP) and *vice versa* and in both cases establishes a one-to-one correspondence between extensions under the main acceptability semantics (except for semi-stable semantics where the correspondence is not completely full) of an AFN and particular cases of 3-valued stable models of normal LPs.

Keywords: Abstract argumentation, bipolarity, acceptability semantics, linking argumentation and logic programming

1. Introduction

In the last decades, formal argumentation has become an attractive research field in artificial intelligence (AI) (see e.g. [15,84]). It provides a form of reasoning based on the construction and the evaluation of arguments in favor or against a given claim. Argumentation-based models are proposed in different AI domains such as defeasible reasoning [79,87] and multi-agent systems [17,64,82,84]. Moreover, the argumentation approach has been used to provide solutions to several problems including decision making [10,12], negotiation [11,13], opinion analysis [18], practical reasoning [9], critical thinking [63], ontology alignment [89,92], statistical modeling [86], etc. (see [65] for more practical applications of formal argumentation).

Roughly speaking, works on argumentation may be divided into two classes. The first one is interested in the internal structure of arguments. We find in this class approaches that instantiate arguments from knowledge bases expressed in propositional logic [4–6,17,61], in conditional logic [16], in description logic [94], in rule-based systems [7,68] or in logic programming [30,43]. The interactions between arguments are expressed by attack relations (defeaters, undercuts, rebuttals, etc.) induced from the arguments' structure. The second class is that of abstract approaches which consider arguments as atomic entities and do not care about their internal structure. It focuses rather on the interaction between arguments and aims at drawing plausible conclusions according to some acceptability principle. Most of

*Corresponding author. E-mail: farid.nouioua@gmail.com.

the works in this class are developed around Dung's abstract model [53] where the unique interaction between arguments is a negative one captured by an attack relation. Extensive research has been done to generalize this model (see e.g. [8,19,21,22,38–41,44,46,47,69–72]).

Besides, one interesting research topic, which goes back to Dung's seminal work [53], is to explore links between Argumentation Frameworks (AFs) and Logic Programs (LPs) [29,30,73]. One important research line in this context (see e.g.. [29,30]) consists in representing an LP as a Dung system following an instantiation process similar to that used in rule-based systems [7,68] and then different 3-valued semantics of an LP are related to well-known acceptability semantics of the corresponding AF.

Some relatively recent works have addressed frameworks that include, in addition to an attack relation, positive interactions between arguments represented by a support relation. Dung AFs define an implicit positive interaction between arguments by means of the notion of defense: an argument defends another one if it attacks all its attackers. But some positive influences between arguments may not be reduced to the notion of defense and this motivates the study of new explicit support relations. Notice that in [77], the need for bipolar approaches has been empirically assessed. For our purpose in this work, let us illustrate this idea by the following example: (another example following the same principle is given in [45] inspired from [82] and [39]): Consider the following exchange of arguments during a meeting in a computer science department to organize exams:

(A): Only the basic notions of automata theory have been presented to students. Thus, the exam of this unit cannot be taken shortly.

(B): Normally, all the exams are taken during the same week and the exams of all the other teaching units are expected to be soon.

(C): The teacher of the automata theory unit was absent several times.

Clearly, arguments (A) and (B) attack each other. In contrast, the argument (C) supports (A) since it brings some information in favor of it. However, perceiving this support as a defense of (A) by (C) against (B) does not make sense since it is counter-intuitive to consider an attack from (C) to (B).

The work in [38] is an early attempt to explicitly represent a support relation in abstract argumentation. It uses an unspecified meaning of the support relation without considering additional constraints. This generality leads in some contexts to counter-intuitive conclusions since the correct interactions between attacks and supports depend strongly on the exact meaning of the support. According to the exact meaning given to support, several approaches have been proposed. The evidential support approach [72] limits acceptance to arguments supported by some evidence provided either directly from the environment (*prima facie* arguments) or from other supported arguments (*standard* arguments). In the deductive support approach, a supports b if the acceptance of a suffices to accept b. The abstract dialectical frameworks [22] extend Dung AFs by generalizing the acceptability conditions of an argument according to the acceptability of other arguments related to it. Finally, the backing relation approach [43,44] captures the meaning of support used in Toulmin's model of argumentation (see [45] for a survey on the different approaches on support in argumentation and Section 6 for a discussion of these approaches and some of their extensions).

Several extensions and applications of bipolar AFs have been proposed in the literature. We can cite the use of bipolar AFs for text exploration [28], for detecting bipolar relations from texts [26], for supporting users [27] and ranking comment sorting policies [93] in inline debate and for social networks analysis [60].

In this paper we focus on Argumentation Frameworks with Necessities (AFNs) introduced in [70,71] where the support relation has the meaning of necessity and relates sets of arguments to single arguments.

On the one hand this paper synthesizes and extends the two conference papers [70,71] (Sections 3 and 4) by introducing new concepts and detailed proofs of results (given in the Appendix). On the other hand, the paper further investigates the relationship between AFNs and LPs under 3-valued semantics.

We show in particular that the interest of using the particular meaning of necessity is twofold: First, it allows us to extend in a natural way several results concerning Dung AFs, namely the main acceptability semantics and their relationships. Hence, to directly draw conclusions from an AFN, it is not necessary to use an intermediate Dung AF (even if such an option remains available, see Section 4) or to borrow techniques from other domains such as logic programming. Moreover, the proposed framework is a proper generalization of Dung AFs in the sense that if no supports are present, the new definitions and results collapse with the classical ones of Dung AFs. Second, we show that the proposed framework is strongly related with logic programming under 3-valued semantics. We highlight in particular that thanks to the necessity relation, an easy and immediate translation of an LP into an AFN and *vice versa* is obtained and may be used instead of the usual but relatively complex instantiation process of arguments from an LP. In summary, the present work brings answers to the following main questions:

- How to generalize the main acceptability semantics of Dung AFs to AFNs by accommodating directly the necessity relation instead of translating the AFN into a Dung AF or using techniques from other formalisms?
- How to generalize the labelling characterization to the case of AFNs?
- How to extract a meta-argumentation model having the structure of a Dung AF from any AFN? What is the impact of using a necessity relation that relates sets of arguments to single arguments instead of a binary support relation as it is the case in most of existing works?
- How to instantiate an AFN from an LP and how to represent an AFN as an LP? In both cases, how to exploit the necessity relation to simplify the translation and how are the acceptability semantics of an AFN related to 3-valued semantics of the corresponding LP?

The rest of the paper is organized as follows. Section 2 recalls basic notions about Dung AFs and main concepts of LPs under 3-valued semantics. Section 3 presents AFNs. It introduces new notions regarding the necessity relation, generalizes the main notions of Dung AFs to the new context and presents in detail the main acceptability semantics of AFNs. Then, a labelling characterization of acceptability semantics for AFNs is presented. It extends the existing characterization for Dung AFs to take into account jointly both the attack and the necessity relations. Section 4 discusses the representation of an AFN as a meta Dung AF so that a one-to-one correspondence is established between acceptability semantics of an AFN and that of the corresponding meta Dung AF. Section 5 is devoted to a deep analysis of the links between the acceptability semantics of AFNs and 3-valued semantics of LPs. Finally, in Section 6, we discuss related work and give some perspectives for possible future work.

2. Preliminaries

2.1. Dung AFs

A Dung AF is an abstract argumentation model based on a set of arguments and the attacks between them.

Definition 2.1 (Argumentation framework)**.** A Dung AF is a pair $\mathcal{H} = \langle \mathcal{A}, \mathcal{R} \rangle$ where $\mathcal{A}$ is a set of arguments and $\mathcal{R} \subseteq \mathcal{A} \times \mathcal{A}$ is a binary attack relation.

Throughout the paper, we use the infix $a\ \mathcal{R}\ b$ to denote the attack $(a, b) \in \mathcal{R}$. Moreover, for $\mathcal{E} \subseteq \mathcal{A}$ and $a \in \mathcal{A}$, we abuse notation and write $a\ \mathcal{R}\ \mathcal{E}$ (resp. $\mathcal{E}\ \mathcal{R}\ a$) if there is $b \in \mathcal{E}$ such that $a\ \mathcal{R}\ b$ (resp. $b\ \mathcal{R}\ a$). For $\mathcal{E}, \mathcal{E}' \subseteq \mathcal{A}$, we similarly write $\mathcal{E}\ \mathcal{R}\ \mathcal{E}'$ if there is $a \in \mathcal{E}$ and $b \in \mathcal{E}'$ s.t. $a\ \mathcal{R}\ b$. Finally we denote by $\mathcal{E}^+$ the set of arguments attacked by $\mathcal{E}$, i.e., $\mathcal{E}^+ = \{a \in \mathcal{A} \mid \mathcal{E}\ \mathcal{R}\ a\}$.

Intuitively, $a\ \mathcal{R}\ b$ means that accepting a blocks the acceptance of b. In presence of various interacting arguments, one needs to know what are the sets of arguments that may be accepted collectively, called extensions. Several principles may be used as a basis to determine the extensions of a framework. Such principles are called acceptability semantics in the abstract argumentation literature. Acceptability semantics are defined on the basis of some elementary concepts that we sum up in Definition 2.2.

Definition 2.2 (Conflict-freeness, defense, characteristic function, admissibility)**.** Let $\mathcal{H} = \langle \mathcal{A}, \mathcal{R} \rangle$ be an AF and $\mathcal{E} \subseteq \mathcal{A}$. $\mathcal{E}$ is conflict-free iff $\nexists a, b \in \mathcal{E}$ s.t. $a\ \mathcal{R}\ b$; $\mathcal{E}$ defends an argument a iff $\forall b \in \mathcal{A}$, if $b\ \mathcal{R}\ a$, then $\mathcal{E}\ \mathcal{R}\ b$; the characteristic function $\mathcal{F}_\mathcal{H}$ is defined by: $\mathcal{F}_\mathcal{H} : 2^\mathcal{A} \to 2^\mathcal{A}$ s.t. for $\mathcal{E} \subseteq \mathcal{A}$, $\mathcal{F}_\mathcal{H}(\mathcal{E}) = \{a \in \mathcal{A} \mid \mathcal{E} \text{ defends } a\}$; $\mathcal{E}$ is admissible iff $\mathcal{E}$ is conflict-free and $\forall a \in \mathcal{E}$, $\mathcal{E}$ defends a.

The main acceptability semantics[1] of Dung AFs are defined as follows:

Definition 2.3 (Acceptability semantics)**.** Let $\mathcal{H} = \langle \mathcal{A}, \mathcal{R} \rangle$ be an AF and $\mathcal{E} \subseteq \mathcal{A}$. $\mathcal{E}$ is a complete extension iff it is admissible and contains all the elements it defends (i.e., $\mathcal{F}_\mathcal{H}(\mathcal{E}) = \mathcal{E}$); $\mathcal{E}$ is the grounded extension iff it is the $\subseteq$-minimal complete extension; $\mathcal{E}$ is a preferred extension iff it is a $\subseteq$-maximal complete extension; $\mathcal{E}$ is a stable extension iff it is conflict-free and attacks all the arguments outside it (i.e., $\mathcal{E}^+ = \mathcal{A} \setminus \mathcal{E}$); $\mathcal{E}$ is a semi-stable extension iff it is a complete extension s.t. $\mathcal{E} \cup \mathcal{E}^+$ is $\subseteq$-maximal.

Figure 1 depicts the relations between acceptability semantics (an arrow from a semantics s to a semantics s' means that each s-extension is a s'-extension).

Example 1. Consider the Dung AF $\mathcal{H} = \langle \mathcal{A}, \mathcal{R} \rangle$ depicted in Fig. 2.

The admissible sets of $\mathcal{H}$ are: $\emptyset$, $\{a\}$, $\{b\}$ and $\{b, d\}$. All of them are complete extensions except $\{b\}$. The grounded extension of $\mathcal{H}$ is $\emptyset$. $\mathcal{H}$ has two preferred extensions: $\{a\}$ and $\{b, d\}$. Only $\{b, d\}$ is a stable extension and hence is also semi-stable.

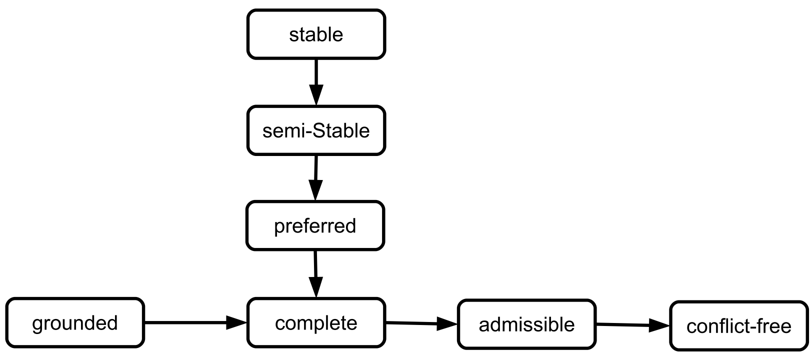

Fig. 1. Relations between AF semantics.

[1] All the presented semantics have been introduced in [53] except the semi-stable semantics which has been introduced in [31].

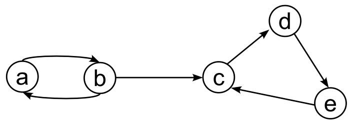

Fig. 2. A Dung AF $\mathcal{H} = \langle \mathcal{A}, \mathcal{R} \rangle$.

2.2. *Logic programs*

LPs represent a main knowledge representation formalism that has been extensively studied in AI. Indeed a large body of work has been developed around LPs and their semantics. In this paper we focus on normal LPs. A (propositional) normal LP Π is a set of rules of the form:

$$a_0 \leftarrow a_1, \ldots, a_m, \text{not } a_{m+1}, \ldots, \text{not } a_n \quad \text{with } 0 \leqslant m \leqslant n$$

a_i $(0 \leqslant a_i \leqslant n)$ are atoms. We read such a rule as follows: if $a_1 \ldots a_m$ are true and none of the atoms $a_{m+1} \ldots a_n$ is true then deduce that a_0 is true. For a rule r, we use the following notations: $\texttt{Head}(r) = a_0$, $\texttt{Body}^+(r) = \{a_1, \ldots, a_m\}$ and $\texttt{Body}^-(r) = \{a_{m+1}, \ldots, a_n\}$. More generally, we write: $\texttt{Head}(\Pi) = \{\texttt{Head}(r) | r \in \Pi\}$, $\texttt{Body}^+(\Pi) = \bigcup_{r\in\Pi} \texttt{Body}^+(r)$ and $\texttt{Body}^-(\Pi) = \bigcup_{r\in\Pi} \texttt{Body}^-(r)$. The Herbrand base of an LP Π denoted $\texttt{HB}_\Pi$ is the set of all atoms present in Π. An LP Π is said to be basic (or positive) if $\texttt{Body}^-(\Pi) = \emptyset$.

We present here a general setting based on 3-valued interpretations and capturing a wide range of semantics for LPs including the well-known bi-valued stable semantics defined in [58] by the so-called Gelfond-Lifschitz reduct.

Definition 2.4 (3-valued interpretation)**.** A 3-valued interpretation I is a pair $I = \langle T, F \rangle$ where T, F are disjoint subsets of $\texttt{HB}$. T (resp. F) stands for true (resp. false) atoms. The truth value of the remaining atoms is undefined. A 3-valued interpretation I can be equivalently defined as a function that associates to each atom a truth value in $\{\mathbf{t}, \mathbf{f}, \mathbf{u}\}$ s.t.: $a \in T$ iff $I(a) = \mathbf{t}$, $a \in F$ iff $I(a) = \mathbf{f}$ and $a \in \texttt{HB} \setminus (T \cup F)$ iff $I(a) = \mathbf{u}$. In the sequel, we use according to the context, one or the other of these two equivalent definitions.

We consider the ordering $\leqslant$ over the set $\{\mathbf{f}, \mathbf{t}, \mathbf{u}\}$ s.t. $\mathbf{f} \leqslant \mathbf{u} \leqslant \mathbf{t}$ and $\forall v \in \{\mathbf{f}, \mathbf{t}, \mathbf{u}\}$, $v \leqslant v$. The extension $\widehat{I}$ of a 3-valued interpretation I is defined as follows: $\widehat{I}(a) = I(a)$ if a is an atom; $\widehat{I}(\text{not } a) = \mathbf{f}$ (resp. $\mathbf{t}$, $\mathbf{u}$) if $I(a) = \mathbf{t}$ (resp. $\mathbf{f}$, $\mathbf{u}$); $\widehat{I}(A_1, \ldots, A_n) = \mathbf{min}(\widehat{I}(A_1), \ldots, \widehat{I}(A_n))$ where each A_i is either an atom or an expression not a_i with a_i an atom. Finally, for any rule of the form $r : a_0 \leftarrow a_1, \ldots, a_m, \text{not } a_{m+1}, \ldots, \text{not } a_n$, we have: $\widehat{I}(r) = \mathbf{t}$ if $\widehat{I}(a_1, \ldots, a_m, \text{not } a_{m+1}, \ldots, \text{not } a_n) \leqslant \widehat{I}(a_0)$ and $\widehat{I}(r) = \mathbf{f}$ otherwise.

Definition 2.5 (Model)**.** A 3-valued interpretation I is a model of an LP Π iff $\forall r \in \Pi$, $\widehat{I}(r) = \mathbf{t}$.

Given an LP Π, the operator Ψ takes a 3-valued interpretation I and gives its "immediate consequences" as follows. For every atom $a \in \texttt{HB}$:

- $\Psi(I)(a) = \mathbf{t}$ iff there is a rule $a \leftarrow a_1, \ldots, a_m, \text{not } a_{m+1}, \ldots, \text{not } a_n$ in Π s.t. $I(a_i) = \mathbf{t}$ for all i s.t. $1 \leqslant i \leqslant m$ and $I(a_j) = \mathbf{f}$ for all j s.t. $m + 1 \leqslant j \leqslant n$.

- $\Psi(I)(a) = \mathbf{u}$ iff $\Psi(I)(a) \neq \mathbf{t}$ and there is a rule $a \leftarrow a_1, \ldots, a_m, \text{not } a_{m+1}, \ldots, \text{not } a_n$ in Π s.t. $I(a_i) \neq \mathbf{f}$ for all i s.t. $1 \leqslant i \leqslant m$ and $I(a_j) \neq \mathbf{t}$ for all j s.t. $m+1 \leqslant j \leqslant n$.
- $\Psi(I)(a) = \mathbf{f}$ otherwise.

It has been shown (see [83]) that a positive LP Π has a unique least Herbrand model. It is worth mentioning that minimality here is w.r.t. the relation $\preceq$ over 3-valued interpretations defined as follows: $\langle T, F\rangle \preceq \langle T', F'\rangle$ iff $T \subseteq T'$ and $F' \subseteq F$. Thus, intuitively a model I_1 is smaller than a model I_2 if I_1 contains "less truth than" I_2. Moreover, this least model is exactly the least fixpoint of the operator Ψ defined above[2] and can be obtained by successive application of the operator Ψ starting from the interpretation $I_0 = \langle \emptyset, \mathtt{HB}\rangle$.

Definition 2.6 (Extended G/L reduct)**.** Let Π be an LP and I be a 3-valued interpretation. The extended Gelfond-Lifschitz (G/L for short) reduct of Π w.r.t. I is the positive LP denoted Π^I obtained by replacing in every rule of Π, every expression not a s.t. $I(a) = \mathbf{f}$ (resp. $I(a) = \mathbf{t}$, $I(a) = \mathbf{u}$) by the constant $\mathbf{t}$ (resp. $\mathbf{f}$, $\mathbf{u}$). Let J be the unique least model of Π^I, we define the operator Γ by $\Gamma(I) = J$.

Like in the approach of [58,59] for bi-valued stable models, the 3-valued fixpoints correspond to the 3-valued stable models called also P-stable models. Various other semantics are defined on the basis of P-stable semantics.[3]

Definition 2.7 (Different kinds of models)**.** Let Π be an LP and $I = \langle T, F\rangle$ be a 3-valued interpretation of Π. Then:

- I is a P-stable model of Π iff $\Gamma(I) = I$.
- I is a well-founded model of Π iff I is a P-stable model having the $\subseteq$-minimal set T among all P-stable models of Π.
- I is an M-stable model of Π iff I is a P-stable model having the $\subseteq$-maximal set T among all P-stable models of Π.
- I is a stable model of Π iff I is a P-stable model s.t. $T \cup F = \mathtt{HB}$ (no atom is undefined).
- I is an L-stable model of Π iff I is a P-stable model having the $\subseteq$-maximal set $T \cup F$ among all P-stable models of Π.

Example 2. Let us consider the LPs Π_1, Π_2, Π_3 and Π_4 (see Fig. 3):

(Π_1)	(Π_2)	(Π_3)	(Π_4)
(a) $p \leftarrow q$	(a) $p \leftarrow q$	(a) $p \leftarrow not\ q$	(a) $p \leftarrow not\ q$
(b) $q \leftarrow q$	(b) $q \leftarrow not\ p$	(b) $q \leftarrow not\ p$	(b) $p \leftarrow q$
(c) $s \leftarrow not\ q$	(c) $q \leftarrow not\ s$	(c) $s \leftarrow q,\ not\ x$	(c) $q \leftarrow not\ p$
	(d) $s \leftarrow$	(d) $y \leftarrow not\ s$	(d) $q \leftarrow p$
		(e) $x \leftarrow not\ y$	
		(f) $v \leftarrow w,\ not\ x$	
		(g) $w \leftarrow not\ w$	

Fig. 3. Examples of LPs.

[2] I is a fixed-point of Ψ iff $\Psi(I) = I$.

[3] Except for P-stable semantics which defines a P-stable model as a 3-valued interpretation I, in the original definitions of the other semantics, the corresponding models are defined as the set T of true atoms of a particular P-stable model $I = \langle T, F\rangle$. For the sake of homogeneity we define in this paper all kinds of models as 3-valued interpretations.

Π_1 has one P-stable model $I = \langle T = \{s\}, F = \{p, q\}\rangle$. Indeed the extended G/L reduct of Π_1 w.r.t. I is $\Pi_1^I = \{p \leftarrow q; q \leftarrow p; s \leftarrow \mathbf{t}\}$. Starting from $I_0 = \langle\emptyset, \{p, q, s\}\rangle$, we have $\Psi(I_0) = I$, $\Psi(I) = I$. Thus, $\Gamma(I) = I$ which means that I is a P-stable model of Π_1. It is easy to check that for all 3-valued interpretation $I' \neq I$ of Π_1, $\Gamma(I') \neq I'$, i.e., I is the unique P-stable model of Π_1 which is also its unique well-founded, M-stable and L-stable model. Since $\mathtt{HB}_{\Pi_1} = \{p, q, s\} = T \cup F$, I is the unique stable model of Π_1.

Following the same method, one can check that Π_2 has one P-stable model $I = \langle T = \{s\}, F = \emptyset\rangle$ which is also its unique well-founded, M-stable and L-stable model. Since $\mathtt{HB}_{\Pi_2} = \{p, q, s\} \neq T \cup F$, I is not a stable model of Π_2.

Π_3 has three P-stable models: $I_1 = \langle T_1 = \emptyset, F_1 = \emptyset\rangle$, $I_2 = \langle T_2 = \{q\}, F_2 = \{p\}\rangle$ and $I_3 = \langle T_3 = \{p, y\}, F_3 = \{q, s, x\}\rangle$. The well-founded model of Π_3 is I_1. I_2 and I_3 are the two M-stable models of Π_3 and only I_3 is an L-stable model of Π_3. Since $\mathtt{HB}_{\Pi_3} = \{p, q, s, y, x, v, w\} \neq T_i \cup F_i$ (for $i \in \{1, 2, 3\}$), Π_3 has no stable model.

Π_4 has one P-stable model $I = \langle T = \emptyset, F = \emptyset\rangle$ which is also its unique well-founded, M-stable and L-stable model. Since $\mathtt{HB}_{\Pi_4} = \{p, q\} \neq T \cup F$, Π_4 has no stable model.

3. Argumentation frameworks with necessities

This section introduces AFNs, a bipolar generalization of Dung AFs where the support relation has the meaning of necessity. We show how to extend the basic concepts used in Dung AFs to accommodate the new support relation. We show then how to use the new basic concepts to generalize the main acceptability semantics to AFNs. The proposed approach has the advantage to keep the same properties and relationships for the acceptability semantics as in the classical Dung approach. Moreover, the new semantics represent proper generalizations of Dung semantics, i.e., in an AFN where the necessity relation is empty the new semantics collapse to the classical ones.

3.1. Basic concepts

An AFN extends classical Dung AF with a necessity relation $\mathcal{N}$ that relates sets of arguments to single arguments.

Definition 3.1 (Argumentation framework with necessities)**.** An AFN is a tuple $\mathcal{G} = \langle\mathcal{A}, \mathcal{R}, \mathcal{N}\rangle$ where $\mathcal{A}$ is a set of arguments, $\mathcal{R} \subseteq \mathcal{A} \times \mathcal{A}$ is a binary attack relation and $\mathcal{N} \subseteq 2^{\mathcal{A}} \times \mathcal{A}$ is a necessity relation.

The attack relation $\mathcal{R}$ is interpreted as usual: $a\ \mathcal{R}\ b$ means that the acceptance of b requires the non acceptance of a. The new relation $\mathcal{N}$ is interpreted analogously but in a positive manner as follows: $E\ \mathcal{N}\ b$ means that the acceptance of b requires the acceptance of at least an argument of E. When all the necessary sets are singletons, $\mathcal{N}$ becomes a binary relation like $\mathcal{R}$. The general case captures the fact that an argument may satisfy a requirement by different possible combinations of arguments, instead of one possible way .[4]

[4]Notice that a similar generalization of the attack relation is also possible by considering relations of the form $E\ \mathcal{R}\ a$ where the argument a is attacked by the set of arguments E but not by a subset E' of E, unless there is another explicit attack relation: $E'\ \mathcal{R}\ a$. There is no substantial difficulty to generalize the framework to this extended setting.

In presence of the necessity relation, conflict-freeness is no more the minimal requirement for any extension. It has to be reinforced by two additional requirements w.r.t. necessity relation. The first requirement is closure under $\mathcal{N}^{-1}$. Intuitively, a set of arguments is closed under $\mathcal{N}^{-1}$ if it satisfies the necessities of each of its arguments.

Definition 3.2 (closure under $\mathcal{N}^{-1}$)**.** Let $\mathcal{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{N} \rangle$ be an AFN and $\mathcal{E} \subseteq \mathcal{A}$. $\mathcal{E}$ is closed under $\mathcal{N}^{-1}$ iff for each argument $a \in \mathcal{E}$, if $E\ \mathcal{N}\ a$ for some $E \subseteq \mathcal{A}$, then $E \cap \mathcal{E} \neq \emptyset$.

A second requirement that must be satisfied in any extension is the absence of self-supported cycles, i.e., cycles of necessity links .[5]

Definition 3.3 (N-cycle freeness)**.** Let $\mathcal{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{N} \rangle$ be an AFN, $\mathcal{E} \subseteq \mathcal{A}$ and $a \in \mathcal{E}$. We say that a is N-Cycle-Free in $\mathcal{E}$ iff for all $E \subseteq \mathcal{A}$ s.t. $E\ \mathcal{N}\ a$, we have either $E \cap \mathcal{E} = \emptyset$ or there is $b \in \mathcal{E} \cap E$ s.t. b is N-Cycle-Free in $\mathcal{E}$. $\mathcal{E}$ is N-Cycle-Free iff every $a \in \mathcal{E}$ is N-Cycle-Free in $\mathcal{E}$.

The combination of the two previous requirements gives rise to the notion of coherence:

Definition 3.4 (Coherence)**.** Let $\mathcal{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{N} \rangle$ be an AFN and $\mathcal{E} \subseteq \mathcal{A}$. $\mathcal{E}$ is coherent iff it is closed under $\mathcal{N}^{-1}$ and N-Cycle-Free.

Intuitively, in a coherent set $\mathcal{E}$, the necessities of each argument are satisfied and no risk of a deadlock due to necessity cycles is present.

The notion of coherence may be equivalently characterized by using the notion of a *powerful* argument. Intuitively, an argument a is powerful in a set of arguments $\mathcal{E}$ if it is always possible to find a sequence of distinct arguments ending by a s.t. the necessities of every argument of the sequence are satisfied by the arguments that precede it.

Definition 3.5 (Powerful argument)**.** Let $\mathcal{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{N} \rangle$ be an AFN and $\mathcal{E} \subseteq \mathcal{A}$. An argument $a \in \mathcal{A}$ is powerful in $\mathcal{E}$ iff $a \in \mathcal{E}$ and there is a sequence $a_0, \ldots, a_k$ of elements of $\mathcal{E}$ s.t. $a_k = a$; there is no $E \subseteq \mathcal{A}$ s.t. $E\ \mathcal{N}\ a_0$ and for $1 \leqslant i \leqslant k$: for all $E \subseteq \mathcal{A}$, if $E\ \mathcal{N}\ a_i$, then $E \cap \{a_0, \ldots, a_{i-1}\} \neq \emptyset$.

Coherent sets are characterized in terms of powerful arguments as follows:

Proposition 3.6. *Let $\mathcal{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{N} \rangle$ be an AFN and $\mathcal{E} \subseteq \mathcal{A}$. $\mathcal{E}$ is coherent iff each $a \in \mathcal{E}$ is powerful in $\mathcal{E}$.*

The following proposition gives two equivalent characterizations of non powerful arguments w.r.t. to a set of arguments:

Proposition 3.7. *Let $\mathcal{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{N} \rangle$ be an AFN, $\mathcal{E} \subseteq \mathcal{A}$ and $a \in \mathcal{E}$.*
a is not powerful in $\mathcal{E}$ iff there is no coherent subset $\mathcal{C}$ of $\mathcal{E}$ s.t. $a \in \mathcal{C}$
iff $\exists E \subseteq \mathcal{A}$ s.t. $E\ \mathcal{N}\ a$ and $\forall E \subseteq \mathcal{A}$ s.t. $E\ \mathcal{N}\ a$, $\forall b \in E \cap \mathcal{E}$, b is not powerful in $\mathcal{E}$.

Putting together conflict-freeness and coherence results in the notion of *strong coherence* which represents the new minimal requirement that any extension has to satisfy:

[5]Such cycles exist in LPs but not in Dung AFs because they do not contain any support relation.

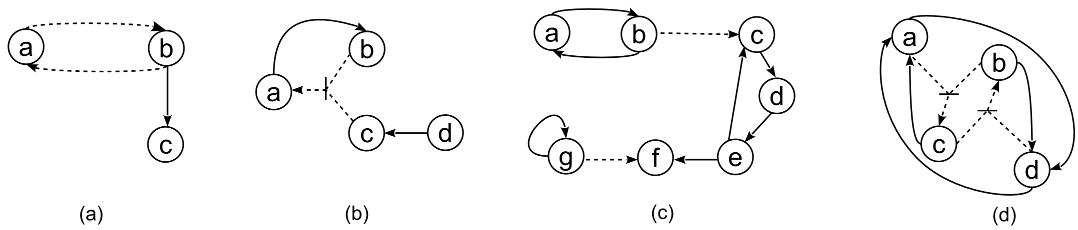

Fig. 4. Four AFNs: (a) $\mathcal{G}_1$, (b) $\mathcal{G}_2$, (c) $\mathcal{G}_3$, (d) $\mathcal{G}_4$.

Definition 3.8 (Strong coherence)**.** Let $\mathcal{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{N} \rangle$ be an AFN and $\mathcal{E} \subseteq \mathcal{A}$. $\mathcal{E}$ is strongly coherent iff it is coherent and conflict-free.

Example 3. Let us consider the four AFNs $\mathcal{G}_i = \langle \mathcal{A}_i, \mathcal{R}_i, \mathcal{N}_i \rangle$ ($1 \leqslant i \leqslant 4$) depicted in Fig. 4 where continuous (resp. dashed) arrows represent attacks (resp. necessities).

For $\mathcal{G}_1$, the only coherent sets are $\emptyset$ and $\{c\}$. In particular the set $\{a, b\}$ is closed under $\mathcal{N}^{-1}$ but not N-Cycle-Free, hence $\{a, b\}$ is not coherent. The coherent sets of $\mathcal{G}_2$ are those sets of arguments that contain b or c whenever they contain a. The coherent sets of $\mathcal{G}_3$ are those sets of arguments that contain b whenever they contain c and contain g whenever they contain f. The coherent sets of $\mathcal{G}_4$ are those sets of arguments that contain b or a whenever they contain c and contain c or d whenever they contain b except the set $\{b, c\}$. Indeed, the set $\{b, c\}$ is closed under $\mathcal{N}^{-1}$ but not N-Cycle-Free, whereas the sets $\{b\}$, $\{c\}$, $\{a, b\}$, $\{c, d\}$ are N-Cycle-Free but not closed under $\mathcal{N}^{-1}$ and hence they are not coherent. All the other sets of arguments not including $\{b, c\}$ are coherent.

For each AFN $\mathcal{G}_i$, the strongly coherent sets are limited to coherent sets that are also conflict-free.

The second ingredient in the generalization of acceptability semantics to AFNs is to redefine the notion of defense.

Definition 3.9 (Defense in AFNs)**.** Let $\mathcal{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{N} \rangle$ be an AFN, $\mathcal{E} \subseteq \mathcal{A}$ and $a \in \mathcal{A}$. We say that $\mathcal{E}$ defends a iff $\mathcal{E} \cup \{a\}$ is coherent and for all $b \in \mathcal{A}$, if $b \mathrel{\mathcal{R}} a$ then for every coherent subset $\mathcal{C} \subseteq \mathcal{A}$ s.t. $b \in \mathcal{C}$, $\mathcal{E} \mathrel{\mathcal{R}} \mathcal{C}$.

It is worth noticing that the obligation of counter-attacking is limited to those arguments that belong to at least one coherent set of arguments. This means that the attacks coming from incoherent sets of arguments are not effective and need not be counter-attacked. Based on the new definition of defense, the characteristic function of an AFN is defined exactly as in Dung AFs:

Definition 3.10 (Characteristic function of AFNs)**.** Let $\mathcal{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{N} \rangle$ be an AFN and $\mathcal{E} \subseteq \mathcal{A}$. The characteristic function of $\mathcal{G}$ is defined by $\mathcal{F}_\mathcal{G} : 2^\mathcal{A} \to 2^\mathcal{A}$ with $\mathcal{F}_\mathcal{G}(\mathcal{E}) = \{a \mid \mathcal{E} \text{ defends } a\}$.

Finally, the last ingredient we need in generalizing the acceptability semantics to AFNs is the notion of arguments deactivated by a given set of arguments, which replaces the set of arguments attacked by a set of arguments.

Definition 3.11 (Deactivated arguments)**.** Let $\mathcal{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{N} \rangle$ be an AFN and $\mathcal{E} \subseteq \mathcal{A}$ be a strongly coherent subset of arguments. The set of arguments deactivated by $\mathcal{E}$ is $\mathcal{E}^d = \{a \mid \text{if } \mathcal{C} \subseteq \mathcal{A} \text{ is a coherent subset s.t. } a \in \mathcal{C} \text{ then } \mathcal{E} \mathrel{\mathcal{R}} \mathcal{C}\}$.

The set $\mathcal{E}^d$ includes, in addition to arguments deactivated because of a direct attack from $\mathcal{E}$ against them, the arguments that are not powerful in $\mathcal{A}$[6] as well as the arguments that $\mathcal{E}$ "indirectly" attacks by making impossible to accept arguments from at least one set of arguments that is necessary for them.

3.2. Acceptability semantics for AFNs

Now we are ready to define the different acceptability semantics of AFNs.

Definition 3.12 (Acceptability semantics for AFNs)**.** Let $\mathcal{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{N} \rangle$ be an AFN and $\mathcal{E} \subseteq \mathcal{A}$.

- $\mathcal{E}$ is an **admissible set** iff it is strongly coherent and $\forall a \in \mathcal{E}$, $\mathcal{E}$ defends a.
- $\mathcal{E}$ is a **complete extension** iff it is admissible and $\forall a \in \mathcal{A}$, if $\mathcal{E}$ defends a then $a \in \mathcal{E}$.
- $\mathcal{E}$ is the **grounded extension** iff $\mathcal{E}$ is the least fixpoint of $\mathcal{F}_\mathcal{G}$. It is obtained by the repetitive application of $\mathcal{F}_\mathcal{G}$ starting from $\emptyset$ until a fixpoint is reached.
- $\mathcal{E}$ is a **preferred extension** iff $\mathcal{E}$ is a maximal (w.r.t. set inclusion) admissible set.
- $\mathcal{E}$ is a **stable extension** of $\mathcal{G}$ iff it is a strongly coherent subset of $\mathcal{A}$ s.t. for all $a \in \mathcal{A} \setminus \mathcal{E}$, a is deactivated by $\mathcal{E}$.
- $\mathcal{E}$ is a **semi-stable** extension iff it is a complete extension and $\mathcal{E} \cup \mathcal{E}^d$ is maximal (w.r.t set inclusion).

Now, the following Theorem shows that the main properties and relationships that hold for Dung acceptability semantics continue to hold for AFNs, by simply using strong coherence instead of conflict-freeness and deactivated arguments instead of attacked arguments.

Theorem 3.13. *Let $\mathcal{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{N} \rangle$ be an AFN and $\mathcal{E} \subseteq \mathcal{A}$ be a strongly coherent set.*

(1) *$\mathcal{E}$ is an admissible set iff $\mathcal{E} \subseteq \mathcal{F}_\mathcal{G}(\mathcal{E})$ (characterization of admissible sets using the characteristic function $\mathcal{F}_\mathcal{G}$).*
(2) *$\mathcal{E}$ is a complete extension iff $\mathcal{E} = \mathcal{F}_\mathcal{G}(\mathcal{E})$ (complete extensions are exactly the fixpoints of the characteristic function $\mathcal{F}_\mathcal{G}$).*
(3) *$\mathcal{E}$ is the grounded extension of $\mathcal{G}$ iff it is the least (w.r.t. set inclusion) complete extension of $\mathcal{G}$.*
(4) *There is at least one preferred extension for $\mathcal{G}$; every preferred extension is a complete extension but not vice versa; $\mathcal{E}$ is a preferred extension iff it is a maximal (w.r.t. set inclusion) complete extension.*
(5) *If $\mathcal{E}$ is a stable extension then $\mathcal{E}$ is a semi-stable extension but not vice versa; if $\mathcal{E}$ is a semi-stable extension, then $\mathcal{E}$ is a preferred extension but not vice versa.*
(6) *If $\mathcal{E}$ is a stable extension, then $\mathcal{E}$ is a preferred extension but not vice versa; there may be zero, one or several stable extensions for $\mathcal{G}$.*

Example 3 (Cont)**.** We continue with the AFNs $\mathcal{G}_1, \ldots, \mathcal{G}_4$.

$\mathcal{G}_1$ has two admissible sets: $\emptyset$ and $\{c\}$. Indeed, $\{c\}$ defends itself since the only attacker of c is b and there is no coherent set containing b. We have: $\mathcal{F}_{\mathcal{G}_1}(\emptyset) = \{c\}$, $\mathcal{F}_{\mathcal{G}_1}(\{c\}) = \{c\}$. Thus $\{c\}$ is the unique complete extension of $\mathcal{G}_1$ which is also its unique grounded, preferred and semi-stable extension. Moreover, $\{c\}$ is also the unique stable extension of $\mathcal{G}_1$ since the set of arguments deactivated by $\{c\}$ is $\{c\}^d = \{a, b\} = \mathcal{A} \setminus \{c\}$ (Indeed, a and b are not powerful in $\mathcal{A}$).

[6]Every argument which is not powerful in $\mathcal{A}$ does not belong to any coherent set. This means that such arguments always verify the condition of Definition 3.11 and hence, are deactivated by any set of arguments $\mathcal{E}$.

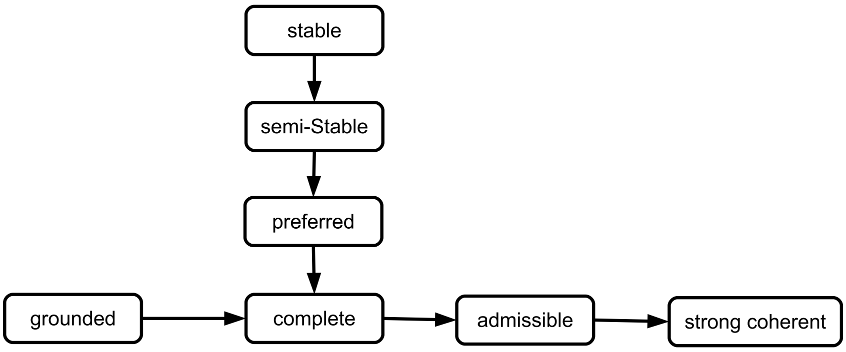

Fig. 5. Relations between AFN semantics.

$\mathcal{G}_2$ has two admissible sets: $\emptyset$ and $\{d\}$. $\{d\}$ defends itself since d has no attackers. No other strongly coherent set of $\mathcal{G}_2$ is admissible. For instance, $\{b, d\}$ is not admissible because a attacks b and $\{a, b\}$ is a coherent set containing a but not attacked by $\{b, d\}$. We have: $\mathcal{F}_{\mathcal{G}_2}(\emptyset) = \{d\}$, $\mathcal{F}_{\mathcal{G}_2}(\{d\}) = \{d\}$. Thus $\{d\}$ is the unique complete extension of $\mathcal{G}_2$ which is also its unique grounded, preferred and semi-stable extension. However $\{d\}$ is not a stable extension since $\{d\}^d = \{c\} \neq \mathcal{A} \setminus \{d\}$.

The admissible sets of $\mathcal{G}_3$ are: $\emptyset$, $\{a\}$, $\{b\}$ and $\{a, d\}$. Let us take for instance $\{a, d\}$: a is attacked by b and a attacks b (hence a attacks any coherent set containing b) and d is attacked by c but any coherent set containing c contains b and hence is attacked by a. We have: $\mathcal{F}_{\mathcal{G}_3}(\emptyset) = \emptyset$, $\mathcal{F}_{\mathcal{G}_3}(\{a\}) = \{a, d\}$, $\mathcal{F}_{\mathcal{G}_3}(\{b\}) = \{b\}$, $\mathcal{F}_{\mathcal{G}_3}(\{a, d\}) = \{a, d\}$. It follows that $\mathcal{G}_3$ has three complete extensions: $\emptyset$, $\{b\}$ and $\{a, d\}$. The grounded extension of $\mathcal{G}_3$ is $\emptyset$. $\mathcal{G}_3$ has two preferred extensions that are $\{b\}$ and $\{a, d\}$. $\mathcal{G}_3$ has no stable extension since $\{b\}^d = \{a\} \neq \mathcal{A} \setminus \{b\}$ and $\{a, d\}^d = \{b, c, e\} \neq \mathcal{A} \setminus \{a, d\}$. We have: $\{b\}^d \cup \{b\} = \{a, b\}$ and $\{a, d\}^d \cup \{a, d\} = \{a, b, c, d, e\}$. Since $\{b\}^d \cup \{b\} \subset \{a, d\}^d \cup \{a, d\}$, $\mathcal{G}_3$ admits $\{a, d\}$ as a unique semi-stable extension.

The only admissible set of $\mathcal{G}_4$ is $\emptyset$. Namely, the strongly coherent set $\{a\}$ is not admissible because c attacks a and $\{a, c\}$ is a coherent set containing c but not attacked by $\{a\}$. A similar reasoning is valid for the non admissibility of $\{d\}$. We have: $\mathcal{F}_{\mathcal{G}_4}(\emptyset) = \emptyset$. Thus, $\emptyset$ is the only complete extension of $\mathcal{G}_4$ which is also its unique grounded, preferred and semi-stable extension. $\mathcal{G}_4$ has no stable extension.

The relations between AFN acceptability semantics are depicted in Fig. 5. We can notice that these relations are the same as those connecting Dung AF acceptability semantics except that AFN semantics are based on strong coherence instead of conflict-freeness as a minimal requirement for all acceptability semantics.

3.3. Labelling characterization of AFNs

The labelling approach has been proposed as an elegant characterization of acceptability semantics of Dung AFs (see e.g. [32,67]). In this approach, each argument receives a label indicating its status: accepted, rejected or undefined. Extensions under a given semantics are then characterized by labellings fulfilling particular conditions that depend on the used semantics. In this section, we show how to take into account the necessity relation in order to adapt this approach to the case of AFNs. Let us start by recalling the notion of labelling:

Definition 3.14 (Labelling)**.** Let $\mathcal{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{N} \rangle$ be an AFN. A labelling is a function $\mathcal{L} : \mathcal{A} \longrightarrow \{\mathtt{in}, \mathtt{out}, \mathtt{undec}\}$. We put $\text{in}(\mathcal{L}) = \{a \in \mathcal{A} | \mathcal{L}(a) = \text{in}\}$, $\text{out}(\mathcal{L}) = \{a \in \mathcal{A} | \mathcal{L}(a) = \text{out}\}$ and $\text{undec}(\mathcal{L}) = \{a \in \mathcal{A} | \mathcal{L}(a) = \text{undec}\}$ and we write a labelling $\mathcal{L}$ as a triplet $(\text{in}(\mathcal{L}), \text{out}(\mathcal{L}), \text{undec}(\mathcal{L}))$.

In a given labelling, the label attributed to an argument may or may not be justified. For example, if all the attackers of an argument a are labelled `out` and each set E necessary for a contains at least an argument labelled `in`, it would not be justified that a be labelled `out`. This leads to the notion of legal labelling:

Definition 3.15 (Legal labelling)**.** Let $\mathcal{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{N} \rangle$ be an AFN, $\mathcal{L}$ be a labelling and a be an argument.

- a is legally `in` iff a is labelled `in` and the two following conditions hold:
  (1) $\forall b \in \mathcal{A}$, if $b\mathcal{R}a$ then $b \in \text{out}(\mathcal{L})$ (all attackers of a are labelled `out`) and
  (2) $\forall E \subseteq \mathcal{A}$, if $E\mathcal{N}a$ then $E \cap \text{in}(\mathcal{L}) \neq \emptyset$ (at least one argument from each necessary set for a is labelled `in`).
- a is legally `out` iff a is labelled `out` and at least one of the two following conditions holds:
  (1) either $\exists b \in \mathcal{A}$ s.t. $b\mathcal{R}a$ and $b \in \text{in}(\mathcal{L})$ (at least one attacker of a is labelled `in`) or
  (2) $\exists E \subseteq \mathcal{A}$, s.t. $E\mathcal{N}a$ and $E \subseteq \text{out}(\mathcal{L})$ (all the arguments of a necessary set for a are labelled `out`).
- a is legally `undec` iff a is labelled `undec` and the following conditions hold:
  (1) $\forall b \in \mathcal{A}$, if $b\mathcal{R}a$ then $b \notin \text{in}(\mathcal{L})$ (no attacker of a is labelled `in`) and
  (2) $\forall E \subseteq \mathcal{A}$, if $E\mathcal{N}a$ then $E \nsubseteq \text{out}(\mathcal{L})$ (not all the arguments of any necessary set for a are labelled `out`) and
  (3) either $\exists b \in \mathcal{A}$ s.t. $b\mathcal{R}a$ and $b \notin \text{out}(\mathcal{L})$ or $\exists E \subseteq \mathcal{A}$ s.t. $E\mathcal{N}a$ and $E \cap \text{in}(\mathcal{L}) = \emptyset$ (either at least one attacker of a is not labelled `out` or at least one necessary set for a does not contain any argument that is labelled `in`).

Notice that for $\mathcal{N} = \emptyset$, we find exactly the original definitions of legal labels given in [67]. In addition to legality of labels, the presence of necessity relation imposes two further constraints. Any argument which is not powerful in $\mathcal{A}$ does not belong to any extension and must be labelled `out` and since each extension $\mathcal{E}$ under any semantics must be coherent, the set of `in` arguments of any labelling characterizing any acceptability semantics for an AFN must be coherent. Labellings that satisfy these constraints are called *safe* labellings.

Definition 3.16 (Safe labelling)**.** We say that a labelling $\mathcal{L}$ is *safe* iff the set $\text{in}(\mathcal{L})$ is coherent and for each $a \in \mathcal{A}$: if a is not powerful in $\mathcal{A}$ then $a \in \text{out}(\mathcal{L})$.

Once the notion of labelling is extended to the necessity relation, the different kinds of labellings are defined as usual except that they must always be safe.

Definition 3.17 (Different kinds of labellings)**.** A labelling $\mathcal{L}$ is:

- *admissible* iff $\mathcal{L}$ is safe and without arguments that are illegally `in` and without arguments that are illegally `out`;
- *complete* iff $\mathcal{L}$ is admissible and without arguments that are illegally `undec`;
- *grounded* iff $\mathcal{L}$ is complete and $\text{in}(\mathcal{L})$ is $\subseteq$-minimal;

- *preferred* iff $\mathcal{L}$ is complete and in($\mathcal{L}$) is $\subseteq$-maximal;
- *stable* iff $\mathcal{L}$ is complete and undec($\mathcal{L}$) $= \emptyset$ and
- *semi-stable* iff $\mathcal{L}$ is complete and undec($\mathcal{L}$) is $\subseteq$-minimal.

Notice that since admissible labellings must be safe, all other kinds of labellings (complete, preferred, grounded, stable and semi-stable) must be safe too because all of them are admissible. Besides, admissible labellings only require that every argument which is labelled in or out must be legal but tolerate the illegality of arguments that are labelled undec. However, complete labellings require the legality of all arguments: Every argument which is labeled in (resp. out, undec) must be legally in (resp. out, undec). Hence, Since all other kinds of labellings (grounded, preferred, stable and semi-stable) are also complete, they require the legality of all their arguments.

For Dung AFs (i.e. an AFN where $\mathcal{N} = \emptyset$), any set of arguments is safe. In this case, we obtain exactly the classical definitions for legally `in`, `out` and `undec` arguments and for the different kinds of labellings. The relationship between labellings and acceptability semantics for AFNs is given as follows.

Theorem 3.18. *Let $\mathcal{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{N} \rangle$ be an AFN, $\mathcal{E} \subseteq \mathcal{A}$ and $\mathcal{L}$ be a labelling of $\mathcal{G}$.*

- *If $\mathcal{E}$ is an admissible set of $\mathcal{G}$ then the labelling $\mathcal{L} = (\mathcal{E}, \mathcal{E}^d, \mathcal{A} \setminus (\mathcal{E} \cup \mathcal{E}^d))$ is an admissible labelling of $\mathcal{G}$. Inversely, if $\mathcal{L}$ is an admissible labelling of $\mathcal{G}$ then $\mathcal{E} = in(\mathcal{L})$ is an admissible set of $\mathcal{G}$ and $out(\mathcal{L}) \subseteq \mathcal{E}^d$.*
- *If $\mathcal{E}$ is a complete (resp. the grounded, a preferred, a stable, a semi-stable) extension of $\mathcal{G}$ then the labelling $\mathcal{L} = (\mathcal{E}, \mathcal{E}^d, \mathcal{A} \setminus (\mathcal{E} \cup \mathcal{E}^d))$ is a complete (resp. the grounded, a preferred, a stable, a semi-stable extension) labelling of $\mathcal{G}$. Inversely, if $\mathcal{L}$ is a complete (resp. the grounded, a preferred, a stable, a semi-stable extension) labelling of $\mathcal{G}$ then $\mathcal{E} = in(\mathcal{L})$ is a complete (resp. the grounded, a preferred, a stable, a semi-stable) extension of $\mathcal{G}$ and $out(\mathcal{L}) = \mathcal{E}^d$.*

Example 3 (Cont)**.** Let us consider again our four AFNs: $\mathcal{G}_1 \ldots \mathcal{G}_4$.

Consider the labellings: $\mathcal{L}_1 = (\{c\}, \{b\}, \{a\})$, $\mathcal{L}_2 = (\emptyset, \emptyset, \{a, b, c\})$, $\mathcal{L}_3 = (\emptyset, \{a, b\}, \{c\})$, and $\mathcal{L}_4 = (\{c\}, \{a, b\}, \emptyset)$ for $\mathcal{G}_1$. In $\mathcal{L}_1$ c is legally in because $\mathcal{L}_1(b) =$ out but b is illegally out. Moreover, $\mathcal{L}_1$ is not safe because a is not powerful in $\mathcal{A}$ but $a \notin$ out($\mathcal{L}_1$). Thus, $\mathcal{L}_1$ is not admissible. $\mathcal{L}_2$ is not safe for the same reason and thus, is not admissible. In $\mathcal{L}_3$, a and b are legally out but c is illegally undec. $\mathcal{L}_3$ is admissible but not complete. In $\mathcal{L}_4$ c is legally in and a and b are legally out. Moreover, $\mathcal{L}_4$ is safe and thus it is admissible and complete (no argument is illegally undec). In summary, $\mathcal{L}_3$ and $\mathcal{L}_4$ are the admissible labellings of $\mathcal{G}_1$ and $\mathcal{L}_4$ is its unique complete labelling which is also its unique grounded and preferred labelling. Moreover, since undec($\mathcal{L}_4$) $= \emptyset$, $\mathcal{L}_4$ is also the unique stable and semi-stable labelling of $\mathcal{G}_1$.

$\mathcal{L}_1 = (\emptyset, \emptyset, \{a, b, c, d\})$ and $\mathcal{L}_2 = (\{d\}, \{c\}, \{a, b\})$ are the admissible labellings of $\mathcal{G}_2$. $\mathcal{L}_2$ is the only complete labelling of $\mathcal{G}_2$ (d is illegally undec in $\mathcal{L}_1$) which is also its unique grounded, preferred and semi-stable labelling. However, since undec($\mathcal{L}_2$) $\neq \emptyset$, $\mathcal{L}_4$ is not stable.

$\mathcal{L}_1 = (\emptyset, \emptyset, \{a, b, c, d, e, f, g\})$, $\mathcal{L}_2 = (\{a\}, \{b, c\}, \{d, e, f, g\})$, $\mathcal{L}_3 = (\{b\}, \{a\}, \{c, d, e, f, g\})$ and $\mathcal{L}_4 = (\{a, d\}, \{b, c, e\}, \{f, g\})$ are the admissible labellings of $\mathcal{G}_3$. Among them, only $\mathcal{L}_2$ is not complete (d is illegally undec in $\mathcal{L}_2$). The grounded labelling is $\mathcal{L}_1$, the preferred labellings are $\mathcal{L}_3$ and $\mathcal{L}_4$. No complete labelling has an empty set of undec arguments, thus no labelling is stable. The only semi-stable stable labelling (which minimizes undec) is $\mathcal{L}_4$.

$\mathcal{G}_4$ admits $\mathcal{L} = (\emptyset, \emptyset, \{a, b, c, d\})$ as the unique admissible, complete, grounded, preferred and semi-stable labelling. $\mathcal{G}_4$ has no stable labelling.

For each of the previous AFNs, $\mathcal{L}$ is an s-labelling ($s \in$ {admissible, complete, grounded, preferred, stable, semi-stable}) if and only if in($\mathcal{L}$) is an s-extension.

4. AFNs and Dung AFs

A Dung AF is simply a particular case of AFN where the necessity relation is empty.

Theorem 4.1. *Let $\mathcal{H} = \langle \mathcal{A}, \mathcal{R} \rangle$ be a Dung AF. We define the AFN $\mathcal{G}_{\mathcal{H}}$ by $\mathcal{G}_{\mathcal{H}} = \langle \mathcal{A}, \mathcal{R}, \emptyset \rangle$. Let $\mathcal{E} \subseteq \mathcal{A}$. $\mathcal{E}$ is an admissible set (resp. complete, grounded, preferred, stable, semi-stable extension) of $\mathcal{H}$ iff $\mathcal{E}$ is an admissible set (resp. complete, grounded preferred, stable, semi-stable extension) of $\mathcal{G}_{\mathcal{H}}$.*

Let us now consider the opposite issue, i.e., representing an AFN as an AF.

Given an AFN $\mathcal{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{N} \rangle$, a first question we are interested in is to know if it is always possible to find a Dung AF with exactly the same arguments and which contains all the information encoded in $\mathcal{G}$. It has been shown in [71] that the answer is positive when the necessity relation is binary (for AFNs where if $E\mathcal{N}a$ then E is a singleton). The idea is to add the implicit attacks that result from the interaction between attacks and necessities as follows: if a attacks b and b is necessary for c then a attacks indirectly c and if a requires b and b attacks c then a attacks indirectly c.

We show here that the answer is negative in the general case and one may need a greater number of arguments to encode all the information of an AFN in an AF. To show this, let us take the AFN $\mathcal{G}_2$ of Example 3 and let us suppose that $\mathcal{F} = \langle \mathcal{A}, \mathcal{R}' \rangle$ is an AF encoding the same information as $\mathcal{G}_2$. It is clear that (a, b), (d, c) are in $\mathcal{R}'$. The AF $\langle \mathcal{A}, \{(a, b), (d, c)\} \rangle$ does not have the same extensions for all the considered semantics. Apart from these two attacks, any other possible attack from an argument x to an argument y ($x, y \in \mathcal{A}$) is not present directly or indirectly in $\mathcal{G}_2$. In particular we cannot say that d attacks a because a may be obtained either by having c or b and d attacks only c. The solution is to represent separately the two different ways to obtain a (by providing b and by providing c) as two meta arguments, say A_1 and A_2. Only the second meta argument, involving a and c, is attacked by d. More generally, the notion of a meta-argument is defined as follows:

Definition 4.2 (Meta-argument)**.** Let $\mathcal{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{N} \rangle$ be an AFN and $a \in \mathcal{A}$. A meta-argument associated to a is a minimal (w.r.t. set inclusion) coherent set $\mathcal{C} \subseteq \mathcal{A}$ containing a (no subset of $\mathcal{C}$ containing a is coherent).

The meta Dung AF representing an AFN is then defined as follows:

Definition 4.3 (Meta AF representing an AFN)**.** Let $\mathcal{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{N} \rangle$ be an AFN. The Dung AF representing $\mathcal{G}$ is $\mathcal{H}_{\mathcal{G}} = \langle \mathcal{A}', \mathcal{R}' \rangle$ where: $\mathcal{A}'$ is the set of all meta-arguments associated to all arguments in $\mathcal{A}$ and for $\mathcal{C}_1, \mathcal{C}_2 \in \mathcal{A}'$, $\mathcal{C}_1 \ \mathcal{R}' \ \mathcal{C}_2$ iff there is $a \in \mathcal{C}_1$ and there is $b \in \mathcal{C}_2$ s.t. $a\mathcal{R}b$.

It is worth noticing that by construction of meta-arguments, any argument which is not powerful in $\mathcal{A}$ is ruled-out.

Proposition 4.4. *Let $\mathcal{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{N} \rangle$ be an AFN, $\mathcal{H}_{\mathcal{G}} = \langle \mathcal{A}', \mathcal{R}' \rangle$ be its corresponding meta AF and $a \in \mathcal{A}$. If a is not powerful in $\mathcal{A}$, then $\nexists \mathcal{C} \in \mathcal{A}'$ s.t. $a \in \mathcal{C}$.*

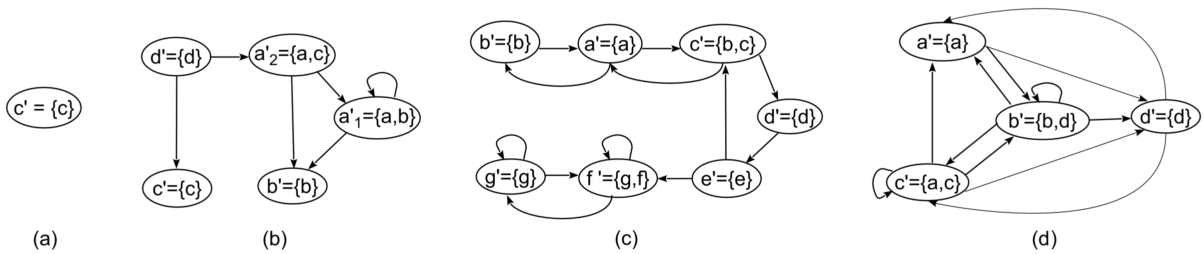

Fig. 6. The Dung AFs associated to the four AFNs $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}_4$.

Example 3 (Cont)**.** Each AFN $\mathcal{G}_i = \langle \mathcal{A}_i, \mathcal{R}_i, \mathcal{N}_i \rangle$ is translated into $\mathcal{H}_{\mathcal{G}_i} = \langle \mathcal{A}'_i, \mathcal{R}'_i \rangle$ for $i \in \{1, 2, 3, 4\}$.

For $\mathcal{H}_{\mathcal{G}_1}$, the arguments a and b are not powerful in $\mathcal{A}$ and hence they do not give rise to any meta-argument. The argument c gives rise to the unique meta-argument $c' = \{c\}$. Accordingly, $\mathcal{A}'_1 = \{c'\}$ and $\mathcal{R}'_1 = \emptyset$ (see Fig. 6-(a)).

For $\mathcal{H}_{\mathcal{G}_2}$, the argument a gives rise to two meta-arguments: $a'_1 = \{a, b\}$ and $a'_2 = \{a, c\}$. b (resp. c, d) gives rise to the unique meta-argument $b' = \{b\}$ (resp. $c' = \{c\}$, $d' = \{d\}$). Thus, $\mathcal{A}'_2 = \{a'_1, a'_2, b', c', d'\}$. $\mathcal{R}'_2$ is depicted in Fig. 6-(b).

For $\mathcal{H}_{\mathcal{G}_3}$, the argument c (resp. f) gives rise to the meta-arguments: $c' = \{b, c\}$ (resp. $f' = \{f, g\}$). The argument a (resp. b, d, e, g) gives rise to the unique meta-argument $a' = \{a\}$ (resp. $b' = \{b\}$, $d' = \{d\}$, $e' = \{e\}$, $g' = \{g\}$). Thus $\mathcal{A}'_3 = \{a', b', c', d', e', f', g'\}$. $\mathcal{R}'_3$ is depicted in Fig. 6-(c).

For $\mathcal{H}_{\mathcal{G}_4}$, the argument b (resp. c) gives rise to the meta-argument: $b' = \{b, d\}$ (resp. $c' = \{a, c\}$). The argument a (resp. d) gives rise to the meta-argument $a' = \{a\}$ (resp. $d' = \{d\}$). Note that $\{b, c\}$ is not a meta-argument since it is not coherent (it is not N-Cycle-Free). Thus, $\mathcal{A}'_4 = \{a', b', c', d'\}$. $\mathcal{R}'_4$ is depicted in Fig. 6-(d).

The following result shows that there is a full correspondence between the extensions of an AFN $\mathcal{G}$ and those of the corresponding meta Dung AF $\mathcal{H}_{\mathcal{G}}$ under all the considered semantics.

Theorem 4.5. *Let $\mathcal{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{N} \rangle$ be an AFN, $\mathcal{H}_{\mathcal{G}} = \langle \mathcal{A}', \mathcal{R}' \rangle$ be its corresponding Dung AF.*

- *If $\mathcal{E} \subseteq \mathcal{A}$ is an admissible set (resp. a complete, the grounded, a preferred, a stable, a semi-stable extension) of $\mathcal{G}$ then $\Gamma_{\mathcal{E}} = \{\mathcal{C}_a \mid a \in \mathcal{E}, \mathcal{C}_a$ is a meta-argument associated to a and $\mathcal{C}_a \subseteq \mathcal{E}\}$ is an admissible set (resp. a complete, the grounded, a preferred, a stable, a semi-stable extension) of $\mathcal{H}_{\mathcal{G}}$.*
  Inversely, if Γ is an admissible set (resp. a complete, the grounded, a preferred, a stable, a semi-stable extension) of $\mathcal{H}_{\mathcal{G}}$, then $\mathcal{E}_{\Gamma} = \bigcup_{\mathcal{C} \in \Gamma} \mathcal{C}$ is an admissible set (resp. a complete, the grounded, a preferred, a stable, a semi-stable extension) of $\mathcal{G}$.

Example 3 (Cont)**.** It is easy to check that for any of the considered acceptability semantics, the extensions of each $\mathcal{G}_i$ ($1 \leqslant \mathcal{G}_i \leqslant 4$) correspond exactly (in the sense of theorem 4.5) to the extensions of $\mathcal{H}_{\mathcal{G}_i}$ under the same semantics.

It is worth noticing that by using the translation described above there are AFNs whose corresponding AFs contain a number of arguments that is exponential with respect to the number of arguments in the initial AFN. To illustrate this, let us take the example of the AFN $\mathcal{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{N} \rangle$ where $\mathcal{A} = \{a\} \cup \mathcal{A}_1 \cup \ldots \mathcal{A}_n$, each $\mathcal{A}_i$ contains p arguments ($p > 1$) and $\mathcal{A}_i \mathcal{N} a$ (for $1 \leqslant i \leqslant n$). Let $\mathcal{H}$ be the corresponding AF. The number of arguments in $\mathcal{G}$ is $1 + p \times n$. Each set $\{a, b_1, \ldots b_n\}$ s.t. $b_i \in \mathcal{A}_i$ for

$1 \leqslant i \leqslant n$ is a minimal coherent set containing a, i.e., is a meta argument in $\mathcal{H}$. The number of meta arguments corresponding to a is p^n and each argument x of $\mathcal{A} \setminus \{a\}$ gives rise to one meta argument $(\{x\})$. The total number of the meta arguments is then $p^n + p \times n$.

This means that even if the information present in an AFN may always be encoded by a Dung AF, the use of an AFN in general, may allow a representation that is significantly more concise than that obtained by moving to the corresponding AF using the translation described in this section. The question of whether there exists alternative translations from AFNs to AFs that provide more concise representations reminds open and will be addressed in future work.

5. AFNs and logic programs

Most of the works done in the domain of connecting LPs to abstract argumentation use Dung AFs as an abstract argumentation formalism (see [30,33,53,55,56]).

This section addresses the issue of connecting AFNs and LPs under 3-valued semantics. We consider both the representation of an LP as an AFN and *vice versa*. In both cases, we establish the correspondences between acceptability semantics of AFNs and different kinds of partial stable models of LPs.

5.1. From a logic program to an AFN

We present in this section a straightforward representation of any LP as an AFN where each rule of the LP is represented as an argument in the AFN. Accordingly, the proposed instantiation is no more based on a complex process that constructs arguments from the knowledge base by combining sets of rules (see e.g. [7,30,33]). This shows in particular the usefulness of AFN as an abstract reasoning tool to be used for knowledge bases expressed by LPs.

Before discussing the representation of LPs as AFNs, let us first consider some technical requirements that will make easier the subsequent development. We present a kind of pre-processing that is performed on any LP Π to produce another LP Π' which has exactly the same models as Π under any semantics but is more suitable to be represented as an AFN. This pre-processing is based on the remark that: (1) if a rule in an LP has in its positive body an atom that never appears as a head of any rule, then this rule may never be applied and may be removed from the LP; (2) if a rule in an LP has in its negative body an atom that never appear as a head of any rule, then this rule may be simplified by removing this part of its negative body. The pre-processing step consists in repeatedly applying (1) and (2) until reaching a fixpoint, i.e., until in the resulting LP Π', any atom in the body of any rule appears as a head in at least one rule, i.e., the Herbrand base of Π' is $\mathtt{HB}_{\Pi'} = \mathtt{Head}(\Pi')$. Notice that because each step of the pre-processing process can only remove rules and/or negative atoms, it is sure to have a fixpoint.

Theorem 5.1. *Let Π be an LP and Π' be the LP which is the fixpoint obtained from Π by repeatedly removing every rule r s.t. $\mathtt{Body}^+(r) \nsubseteq \mathtt{Head}(\Pi)$ and every expression not a s.t. $a \notin \mathtt{Head}(\Pi)$. Then, a 3-valued interpretation $I = \langle T, F\rangle$ is a P-stable model of Π if and only if $I' = \langle T, F'\rangle$ is a P-stable model of Π' where $F' = F \cap \mathtt{HB}_{\Pi'}$.*

$\hat{I}$ As a result of the previous theorem, the same result continues to hold for all the other models that we consider in this paper.

Corollary 5.2. *Let Π be an LP and Π' be the LP obtained from Π by the method described in Theorem 5.1. Then, a 3-valued interpretation I is a well-founded (resp. M-stable, stable, L-stable) model of Π if and only if $I' = \langle T, F' \rangle$ is a well-founded (resp. M-stable, stable, L-stable) model of Π' where $F' = F \cap \mathtt{HB}_{\Pi'}$.*

Now, without loss of generality, we consider in the rest of the paper only LPs that have already been pre-processed. That is, any LP Π used in what follows is s.t.: $\bigcup_{r \in \Pi} \mathtt{Body}^+(r) \subseteq \mathtt{Head}(\Pi)$ and $\bigcup_{r \in \Pi} \mathtt{Body}^-(r) \subseteq \mathtt{Head}(\Pi)$. Let us call this class of LPs *AFN-logic programs (AFN-LPs)*. This is the class of LPs that are directly translatable into AFNs by our proposed approach. Notice that every LP can be transformed into an AFN-Program preserving its semantics (see Theorem 5.1).

Two kinds of interactions are possible between the rules of an LP. To see that, let Π be an LP and r be a rule of Π, then r is blocked if one of the atoms of its negative body is inferred. So, r is attacked by any rule whose head is present in $\mathtt{Body}^-(r)$. On the other hand, if a is an atom of $\mathtt{Body}^+(r)$, then r cannot be applied unless at least one rule of Π whose head is a is applied. This corresponds exactly to the meaning of the necessity relation in an AFN. Accordingly, the translation of an LP into an AFN does not need any complex construction of arguments since the rules themselves can serve as arguments.[7]

Definition 5.3 (The AFN representing an LP)**.** Let $\Pi = \{r_1, \ldots, r_n\}$ be an LP. The AFN representing Π is defined by $\mathcal{G}_\Pi = \langle \Pi, \mathcal{R}_\Pi, \mathcal{N}_\Pi \rangle$ where:

- if $E \subseteq \Pi$ is a subset of rules having the same head (denoted $\mathtt{Head}(E)$) and $r \in \Pi$ is a rule s.t. $\mathtt{Head}(E) \in \mathtt{Body}^+(r)$, then we put $E \, \mathcal{N}_\Pi \, r$;
- if $r, r' \in \Pi$ are two rules s.t. $\mathtt{Head}(r) \in \mathtt{Body}^-(r')$ then we put $r \, \mathcal{R}_\Pi \, r'$.

Before giving the results that relate the models of an LP and the extensions of the corresponding AFN, we need two further definitions allowing one to extract a labelling from a 3-valued interpretation and *vice versa*.

Definition 5.4 (From a labelling of $\mathcal{G}_\Pi$ to a 3-valued interpretation of Π)**.** Let $\Pi = \{r_1, \ldots, r_n\}$ be an LP and $\mathcal{G}_\Pi = \langle \Pi, \mathcal{R}_\Pi, \mathcal{N}_\Pi \rangle$ its corresponding AFN. Let $\mathcal{L} = \langle \mathrm{IN}, \mathrm{OUT}, \mathrm{UND} \rangle$ be a labelling of $\mathcal{G}_\Pi$. The 3-valued interpretation associated to $\mathcal{L}$ is denoted $\mathtt{Int}(\mathcal{L})$ and defined as follows. For every $a \in \mathtt{HB}_\Pi$:

- if there is a rule $r \in \Pi$ s.t. $a = \mathtt{Head}(r)$ and $\mathcal{L}(r) = \mathtt{in}$ then a is interpreted as true, i.e., $\mathtt{Int}(\mathcal{L})(a) = \mathbf{t}$.
- if a is not interpreted as true and there is a rule $r \in \Pi$ s.t. $a = \mathtt{Head}(r)$ and $\mathcal{L}(r) = \mathtt{undec}$ then a is interpreted as undefined i.e., $\mathtt{Int}(\mathcal{L})(a) = \mathbf{u}$.
- Otherwise, i.e., if for every rule $r \in \Pi$ s.t. $a = \mathtt{Head}(r)$ we have $\mathcal{L}(r) = \mathtt{out}$ then a is interpreted as false, i.e., $\mathtt{Int}(\mathcal{L})(a) = \mathbf{f}$.

Definition 5.5 (From a 3-valued interpretation of Π to a labelling of $\mathcal{G}_\Pi$)**.** Let $\Pi = \{r_1, \ldots, r_n\}$ be an LP and $\mathcal{G}_\Pi = \langle \Pi, \mathcal{R}_\Pi, \mathcal{N}_\Pi \rangle$ its corresponding AFN. Let $I = \langle T, F \rangle$ be a 3-valued interpretation of Π. The labelling associated to I is denoted $\mathtt{Label}(I)$ and defined as follows. For every $r \in \Pi$:

- if $\mathtt{Body}^+(r) \subseteq T$ and $\mathtt{Body}^-(r) \subseteq F$ then r is labelled $\mathtt{in}$ in $\mathtt{Label}(I)$.
- if $\mathtt{Body}^+(r) \cap F \neq \emptyset$ or $\mathtt{Body}^-(r) \cap T \neq \emptyset$ then r is labelled $\mathtt{out}$ in $\mathtt{Label}(I)$.
- Otherwise, r is labelled $\mathtt{undec}$ in $\mathtt{Label}(I)$.

[7] This explains why we focused on this particular setting where the roles of attacks and supports are not symmetric.

Now we are ready to establish the relationships between 3-valued semantics of an LP and acceptability semantics of the corresponding AFN.

Theorem 5.6. *Let Π be an LP and $\mathcal{G}_\Pi$ be the AFN representing Π.*

- *If $\mathcal{L}$ is a complete (resp. the grounded, a preferred, a stable) labelling of $\mathcal{G}_\Pi$ then $\mathtt{Int}(\mathcal{L})$ is a P-stable (resp. the well-founded, an M-stable, a stable) model of Π. Inversely, if I is a P-stable (resp. the well-founded, an M-stable, a stable) model of Π then $\mathtt{Label}(I)$ is a complete (resp. the grounded, a preferred, a stable) labelling of $\mathcal{G}_\Pi$.*
- *It is possible to find a semi-stable labelling $\mathcal{L}$ of $\mathcal{G}_\Pi$ s.t. $\mathtt{Int}(\mathcal{L})$ is not an L-stable model of Π. Inversely, it is possible to find an L-stable model I of Π s.t. $\mathtt{Label}(I)$ is not a semi-stable labelling of $\mathcal{G}_\Pi$.*

In summary, except for the case of L-stable models and semi-stable semantics, there is a bijection between semantics of an LP and that of the corresponding AFN. It is worth mentioning that this same exception for semi-stable semantics has been encountered in [30] when translating LPs into Dung AFs. Similarly to [30], we can show that L-stable models of an LP are obtained from the complete labellings, not by minimizing first the set of undefined arguments and then take the conclusions of arguments labelled in, but by directly minimizing the set of conclusions of the undefined arguments.

Theorem 5.7. *Let Π be an LP and $\mathcal{G}_\Pi$ be the AFN representing Π.*
If $\mathcal{L} = (IN, OUT, UND)$ is a complete labelling of $\mathcal{G}_\Pi$ that minimizes (w.r.t. set inclusion) the set $\{\mathtt{Head}(r) | r \in UND\}$ then $\mathtt{Int}(\mathcal{L})$ is an L-stable model of Π. Inversely, if $I = \langle T, F\rangle$ is an L-stable model of Π then $\mathtt{Label}(I)$ is a complete labelling of $\mathcal{G}_\Pi$ that minimizes (w.r.t. set inclusion) the set $\{\mathtt{Head}(r) | r \in UND\}$.

Example 2 (Cont)**.** Consider again the LPs Π_1, Π_2, Π_3 and Π_4 given in Example 2. Using Definition 5.3, it is easy to check that the AFN $\mathcal{G}_{\Pi_1}$ (resp. $\mathcal{G}_{\Pi_2}$, $\mathcal{G}_{\Pi_3}$, $\mathcal{G}_{\Pi_4}$) representing the LP Π_1 (resp. Π_2, Π_3, Π_4) corresponds exactly to the AFN $\mathcal{G}_1$ (resp. $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$) depicted in Fig. 4-(a) (resp. Fig. 4-(b), Fig. 4-(c), Fig. 4-(d)).

Recall that Π_1 has one P-stable model $I = \langle\{s\}, \{p, q\}\rangle$ which is also its unique well-founded, M-stable, stable and L-stable model and $\mathcal{G}_1$ has one complete labelling $\mathcal{L} = (\{c\}, \{a, b\}, \emptyset)$ which is also its unique grounded, preferred, stable and semi-stable labelling. We can check that $I = \mathtt{Int}(\mathcal{L})$ and $\mathcal{L} = \mathtt{Label}(I)$.

Π_2 has one P-stable model $I = \langle\{s\}, \emptyset\rangle$ which is also its unique well-founded, M-stable and L-stable model but Π_2 has no stable model. $\mathcal{G}_2$ has one complete labelling $\mathcal{L} = (\{d\}, \{c\}, \{a, b\})$ which is also its unique grounded, preferred and semi-stable labelling but $\mathcal{G}_2$ has no stable labelling. We have: $I = \mathtt{Int}(\mathcal{L})$ and $\mathcal{L} = \mathtt{Label}(I)$.

Π_3 has three P-stable models $I_1 = \langle\emptyset, \emptyset\rangle$, $I_2 = \langle\{q\}, \{p\}\rangle$, $I_3 = \langle\{p, t\}, \{q, s, u\}\rangle$. Its well-founded model is I_1, its preferred models are I_2 and I_3, its unique L-stable model is I_3 and it has no stable model. $\mathcal{G}_3$ has three complete labellings: $\mathcal{L}_1 = (\emptyset, \emptyset, \{a, b, c, d, e, f, g\})$, $\mathcal{L}_2 = (\{b\}, \{a\}, \{c, d, e, f, g\})$ and $\mathcal{L}_3 = (\{a, d\}, \{b, c, e\}, \{f, g\})$. Its grounded labelling is $\mathcal{L}_1$, its preferred labellings are $\mathcal{L}_2$ and $\mathcal{L}_3$, its unique semi-stable labelling is $\mathcal{L}_3$ and it has no stable labelling. We can check that $I_i = \mathtt{Int}(\mathcal{L}_i)$ for $i \in \{1, 2, 3\}$.

Π_4 has one P-stable model $I = \langle\emptyset, \emptyset\rangle$ which is also its unique well-founded, M-stable and L-stable model but Π_4 has no stable model. $\mathcal{G}_4$ has one complete labelling $\mathcal{L} = (\emptyset, \emptyset, \{a, b, c, d\})$ which is also its unique grounded, preferred and semi-stable labelling but $\mathcal{G}_4$ has no stable labelling. We have: $I = \texttt{Int}(\mathcal{L})$ and $\mathcal{L} = \texttt{Label}(I)$.

Notice that as stated in Theorem 5.6, in general, there is no full correspondence between L-stable models of an LP and semi- stable labellings of its representing AFN (see counter-examples in the proof of Theorem 5.6 given in the Appendix).

5.2. *From an AFN to a logic program*

Let us consider now the opposite issue, i.e, the representation of an AFN as an LP. The idea is that each argument a gives rise in the LP to an atom and a rule (r_a) that expresses its acceptability conditions. Intuitively, whenever an argument b attacks a, the expression not b appears in the body of r_a. Similarly, whenever a set of arguments E is necessary for a, we introduce a new atom e which appears in the positive body of r_a and a rule for every argument $x \in E$ telling that e is obtained from x. This translation is formally given as follows:

Definition 5.8 (The LP representing an AFN)**.** Let $\mathcal{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{N} \rangle$ be an AFN. The LP $\Pi_\mathcal{G}$ representing $\mathcal{G}$ is constructed as follows:

- Each argument of $\mathcal{A}$ is considered as an atom in $\Pi_\mathcal{G}$. Moreover, let $E_1, \ldots, E_p$ be all the subsets of $\mathcal{A}$ s.t. for every E_i there is some argument $a \in \mathcal{A}$ with $E_i\ \mathcal{N}\ a$. We associate to each E_i an atom (denoted e_i) in $\Pi_\mathcal{G}$. Thus, the Herbrand base of $\Pi_\mathcal{G}$ is $\texttt{HB}_{\Pi_\mathcal{G}} = \mathcal{A} \cup \mathcal{A}'$ where $\mathcal{A}' = \{e_1, \ldots, e_p\}$.
- For every argument $a \in \mathcal{A}$, let $b_1, \ldots, b_m$ be its attackers and let $E_1, \ldots, E_k$ be all the subsets of $\mathcal{A}$ s.t. $E_i\ \mathcal{N}\ a$ for $0 \leqslant i \leqslant k$. The argument a gives rise in $\Pi_\mathcal{G}$ to the rule: $r_a : a \leftarrow e_1, \ldots, e_k, \text{not } b_1, \ldots, \text{not } b_m$.
- For every atom e_i associated to a set E_i we add $|E_i|$ rules as follows: for every $x \in E_i$, the rule: $e_i \leftarrow x$ is added to $\Pi_\mathcal{G}$.

To extract a 3-valued interpretation from a labelling and *vice versa*, let us define the functions $\texttt{Int}'$ and $\texttt{Label}'$.

Definition 5.9 (From a labelling of $\mathcal{G}$ to a 3-valued interpretation of $\Pi_\mathcal{G}$)**.** Let $\mathcal{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{N} \rangle$ be an AFN and $\Pi_\mathcal{G}$ its corresponding LP. Let $\mathcal{L} = \langle \text{IN}, \text{OUT}, \text{UND} \rangle$ be a labelling of $\mathcal{G}$. The 3-valued interpretation associated to $\mathcal{L}$ is denoted $\texttt{Int}'(\mathcal{L})$ and defined as follows:

- For every $a \in \mathcal{A}$, if $\mathcal{L}(a) = \texttt{in}$ (resp. $\mathcal{L}(a) = \texttt{out}$, $\mathcal{L}(a) = \texttt{undec}$) then $\texttt{Int}'(\mathcal{L})(a) = \mathbf{t}$ (resp. $\texttt{Int}'(\mathcal{L})(a) = \mathbf{f}$, $\texttt{Int}'(\mathcal{L})(a) = \mathbf{u}$).
- For every $e \in \mathcal{A}'$:
  - ∗ if there is a rule $r : e \leftarrow a$ in $\Pi_\mathcal{G}$ s.t. $\mathcal{L}(a) = \texttt{in}$ then e is interpreted as true, i.e., $\texttt{Int}'(\mathcal{L})(e) = \mathbf{t}$.
  - ∗ if e is not interpreted as true and there is a rule $r : e \leftarrow a$ in $\Pi_\mathcal{G}$ s.t. $\mathcal{L}(a) = \texttt{undec}$ then e is interpreted as undefined i.e., $\texttt{Int}'(\mathcal{L})(e) = \mathbf{u}$.
  - ∗ Otherwise, i.e., if for every rule $r : e \leftarrow a$ in $\Pi_\mathcal{G}$, it holds that $\mathcal{L}(a) = \texttt{out}$ then e is interpreted as false, i.e., $\texttt{Int}'(\mathcal{L})(e) = \mathbf{f}$.

Definition 5.10 (From a 3-valued interpretation of $\Pi_\mathcal{G}$ to a labelling of $\mathcal{G}$)**.** Let $\mathcal{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{N} \rangle$ be an AFN and $\Pi_\mathcal{G}$ its corresponding LP. Let $I = \langle T, F \rangle$ be a 3-valued interpretation of $\Pi_\mathcal{G}$. The labelling associated to I is defined by: $\texttt{Label}'(I) = (T \cap \mathcal{A}, F \cap \mathcal{A}, \mathcal{A} \setminus (T \cup F))$.

Now, the following result shows full correspondences between the acceptability semantics of an AFN and the semantics of the corresponding LP.

Theorem 5.11. *Let $\mathcal{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{N} \rangle$ be an AFN and $\Pi_{\mathcal{G}}$ be the LP representing $\mathcal{G}$. If $\mathcal{L}$ is a complete (the grounded, a preferred, a stable, a semi-stable) labelling of $\mathcal{G}$ then $I_{\mathcal{L}} = \mathtt{Int}'(\mathcal{L})$ is a P-stable (the well-founded, an M-stable, a stable, an L-stable) model of $\Pi_{\mathcal{G}}$. Inversely, if $I = \langle T, F \rangle$ is a P-stable (the well-founded, an M-stable, a stable, an L-stable) model of $\Pi_{\mathcal{G}}$ then $\mathcal{L}_I = \mathtt{Label}'(I)$ is a complete (the grounded, a preferred, a stable, a semi-stable) labelling of $\mathcal{G}$.*

Example 3 (Cont)**.** Let us take again our four AFNs $\mathcal{G}_1, \dots, \mathcal{G}_4$. The LP Π'_1 (resp. Π'_2, Π'_3, Π'_4) is obtained from $\mathcal{G}_1$ (resp. $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$) by Definition 5.8. (see Fig. 7).

Π'_1 has one P-stable model $I = \langle \{c\}, \{a, b, e_1, e_2\} \rangle$ which is also its unique well-founded, M-stable, stable and L-stable model. $\mathcal{G}_1$ has one complete labelling $\mathcal{L} = (\{c\}, \{a, b\}, \emptyset)$ which is also its unique grounded, preferred, stable and semi-stable labelling. It is easy to check that $I = \mathtt{Int}'(\mathcal{L})$ and $\mathcal{L} = \mathtt{Label}'(I)$.

Π'_2 has one P-stable model $I = \langle \{d\}, \{c\} \rangle$ which is also its unique well-founded, M-stable and L-stable model but Π'_2 has no stable model. $\mathcal{G}_2$ has one complete labelling $\mathcal{L} = (\{d\}, \{c\}, \{a, b\})$ which is also its unique grounded, preferred and semi-stable labelling but $\mathcal{G}_2$ has no stable labelling. We have: $I = \mathtt{Int}'(\mathcal{L})$ and $\mathcal{L} = \mathtt{Label}'(I)$.

Π'_3 has three P-stable models $I_1 = \langle \emptyset, \emptyset \rangle$, $I_2 = \langle \{b, e_1\}, \{a\} \rangle$ and $I_3 = \langle \{a, d\}, \{b, c, e, e_1\} \rangle$. Its well-founded model is I_1, its preferred models are I_2 and I_3, its unique L-stable model is I_3 and it has no stable model. $\mathcal{G}_3$ has three complete labellings: $\mathcal{L}_1 = (\emptyset, \emptyset, \{a, b, c, d, e, f, g\})$, $\mathcal{L}_2 = (\{b\}, \{a\}, \{c, d, e, f, g\})$ and $\mathcal{L}_3 = (\{a, d\}, \{b, c, e\}, \{f, g\})$. Its grounded labelling is $\mathcal{L}_1$, its preferred labellings are $\mathcal{L}_2$ and $\mathcal{L}_3$, its unique L-stable labelling is $\mathcal{L}_3$ and it has no stable labelling. It is easy to check that $I_1 = \mathtt{Int}'(\mathcal{L}_1)$ and $\mathcal{L}_1 = \mathtt{Label}'(I_1)$; $I_2 = \mathtt{Int}'(\mathcal{L}_2)$ and $\mathcal{L}_2 = \mathtt{Label}'(I_2)$; $I_3 = \mathtt{Int}'(\mathcal{L}_3)$ and $\mathcal{L}_3 = \mathtt{Label}'(I_3)$.

Π'_4 has one P-stable model $I = \langle \emptyset, \emptyset \rangle$ which is also its unique well-founded, M-stable and L-stable model but Π'_4 has no stable model. $\mathcal{G}_4$ has one complete labelling $\mathcal{L} = (\emptyset, \emptyset, \{a, b, c, d\})$ which is also its unique grounded, preferred and semi-stable labelling but $\mathcal{G}_4$ has no stable labelling. We have: $I = \mathtt{Int}'(\mathcal{L})$ and $\mathcal{L} = \mathtt{Label}'(I)$.

(Π'_1)	(Π'_2)	(Π'_3)	(Π'_4)
$(r_a)\ a \leftarrow e_2$	$(r_a)\ a \leftarrow e$	$(r_a)\ a \leftarrow \mathit{not}\ b$	$(r_a)\ a \leftarrow \mathit{not}\ c, \mathit{not}\ d$
$(r_b)\ b \leftarrow e_1$	$(r_b)\ b \leftarrow \mathit{not}\ a$	$(r_b)\ b \leftarrow \mathit{not}\ a$	$(r_b)\ b \leftarrow e_2$
$(r_c)\ c \leftarrow \mathit{not}\ b$	$(r_c)\ c \leftarrow \mathit{not}\ d$	$(r_c)\ c \leftarrow e_1, \mathit{not}\ e$	$(r_c)\ c \leftarrow e_1$
$(r_1)\ e_1 \leftarrow a$	$(r_d)\ d \leftarrow$	$(r_d)\ d \leftarrow \mathit{not}\ c$	$(r_d)\ d \leftarrow \mathit{not}\ a, \mathit{not}\ b$
$(r_2)\ e_2 \leftarrow b$	$(r_1)\ e \leftarrow b$	$(r_e)\ e \leftarrow \mathit{not}\ d$	$(r_1)\ e_1 \leftarrow a$
	$(r_2)\ e \leftarrow c$	$(r_f)\ f \leftarrow e_2, \mathit{not}\ e$	$(r_2)\ e_1 \leftarrow b$
		$(r_g)\ g \leftarrow \mathit{not}\ g$	$(r_3)\ e_2 \leftarrow c$
		$(r_1)\ e_1 \leftarrow b$	$(r_4)\ e_2 \leftarrow d$
		$(r_2)\ e_2 \leftarrow g$	

Fig. 7. LPs obtained from our four AFNs.

6. Related work and perspectives

An early work of bipolarity in abstract argumentation is that about bipolar argumentation frameworks (BAFs) [38,40]. In this work, the meaning of support is left unspecified to keep a high abstraction level. However, the drawback of this proposal is the possibility to find counter-intuitive results in some situations.

The work presented in [19] started from a criticism of BAFs on two points, namely, the loss of admissibility in the extensions obtained from the meta-model using coalitions [40]. The proposed approach develops the so-called deductive support and introduces mediated attacks instead of indirect attacks. The authors show that the admissibility of extensions is then restored. As remarked in [71] then later in [41,45], it turns out that the deductive support is nothing but the inverse of the necessity relation in the case where the latter is binary (relating couples of single arguments). Thus, if we limit ourselves to a binary support relation, instead of imposing the use of only one type of support relation, one can start from a system where the two types are freely expressed and then reduce in a preliminary stage all the relations to one type. Note that in this case, all the results of our paper hold for a deductive support relation D by simply using closure under D instead of closure under N^{-1}. The subsequent development in [19] focuses on the definition of a meta-argumentation model to handle supports and introduce defeasible supports. In our work we propose a similar approach of meta-argumentation for AFNs. Furthermore, we address the question of generalizing the existing Dung acceptability semantics in presence of the necessity relation. Another difference is that our framework uses a more general setting where single arguments may be supported by sets of arguments. Finally our approach takes benefit of the necessity relation in establishing relationships between AFNs and LPs.

Another approach that shares some features with ours is the evidence based argumentation, first introduced in [72]. This approach considers that only arguments that have some evidential support can attack other arguments. The evidential support of an argument comes either directly from the environment (*prima facie* arguments) or from a chain of supports that originates in such *prima facie* arguments (*standard* arguments). A similar idea is present in our work. Indeed, to ensure admissibility of a set, we must guarantee just the response to attacks coming from arguments that are N-Cycle-Free, i.e., those that have no need for a support or that are ultimately supported by arguments that have no need for a support. Thus, AFNs may be seen as a possible concretization of the notion of evidence where all arguments are by default supported unless they are taken in a set which is not N-Cycle-Free. The reader may refer to [41,45,78] for a more detailed comparison between the necessity, the deductive and the evidential supports in the context where only binary support relations are used.

Several recent works generalized Dung AFs to represent recursive attacks (attacks targeting attacks) [14,34,37]. Likewise, bipolar frameworks that represent recursive attacks and support (see e.g. [35,36,42,46,62]) have also be extensively studied. Notice that [46,62] uses necessity support relation while [36,42] use evidential support.

The work developed in [20,22,75] introduced abstract dialectical frameworks (ADF), a powerful generalization of Dung's AFs that formalizes the idea of proof standards, widely studied in legal reasoning domain. This idea is captured in ADFs by linking each argument to a set of arguments (its parents) and introducing the notion of acceptance conditions that determine whether an argument is accepted or not according to the acceptance status of its parents. However, a sub-class of ADFs called bipolar ADFs (BADFs) is identified, where the relation between an argument and a parent plays always one role: either an attack or a support. A main difference between our work and ADFs lies in the method used to generalize stable and admissible semantics. ADFs adapt techniques from logic programming, namely G/L reduct, to avoid necessity cycles. In our work, thanks to the notions of coherence and strong coherence used instead of conflict-freeness, we keep our definitions similar to that in Dung's original AFs. Another point is that in the method we use to encode an LP as a AFN, each rule is represented by an argument which gives an homogeneous view of the meaning of an argument. In [22], a similar homogenous representation using atoms as arguments is proposed but as pointed out in the paper, it leads in general to an ADF which may not be bipolar. To obtain a BADF, one must introduce new arguments designating rules. The resulting representation is then heterogeneous in the sense that arguments may refer to rules or to atoms. Finally, the opposite question, i.e., the representation of ADFs as LPs is not explicitly considered. However, subsequent work has been done to study ADFs in several other directions. We can cite [20] in which the semantics of ADFs are inspired by approximation fixpoint theory, [76] which proposes a probabilistic version of ADFs, [54] which shows how to represent an ADF with only attack relations and [48] which investigates sub-classes of ADFs.

A main notion in argumentation approaches with structured arguments, is that of sub-argument [17, 57,66,81,87]. A sub-argument provides an intermediary conclusion to its super-argument. From this viewpoint, sub-arguments may be seen as supporting their super-arguments. The work proposed in [66] introduces the AFs with sub-arguments (AFS). An AFS extends a Dung AF with two binary relations on arguments: a sub-argument relation and a preference relation. To capture the requirements of the sub-argument relation, the authors introduce the so-called conflict inheritance constraint on the attack relation. It says that if a attacks b then any super-argument of a attacks any super-argument of b. In [45], the authors show that the kind of support provided by sub-arguments has the meaning of necessity. They point out that a rational constraint which relates arguments and sub-arguments is the compositionality principle (see [82]). It says that an argument cannot be accepted unless all its sub-arguments are accepted, i.e., (i) if an argument is accepted then all its sub-arguments are accepted and (ii) if an argument is not accepted then all its super-arguments are not accepted. It turns out that this captures exactly the meaning of (a binary) necessity relation in an AFN: if we have $a \mathcal{N} b$ then if b is accepted then necessarily a is accepted and if a is not accepted then b is not accepted. In the same spirit, the work of [80] points out the strong correspondence between the necessity relation in AFNs and the sub-argument relation in the ASPIC+ system.

The work in [44] introduces the backing-undercutting argumentation framework (BUAF) which extends Dung AFs by incorporating a special binary support relation (backing relation) and a partial order representing a preference relation among arguments. The support relation is intended to represent the backing link considered in Toulmin's model of an argument (see [88,90] for more details about Toulmin's model).[8] Three kinds of attack relations are distinguished: the rebutting, the undercutting and the undermining attacks. Only the undercutting attack can interact with the backing relation since both of them involve the warrant. As in many other approaches to bipolarity in argumentation, the objective in this approach is to produce new indirect attacks that result from the interaction of direct attacks and supports. In this approach, the final negative interactions are called defeats. They take into account the input attack, the backing as well as the preference relations. Let us briefly present the three kinds of defeats present in this approach. The first one is the primary defeat: a primarily defeats c in one of the three following situation: if a rebuts or undermines c and c is not strictly preferred to a; if a undercuts

[8]Roughly speaking, Toulmin's scheme of an argument is constituted of five elements. A data (D) which is the ground which we produce as a support. A claim (C) which is a conclusion based on the data. A warrant (W) which is a rule-like statement that justifies the conclusion of (C) from (D). A qualifier (Q) which reflects a degree of force that data confers on the claim in virtue of the warrant. A backing (B) which explains why the warrant holds and thus brings a support for it. Finally, a rebuttal (R) which represents particular contexts where the claim is challenged.

c and c has no backing argument or if a undercuts c, c has a backing argument b but b is not strictly preferred to a. The second kind of defeat is the implicit defeat: a implicitly defeats b in two possible situations: if b is a backing for c, a undercuts c and b is not strictly preferred to a; or if a is a backing for c, b undercuts c and neither b is strictly preferred to a nor a is strictly preferred to b. The last kind of defeat is the indirect defeat which covers the other kinds of defeats (primary and implicit defeats) and the new defeats that result recursively by chaining backing arguments. Once all the indirect defeats are computed, Dung acceptability semantics are used to evaluate arguments. It has been shown in [45] that some aspects of the backing relation correspond to some aspects of the necessity relation but there is no full correspondence between the two relations.

Constrained AFs [47] extend Dung AFs with propositional constraints on arguments. We notice that AFNs cannot be reduced to constrained AFs where $E\ N\ b$ is replaced by the implication $b \Rightarrow \bigvee_{a \in E}$. To show this, recall that a stable extension of a constrained AF is a stable extension of the corresponding AF that verifies the additional constraints. This does not hold for AFNs. For instance, the AFN $\langle A = \{a, b, c\}, R = \{a, b\}, N = \{(b, c)\}\rangle$ has one stable extension: $\{a\}$, but the constrained AF $\langle A, R, C = c \Rightarrow b\rangle$ has no stable extension (the only stable extension of $\langle A, R\rangle$ is $\{a, c\}$ which does not verify the constraint C).

Linking abstract argumentation to logic programming is an interesting research topic. It goes back to the seminal work of Dung [53]. Some works in this domain have considered the issue of using logic programming, especially answer set programming, to compute the extensions of AFs under different semantics (see e.g. [55,56,85]). The objective of these works differs from ours.

In [1], the authors focuses on the equivalence between Abstract Dialectical Frameworks and logic programs under 3-valued semantics. More precisely this work focuses on a fragment of ADFs, called Attacking dialectical frameworks (ADF$^+$s) and provides a translation from normal LPs to ADF$^+$ such that partial stable, well-founded, regular and stable models of normal LPs are in a one to one correspondence with complete, grounded, preferred and stable extensions of the corresponding ADF$^+$s, respectively.

Another work that is closer to ours is that presented in [29,30]. This work establishes the links between 3-valued LPs and Dung AFs. An LP is represented by a Dung AF where an argument is obtained by chaining a subset of rules of the LP (as in rule-based argumentation systems [7,68]). An argument involves a set of rules, a conclusion (the head of the last used rule), a set of vulnerabilities (the atoms appearing in the negative bodies of the rules involved in the argument) and a set of sub-arguments. Then, an attack relation is defined from an argument a to an argument b if the conclusion of a belongs to the set of vulnerabilities of b. The process results in a Dung AF. The authors show then a one-to-one correspondence between the well-founded (resp. P-stable, M-stable, stable) models of the LP and the grounded (resp. complete, preferred, stable) extensions of the corresponding AF. Only L-stable semantics of LPs and semi-stable semantics are not in full correspondence. Then, it has been shown that when representing an AF as an LP, the picture is complete, i.e., all the correspondences hold including that semi-stable extensions and L-stable model. Notice that [29] generalizes the approach to cope with ideal and eager semantics. Our work may be seen as a generalization of this approach to bipolar setting using the necessity support. More precisely, in representing an LP as an abstract AFN, our work takes benefit from the positive interaction (necessity links) between the rules of an LP to propose a new simpler instantiation method where each rule is represented by an argument. The use of an additional necessity support relation in the argumentation models avoids the construction of complex arguments. In the opposite direction our work allows one to represent as an LP a wider range of argumentation frameworks since Dung AFs are a strict subset of AFNs.

In the same spirit, the work by Alfano et al. [2,3] presents how to translate different kinds of extensions of dung AFs (called AF-based frameworks) into logic programs and shows how the acceptability semantics of such AF-based frameworks are related to particular cases of partial stable models of the corresponding LPs. The considered AF-based frameworks include: Original Dung AFs [53]; bipolar AFs (BAF), namely AFs with necessity support (AFN) [71] and AFs with deductive support (AFD) [19,91]; recursive AFs (Rec-AF) namely, AFs with recursive attack (AFRA) [14] and recursive AFs (RAF) [37] and recursive bipolar AFs (Rec-BAF) namely, attack-support AFs (ASAF) [62], recursive AFs with necessities (RAFN) [35], AFs with recursive attack and deductive support (AFRAD) [3] and recursive AFs with deductive support (RAFD) [3]. Our work differs from that by Alfano et al. [2,3] in two main respects: First, the work in [2,3] considers only the restricted version of AFNs where the necessary support relation is binary and acyclic while our proposal considers a more general framework where the support relation may relate a set of arguments to a single one and no restriction is made about its cyclicality. The second issue is that the work in [2,3] considers only the representation of AF-based frameworks as LPs and does not study the opposite direction (representing an LP as an AF-based framework). In our work, we consider the translation in the two directions. This shows among other things that among the different existing AF-based frameworks, AFN is suitable to represent any normal LP in a direct and simple way.

For a more comprehensive comparative study of the above approaches of support in argumentation as well as some other approaches having some links with the topic,[9] the reader is referred to [45]. It is worth noticing that most of the current bipolar argumentation approaches use binary support relations and turn eventually the bipolar framework to a Dung AF in order to evaluate arguments. Our work goes a step further: it uses a generalized necessity relation that involves sets of arguments as supports, allows one to evaluate arguments directly in the new setting, generalizes the labelling characterization and algorithms to the new context and relates AFNs to LPs.

As stated above, some works (e.g.. [41]) have started to look for similarities and differences between the different approaches of support in abstract argumentation. An interesting future work would be to propose a unified framework able to take into account different kinds of supports and to define a general approach for acceptability semantics to such a framework in the simple and high-order case (recursive attacks and supports). As pointed out in [80], in argumentation approaches with structured arguments, the sub-argument relation is a possible instantiation of the necessity relation. This idea opens the way to a future work on a more general question about how to instantiate different kinds of supports present in the literature and then to propose general postulates that describe their expected behavior as it is done for the attack relation (see e.g.. [61]).

As regards the links with logic programming, we believe that the strong relationship established between AFNs and LPs is a key tool that will enable us to bring advanced results in logic programming into abstract argumentation theory and *vice versa*. For instance, we are interested on works about equilibrium logics that gave logical foundation to stable models [74] and later partial equilibrium logics that generalize the idea to capture well-founded models [23–25]. We believe that variants of partial equilibrium logics may capture other 3-valued models of LPs and we want to exploit the strong links between AFNs and 3-valued semantics to give a logical foundation for Dung AFs, AFNs and possibly other approaches of bipolar argumentation in terms of partial equilibrium logics. In the same spirit, the work in [52] captures the notion of stable model with the notion of minimally specific model of generalized possibilistic logic [51] which is an extension of possibilistic logic (see e.g. [49,50]) that enables one to reason on

[9] namely, [90] considers a reconstruction of Toulmin's ideas using DEFLOG and [43] studies an extension of DELP, where Toulmin's form of support between backings and warrants are considered.

epistemic states instead of merely constrain them. Again, the strong links between AFNs and LPs may be exploited to capture acceptability semantics of Dung AFs and AFNs in the generalized possibilistic logic framework.

Appendix. Proofs

Proof of Proposition 3.6. $\Rightarrow$) Let $\mathcal{E} \subseteq \mathcal{A}$ be a coherent set. To prove that every argument $a \in \mathcal{E}$ is powerful in $\mathcal{E}$ let us construct a sequence of sets of arguments $\mathcal{E}_1, \ldots, \mathcal{E}_m$ s.t.: (1) $\mathcal{E}_1, \ldots \mathcal{E}_m$ is a partition of $\mathcal{E}$; (2) $\forall a \in \mathcal{E}_1, \nexists E \subseteq \mathcal{A}$ s.t. $E\ \mathcal{N}\ a$ (arguments of $\mathcal{E}_1$ does not require any other argument) and (3) for $i > 1$, then $\forall a \in \mathcal{E}_i, \forall E \subseteq \mathcal{A}$ s.t. $E\ \mathcal{N}\ a$, $E \cap \bigcup_{1 \leqslant j<i} \mathcal{E}_i \neq \emptyset$ (the necessities on any argument of $\mathcal{E}_i$ are satisfied by the preceding sets $\mathcal{E}_1, \ldots \mathcal{E}_{i-1}$). Indeed if such a sequence is constructed, then for every argument $a \in \mathcal{E}$, let $a \in \mathcal{E}_i$. It is then clear that a sequence of arguments (not necessarily a minimal one) leading to a may be obtained by simply flattening the sequence $\mathcal{E}_1, \ldots \mathcal{E}_{i-1}$ (replacing each $\mathcal{E}_i$ by the subsequence of its arguments taken in an arbitrary order) and adding a at the end. Now the sequence of sets of arguments $\mathcal{E}_1, \ldots \mathcal{E}_m$ is constructed us follows. $\mathcal{E}_1 = \{a \in \mathcal{E} \mid \nexists E \subseteq \mathcal{A}$ s.t. $E\ \mathcal{N}\ a\}$. We have $\mathcal{E}_1 \neq \emptyset$ for if $\mathcal{E}_1 = \emptyset$ then $\forall a \in \mathcal{E}, \exists E \subseteq \mathcal{A}$ s.t. $E\ \mathcal{N}\ a$. Since $\mathcal{E}$ is closed under $\mathcal{N}^{-1}$, $E \cap \mathcal{E} \neq \emptyset$. But this means that $\mathcal{E}$ is not coherent, contradiction.

Now, let $\mathcal{E}_2 = \{a \in \mathcal{E} \setminus \mathcal{E}_1 \mid \nexists E \subseteq \mathcal{A}$ s.t. $E \cap \mathcal{E}_1 = \emptyset$ and $E\ \mathcal{N}\ a\}$. We have $\mathcal{E}_2 \neq \emptyset$ for if $\mathcal{E}_2 = \emptyset$ then $\forall a \in \mathcal{E} \setminus \mathcal{E}_1, \exists E \subseteq \mathcal{A}$ s.t. $E\ \mathcal{N}\ a$ and $E \cap \mathcal{E}_1 = \emptyset$. Since $\mathcal{E}$ is closed under $\mathcal{N}^{-1}$, $E \cap \mathcal{E} \neq \emptyset$, hence $\mathcal{E} \cap (\mathcal{E} \setminus \mathcal{E}_1) \neq \emptyset$ (because $E \cap \mathcal{E}_1 = \emptyset$), i.e., $\mathcal{E}$ is not coherent, contradiction.

We continue this process of constructing the sets E_i. Since the number of arguments is finite, this process stops necessarily at some set $\mathcal{E}_m$ whose arguments are not necessary to any other argument. Then, by construction, the sequence $\mathcal{E}_1, \ldots \mathcal{E}_m$ satisfies the conditions (1)–(3) mentioned above.

$\Leftarrow$) Let $\mathcal{E} \subseteq \mathcal{A}$ s.t. $\forall a \in \mathcal{E}$, a is powerful in $\mathcal{E}$, i.e., $\forall a \in \mathcal{E}$, there is a sequence of arguments $a_0, \ldots, a_k$ of $\mathcal{E}$ s.t. $a_k = a$; there is no $E \subseteq \mathcal{A}$ s.t. $E\ \mathcal{N}\ a_0$ and for $1 \leqslant i \leqslant k$: for all $E \subseteq \mathcal{A}$, if $E\ \mathcal{N}\ a_i$, then $E \cap \{a_0, \ldots, a_{i-1}\} \neq \emptyset$.

It is straightforward from this definition that $\forall a \in \mathcal{E}$, if $E\ \mathcal{N}\ a$ for some $E \subseteq \mathcal{A}$, then $E \cap \mathcal{E} \neq \emptyset$, i.e., that $\mathcal{E}$ is closed under $\mathcal{N}^{-1}$. From the definition (first item), it is clear that $\exists a \in \mathcal{E}$ s.t. $\forall E \subseteq \mathcal{A}$, $E\ \not\mathcal{N}\ a$. This condition implies that: $\exists a \in \mathcal{E}$ s.t. $\forall E \subseteq \mathcal{A}$, $E\ \not\mathcal{N}\ a$ or $E \cap \mathcal{E} \neq \emptyset$. It is easy to check that this last condition is equivalent to that for N-Cycle-Freeness (see Definition 3.3). $\square$

Proof of Proposition 3.7.

- a is not powerful in $\mathcal{E}$ iff there is no coherent subset $\mathcal{C}$ of $\mathcal{E}$ s.t. $a \in \mathcal{C}$.
  $\Rightarrow$) Let $\mathcal{E} \subseteq \mathcal{A}$ and let $a \in \mathcal{E}$ be a non powerful argument of $\mathcal{E}$. Suppose that there is $\mathcal{C} \subseteq \mathcal{E}$ s.t. $\mathcal{C}$ is coherent and $a \in \mathcal{C}$. From proposition 3.6, it follows that a is powerful in $\mathcal{C}$. But in this case a remains powerful in any superset of $\mathcal{C}$ (it suffices to take in $\mathcal{E}$ the same sequence ending by a taken in $\mathcal{C}$). Contradiction.
  $\Leftarrow$) Let $\mathcal{E} \subseteq \mathcal{A}$ and $a \in \mathcal{E}$ s.t. there is no coherent subset $\mathcal{C}$ of $\mathcal{E}$ containing a. Suppose that a is powerful in $\mathcal{E}$. Then from Definition 3.5, there is a sequence $a_0, \ldots, a_k \in \mathcal{E}$ s.t. $a_k = a$; there is no $E \subseteq \mathcal{A}$ s.t. $E\ \mathcal{N}\ a_0$ and for $1 \leqslant i \leqslant k$: for all $E \subseteq \mathcal{A}$, if $E\ \mathcal{N}\ a_i$, then $E \cap \{a_0, \ldots, a_{i-1}\} \neq \emptyset$. It is then easy to check that the set $\{a_0, \ldots, a_k\}$ is coherent. Contradiction.
- a is not powerful in $\mathcal{E}$ iff $\exists E \subseteq \mathcal{A}$ s.t. $E\ \mathcal{N}\ a$ and $\forall E \subseteq \mathcal{A}$ s.t. $E\ \mathcal{N}\ a$, $\forall b \in E \cap \mathcal{E}$, b is not powerful in $\mathcal{E}$.
  $\Rightarrow$) Let $\mathcal{E} \subseteq \mathcal{A}$ and a an argument which is not powerful in $\mathcal{E}$. If we suppose that $\nexists E \subset \mathcal{A}$ s.t. $E\ \mathcal{N}\ a$, then clearly $\{a\}$ is a coherent set containing a, contradiction. If we suppose that $\exists E \subseteq \mathcal{A}$

s.t. $E \mathcal{N} a$ and $\exists b \in E \cap \mathcal{E}$ s.t. b is powerful in $\mathcal{E}$. Then, there is $\mathcal{C} \subseteq \mathcal{E}$ s.t. $\mathcal{C}$ is coherent and $b \in \mathcal{C}$. It follows that $\mathcal{C} \cup \{a\}$ is a coherent set too. But thus contradicts the fact that a is not powerful in $\mathcal{E}$.
$\Leftarrow$) Let $\mathcal{E} \subseteq \mathcal{A}$ and a an argument s.t.: $\exists E \subseteq \mathcal{A}$ s.t. $E \mathcal{N} a$ and $\forall E \subseteq \mathcal{A}$ s.t. $E \mathcal{N} a$, $\forall b \in E \cap \mathcal{E}$, b is not powerful in $\mathcal{E}$. Suppose that a is powerful in $\mathcal{E}$. Thus, there is $\mathcal{C} \subseteq \mathcal{E}$ s.t. $\mathcal{C}$ is coherent and $a \in \mathcal{C}$. From the definition of coherence, for every $E \subseteq \mathcal{A}$ s.t. $E \mathcal{N} a$, $E \cap \mathcal{C} \neq \emptyset$, i.e., there exists $b \in E \cap \mathcal{E}$ s.t. $b \in \mathcal{C}$. This means that there is a coherent subset of $\mathcal{E}$ containing b ($\mathcal{C}$), i.e., b is powerful in $\mathcal{E}$. Contradiction. $\square$

Proof of Theorem 3.13. Let $\mathcal{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{N} \rangle$ be an AFN and $\mathcal{E} \subseteq \mathcal{A}$ be a strongly coherent set.

(1) $\mathcal{E}$ is an admissible set iff $\mathcal{E} \subseteq \mathcal{F}_{\mathcal{G}}(\mathcal{E})$:
$\Rightarrow$) Suppose that $\mathcal{E}$ is an admissible set. Thus, if $a \in \mathcal{E}$ then $\mathcal{E}$ defends a, i.e. $a \in \mathcal{F}_{\mathcal{G}}(\mathcal{E})$.
$\Leftarrow$) Suppose that $\mathcal{E} \subseteq \mathcal{F}_{\mathcal{G}}(\mathcal{E})$. $\forall a \in \mathcal{E}$ we have $a \in \mathcal{F}_{\mathcal{G}}(\mathcal{E})$, i.e., a is defended by $\mathcal{E}$. Thus $\mathcal{E}$ is strongly coherent and defends all its elements, so it is admissible.

(2) $\mathcal{E}$ is a complete extension iff $\mathcal{E} = \mathcal{F}_{\mathcal{G}}(\mathcal{E})$:
$\Rightarrow$) Suppose that $\mathcal{E}$ is a complete extension. Then, $\mathcal{E}$ is admissible and from proposition point 1. of the current theorem, we have $\mathcal{E} \subseteq \mathcal{F}_{\mathcal{G}}(\mathcal{E})$. Now, let $a \in \mathcal{F}_{\mathcal{G}}(\mathcal{E})$. Then, a is defended by $\mathcal{E}$ but since $\mathcal{E}$ is a complete extension, $a \in \mathcal{E}$. Thus $\mathcal{F}_{\mathcal{G}}(\mathcal{E}) \subseteq \mathcal{E}$, so $\mathcal{F}_{\mathcal{G}}(\mathcal{E}) = \mathcal{E}$.
$\Leftarrow$) Suppose that $\mathcal{E} = \mathcal{F}_{\mathcal{G}}(\mathcal{E})$. Then $\mathcal{E} \subseteq \mathcal{F}_{\mathcal{G}}(\mathcal{E})$ and $\mathcal{E}$ is admissible (from point 1. of the current theorem). Now, let a be an argument defended by $\mathcal{E}$, then $a \in \mathcal{F}_{\mathcal{G}}(\mathcal{E})$. Since $\mathcal{F}_{\mathcal{G}}(\mathcal{E}) \subseteq \mathcal{E}$, $a \in \mathcal{E}$. Thus $\mathcal{E}$ is admissible and contains any argument it defends, hence $\mathcal{E}$ is a complete extension.

(3) $\mathcal{E}$ is the grounded extension of $\mathcal{G}$ iff it is the least (w.r.t. set inclusion) complete extension of $\mathcal{G}$:
Follows from the definition of a grounded extension (Definition 3.12) and the second point of the current.

(4) There is at least one preferred extension for $\mathcal{G}$; every preferred extension is a complete extension but not *vice versa*; $\mathcal{E}$ is a preferred extension iff it is a maximal (w.r.t. set inclusion) complete extension:
The existence of at least one preferred extension follows from the fact that there is always at least one admissible set for any AFN. Indeed, obviously $\emptyset$ is always admissible.
Let $\mathcal{E}$ be a preferred extension and suppose that $\mathcal{E}$ is not complete. Then, there is $a \in \mathcal{A}$ s.t. $\mathcal{E}$ defends a and $a \notin \mathcal{E}$. But, in this case $\mathcal{E} \cup \{a\}$ is admissible which contradicts the fact that $\mathcal{E}$ is a maximal admissible set. Hence, every preferred extension is a complete extension. Let us take a counter-example showing that a complete extension may not be preferred. Let $\mathcal{G} = \langle \mathcal{A} = \{a, b, c\}, \mathcal{R} = \{(a, b), (b, a), (b, c), (c, c)\}, \mathcal{N} = \emptyset \rangle$. $\mathcal{G}$ has three complete extensions: $\emptyset$, $\{a\}$ and $\{b\}$. $\{a\}$ and $\{b\}$ are also preferred but $\emptyset$ is not.
The fact that $\mathcal{E}$ is a preferred extension iff it is a maximal (w.r.t. $\subseteq$) complete extension follows immediately from point 2 of the current theorem.

(5) If $\mathcal{E}$ is a stable extension then $\mathcal{E}$ is a semi-stable extension but not *vice versa*; if $\mathcal{E}$ is a semi-stable extension, then $\mathcal{E}$ is a preferred extension but not *vice versa*:
Let $\mathcal{E} \subseteq \mathcal{A}$ be a stable extension. Then from the definition of stable extensions we have $\mathcal{E}^d = \mathcal{A} \setminus \mathcal{E}$. Thus, $\mathcal{E} \cup \mathcal{E}^d = \mathcal{A}$ which is the maximal possible set of arguments. Hence, $\mathcal{E}$ is a semi-stable extension.
To show that the inverse is false, take $\mathcal{G} = \langle \mathcal{A} = \{a, b, c\}, \mathcal{R} = \{(c, c)\}, \mathcal{N} = \{a, b\} \rangle$. $\{a, b\}$ is a semi-stable extension of $\mathcal{G}$ but not a stable extension.
Before proving that each semi-stable extension is preferred, let us first prove the following lemma.

Lemma 1. *If $\mathcal{E}$ and $\mathcal{E}'$ are two complete extensions s.t. $\mathcal{E} \subset \mathcal{E}'$ then $\mathcal{E}^d \subset \mathcal{E}'^d$.*

Proof of Lemma 1. Let $\mathcal{E}$ and $\mathcal{E}'$ be two complete extensions s.t. $\mathcal{E} \subset \mathcal{E}'$. Thus, any argument deactivated by $\mathcal{E}$ is clearly deactivated by $\mathcal{E}'$, hence $\mathcal{E} \subseteq \mathcal{E}'$. Now, suppose for the sake of contradiction that $\mathcal{E}^d = \mathcal{E}'^d$. Let $a \in \mathcal{E}' \setminus \mathcal{E}$. For every $b \, \mathcal{R} \, a$, $b \in \mathcal{E}'^d$ because $\mathcal{E}'$ defends all its elements. It follows that $b \in \mathcal{E}^d$, i.e. that $\mathcal{E}$ also defends a. But $a \notin \mathcal{E}$ and this contradicts the fact that E is a complete extension. Then, it follows that $\mathcal{E}^d \subset \mathcal{E}'^d$. □

Now, let $\mathcal{E} \subseteq \mathcal{A}$ be a semi-stable extension. Suppose for the sake of contradiction that $\mathcal{E}$ is not a preferred extension. Thus, there is $\mathcal{E}' \subseteq \mathcal{A}$ s.t. $\mathcal{E} \subset \mathcal{E}'$ and $\mathcal{E}'$ is a complete extension. From the above lemma, it holds that $\mathcal{E}^d \subset \mathcal{E}'^d$. It follows that $\mathcal{E} \cup \mathcal{E}^d \subset \mathcal{E}' \cup \mathcal{E}'^d$. Contradiction with the fact that $\mathcal{E}$ is a semi-stable extension.

To show that the inverse does not hold, let us take a counter-example. Let us consider the AFN $\mathcal{G} = \langle \mathcal{A} = \{a, b, c, d\}, \mathcal{R} = \{(a, b), (a, c), (b, a), (c, c), (d, d)\}, \mathcal{N} = \emptyset\rangle$. This system admits $\{a\}$ and $\{b\}$ as preferred extensions. However, only $\{a\}$ is a semi-stable extension. Indeed $\{a\}^d = \{b, c\}$ and $\{b\}^d = \{a\}$. We have that: $\{b\} \cup \{b\}^d \subset \{a\} \cup \{a\}^d$.

(6) If $\mathcal{E}$ is a stable extension, then $\mathcal{E}$ is a preferred extension but not vice versa; there may be zero, one or several stable extensions for $\mathcal{G}$:

If $\mathcal{E}$ is a stable extension, then from (5) $\mathcal{E}$ is a semi-stable extension and from (5) again it is also a preferred extension.

To show that the inverse is not true, let us take the AFN $\mathcal{G} = \langle \mathcal{A} = \{a, b, c\}, \mathcal{R} = \{(a, b), (b, a), (b, c), (c, c)\}, \mathcal{N} = \emptyset\rangle$. $\{a\}$ is a preferred extension but it is not stable.

To show that there may be zero, one or several stable extensions for an AFN, it suffices to find an example for each case: $\mathcal{G}_1 = \langle \mathcal{A} = \{a\}, \mathcal{R} = \{(a, a)\}, \mathcal{N} = \emptyset\rangle$ has no stable extension; $\mathcal{G}_2 = \langle \mathcal{A} = \{a, b\}, \mathcal{R} = \{(a, b)\}, \mathcal{N} = \emptyset\rangle$ admits $\{a\}$ as a unique stable extension and $\mathcal{G}_3 = \langle \mathcal{A} = \{a, b\}, \mathcal{R} = \{(a, b), (b, a)\}, \mathcal{N} = \emptyset\rangle$ has two stable extensions: $\{a\}$ and $\{b\}$. □

Proof of Theorem 3.18. Admissible

$\Rightarrow$) Let $\mathcal{E}$ be an admissible set and prove that $\mathcal{L} = (\mathcal{E}, \mathcal{E}^d, \mathcal{A} \setminus (\mathcal{E} \cup \mathcal{E}^d))$ is an admissible labelling.

Safety of $\mathcal{L}$ follows directly from the coherence of $\mathcal{E}$ and the fact that an argument a is not powerful in $\mathcal{A}$ iff there is no coherent set $\mathcal{C} \subseteq \mathcal{A}$ containing a (see proposition 3.7) which implies that $a \in \mathcal{E}^d$.

Let $a \in \mathcal{E}$ and show that a is legally `in` in $\mathcal{L}$. Let b be an argument s.t. $b \, \mathcal{R} \, a$. Clearly $b \notin \mathcal{E}$ follows from conflict-freeness of $\mathcal{E}$. Suppose that $b \in \mathcal{A} \setminus (\mathcal{E} \cup \mathcal{E}^d)$. Then, there exists a coherent set $\mathcal{C} \subseteq \mathcal{A}$ containing b and $\mathcal{E} \not\mathcal{R} \, \mathcal{C}$ since $b \notin \mathcal{E}^d$. This means that $\mathcal{E}$ does not defend a which contradict the fact that $\mathcal{E}$ is an admissible set. Thus $b \in \mathcal{E}^d$, i.e. $b \in \text{out}(\mathcal{L})$. Now let $E \subseteq \mathcal{A}$ s.t. $E \, \mathcal{N} \, a$. From closure of $\mathcal{E}$ under $\mathcal{N}^{-1}$, it holds that $E \cap \mathcal{E} \neq \emptyset$, i.e. $E \cap \text{in}(\mathcal{L}) \neq \emptyset$. Hence, a is legally `in` in $\mathcal{L}$.

Now, let $a \in \mathcal{E}^d$ and show that a is legally `out` in $\mathcal{L}$ (i.e. either $\mathcal{E} \, \mathcal{R} \, a$ or $\exists E \in \mathcal{A}$ s.t. $E \, \mathcal{N} \, a$ and $E \subseteq \mathcal{E}^d$). To do so, let us suppose that $\forall E \subseteq \mathcal{A}$ s.t. $E \, \mathcal{N} \, a$, $E \nsubseteq \mathcal{E}^d$ and prove that $\mathcal{E} \, \mathcal{R} \, a$. Then we have: $\forall E \subseteq \mathcal{A}$ s.t. $E \, \mathcal{N} \, a$, $\exists b \in E$ s.t. $b \notin \mathcal{E}^d$, i.e. there is a coherent set $\mathcal{C}$ containing b s.t. $\mathcal{E} \not\mathcal{R} \, \mathcal{C}$. Thus, we can construct a coherent set $\mathcal{C}'$ satisfying $\forall E \subseteq \mathcal{A}$ s.t. $E \, \mathcal{N} \, a$, $\mathcal{C}' \cap \mathcal{E} \neq \emptyset$ and $\mathcal{E} \not\mathcal{R} \, \mathcal{C}'$. But then the set $\mathcal{C}' \cup \{a\}$ is coherent too, and since $a \in \mathcal{E}^d$, it holds (from the definition of $\mathcal{E}^d$) that $\mathcal{E} \, \mathcal{R} \, \mathcal{C}'$. Since $\mathcal{E} \not\mathcal{R} \, \mathcal{C}'$, it must be the case that $\mathcal{E} \; R \; \{a\}$. This means that every $a \in \mathcal{E}^d$ is legally `out`. It follows from the preceding elements that $\mathcal{L}$ is an admissible labelling.

$\Leftarrow$) Let $\mathcal{L}$ be an admissible labelling and prove that $\mathcal{E} = \text{in}(\mathcal{L})$ is an admissible extension and $\text{out}(\mathcal{L}) \subseteq \mathcal{E}^d$.

$\mathcal{L}$ is safe and all arguments in in($\mathcal{L}$) (resp. in out($\mathcal{L}$)) are legally `in` (resp. `out`). The coherence of $\mathcal{E}$ follows from the safety of $\mathcal{L}$. $\mathcal{E}$ is conflict-free, for if we suppose the contrary then there exists $a, b \in \mathcal{E}$ s.t. $a\ \mathcal{R}\ b$. This means that b in not legally `in`. Contradiction. Thus, $\mathcal{E}$ is strongly coherent.

Before proving that $\mathcal{E}$ defends all its elements, let us first prove that out($\mathcal{L}$) $\subseteq \mathcal{E}^d$. Let $a \in$ out($\mathcal{L}$), a is legally `out`. Then, two cases are possible: the first case is that a is attacked by some argument b labelled `in`, i.e., $b \in \mathcal{E}$, but in this case, every coherent set $\mathcal{C}$ containing a is attacked by b, hence by $\mathcal{E}$, i.e., $a \in \mathcal{E}^d$. The second case is that there is $E \subseteq \mathcal{A}$ s.t. $E\ \mathcal{N}\ a$ and $E \subseteq$ out($\mathcal{L}$). In this case either there is no coherent set containing a, so $a \in \mathcal{E}^d$ or such coherent sets exists and every coherent set $\mathcal{C}$ containing a contains at least an argument $b \in E$. Since a_1 is labelled `out` and is legally `out` we have again two cases: either $\mathcal{E}\ \mathcal{R}\ a_1$ so $\mathcal{E}\ \mathcal{R}\ \mathcal{C}$ or there is $E_1 \subseteq \mathcal{A}$ s.t. $E_1\ \mathcal{N}\ a_1$ and $E_1 \subseteq$ out($\mathcal{L}$) and so on. We repeat this process. Since the possibility that a does not belong to any coherent set is discarded (in the considered arbitrary coherent set $\mathcal{C}$ the continuation of this reasoning process does not involve necessity cycles) and since the number of arguments is finite, it must exist some a_i s.t. $a_i \in \mathcal{C}$ and $\mathcal{E}\ \mathcal{R}\ a_i$, i.e., each coherent set $\mathcal{C}$ containing a is attacked by $\mathcal{E}$, hence $a \in \mathcal{E}_d$.

Now, for all $a \in \mathcal{E}$, suppose that b is an argument s.t. $b\mathcal{R}a$. since a is legally `in`, b is necessarily labelled out. From the fact that out($\mathcal{L}$) $\subseteq \mathcal{E}^d$ we deduce that $b \in \mathcal{E}^d$, i.e., every coherent set containing b is attacked by $\mathcal{E}$. Thus, $\mathcal{E}$ defends all its elements. Since $\mathcal{E}$ is strongly coherent and defends all its elements, it is an admissible set.

Complete

$\Rightarrow$) Let $\mathcal{E}$ be an complete extension and prove that $(\mathcal{E}, \mathcal{E}^d, \mathcal{A} \setminus (\mathcal{E} \cup \mathcal{E}^d))$ is a complete labelling.

From the previous proof, it follows that every $a \in \mathcal{E}$ (resp. $a \in \mathcal{E}^d$) is legally `in` (resp. legally `out`). It remains to show that every argument $a \in \mathcal{A} \setminus (\mathcal{E} \cup \mathcal{E}^d)$ is legally `undec`.

Let $a \in \mathcal{A} \setminus (\mathcal{E} \cup \mathcal{E}^d)$. Thus, there is a coherent set $\mathcal{C}$ containing a s.t. $\mathcal{E}\ \not\mathcal{R}\ \mathcal{C}$. Let us call this fact (F). Now, for the sake of contradiction let us suppose that a is illegally `undec`. We have three possible cases:

Case 1: $\exists E \subseteq \mathcal{A}$ s.t. $E\ \mathcal{N}\ a$ and $E \subseteq$ out($\mathcal{L}$) $= \mathcal{E}^d$. In this case, every coherent set $\mathcal{C}$ containing a must contain at least an element of E which is also in $\mathcal{E}^d$. By definition of $\mathcal{E}^d$, it holds that $\mathcal{E}\ \mathcal{R}\ \mathcal{C}$. This contradicts (F).

Case 2: $\exists b \in$ in($\mathcal{L}$) $= \mathcal{E}$ s.t. $b\ \mathcal{R}\ a$. Again, it follows that every coherent set $\mathcal{C}$ containing a is attacked by $\mathcal{E}$, which is in contradiction with (F).

Case 3: $\forall b \in \mathcal{A}$, if $b\ \mathcal{R}\ a$ then $b \in$ out($\mathcal{L}$) $= \mathcal{E}^d$ and $\forall E \subseteq \mathcal{A}$, if $E\ \mathcal{N}\ a$ then $E \cap$ in($\mathcal{L}$) $\neq \emptyset$. In this case, clearly $\mathcal{E} \cup \{a\}$ is coherent and for every b that attacks a, it holds that every coherent set containing b is attacked by $\mathcal{E}$, i.e, $\mathcal{E}$ defends a, but $a \notin \mathcal{E}$. This contradicts the fact that $\mathcal{E}$ is a complete extension. It follows that: every argument in $\mathcal{A} \setminus (\mathcal{E} \cup \mathcal{E}^d)$ is legally `undec`.

$\Leftarrow$) Let $\mathcal{L}$ be a complete labelling and prove that $\mathcal{E} =$ in($\mathcal{L}$) is a complete extension and out($\mathcal{L}$) $= \mathcal{E}^d$. From the precedent proof, we have: $\mathcal{E}$ is admissible and out($\mathcal{L}$) $\subseteq \mathcal{E}^d$. It remains to prove that $\mathcal{E}$ contains all the arguments it defends and that $\mathcal{E}^d \subseteq$ out($\mathcal{L}$).

Let us start by proving that $\mathcal{E}^d \subseteq$ out($\mathcal{L}$). Let $a \in \mathcal{E}^d$. If there is no coherent set of arguments containing a, then a is not powerful in $\mathcal{A}$ (see proposition 3.7) and from the safety of $\mathcal{L}$, it holds that $\mathcal{L}(a) =$ `out`. Now, suppose that there exists at least a coherent set containing a. For the sake of contradiction, let $a \notin$ out($\mathcal{L}$). Then, either $a \in$ in($\mathcal{L}$) or $a \in$ undec($\mathcal{L}$). $a \in$ in($\mathcal{L}$) is impossible because in($\mathcal{L}$) $= \mathcal{E}$. If $a \in$ undec($\mathcal{L}$) then a is legally `undec` because $\mathcal{L}$ is a complete labelling. It follows from the definition of legally `undec` arguments that $\mathcal{E}\ \not\mathcal{R}\ a$ and $\forall E \subseteq \mathcal{A}$ s.t. $E\ \mathcal{N}\ a$, $E \nsubseteq$ out($\mathcal{L}$), i.e. any argument in E is labelled either `in` (hence, not attacked by $\mathcal{E}$) or `undec`. By applying in a repetitive way the same reasoning on undec arguments, and since the number of arguments is finite, we construct a coherent set $\mathcal{C}$ containing a and s.t. $\mathcal{E} \not\mathcal{R}\, \mathcal{C}$. This contradicts the fact that $a \in \mathcal{E}^d$. So $E^d \subseteq \text{out}(\mathcal{L})$.

Now let us suppose that there is $a \notin \mathcal{E}$, hence a is labelled either out or undec and $\mathcal{E}$ defends a, i.e., $\mathcal{E} \cup a$ is coherent and $\forall b \in \mathcal{A}$, if $b\ \mathcal{R}\ a$ then $\mathcal{E}$ attacks any coherent set of arguments containing b. It follows from this statement that: $\forall E \subseteq \mathcal{A}$, if $E\ \mathcal{N}\ a$ then $E \cap \mathcal{E} \neq \emptyset$, i.e., $E \cap \text{in}(\mathcal{L}) \neq \emptyset$ and $\forall b \in \mathcal{A}$, if $b\ \mathcal{R}\ a$ then $b \in \mathcal{E}^d$, i.e. $b \in \text{out}(\mathcal{L})$. It is easy to check that under these conditions, a cannot be neither legally `out` nor legally `undec`. This contradicts the fact that $\mathcal{L}$ is a complete labelling. So, $\mathcal{E}$ contains any argument it defends, i.e., $\mathcal{E}$ is a complete extension.

Grounded

$\Rightarrow$) Let $\mathcal{E}$ be the grounded extension and prove that $\mathcal{L} = (\mathcal{E}, \mathcal{E}^d, \mathcal{A} \setminus (\mathcal{E} \cup \mathcal{E}^d))$ is the grounded labelling. $\mathcal{E}$ is the $\subseteq$-minimal complete extension. From the $\Rightarrow$) part of the proof for complete semantics, it follows that $\mathcal{L} = (\mathcal{E}, \mathcal{E}^d, \mathcal{A} \setminus (\mathcal{E} \cup \mathcal{E}^d))$ is a complete labelling. Suppose that $\mathcal{L}$ is not the grounded labelling, i.e., there is a complete labelling $\mathcal{L}'$ s.t. $\text{in}(\mathcal{L}') \subset \text{in}(\mathcal{L}) = \mathcal{E}$. But, from the $\Leftarrow$) part of the proof for complete semantics, it follows that $\text{in}(\mathcal{L}')$ is a complete extension which contradicts the fact that $\mathcal{E}$ is the $\subseteq$-minimal complete extension.

$\Leftarrow$) Let $\mathcal{L}$ be the grounded labelling and prove that $\mathcal{E} = \text{in}(\mathcal{L})$ is the grounded extension and $\text{out}(\mathcal{L}) = \mathcal{E}^d$. From the $\Leftarrow$) part of the proof for complete semantics, we deduce that $\mathcal{E} = \text{in}(\mathcal{L})$ is a complete extension and that $\text{out}(\mathcal{L}) = \mathcal{E}^d$. Suppose that $\exists \mathcal{E}' \subseteq \mathcal{A}$ s.t. $\mathcal{E}' \subset \mathcal{E}$ and $\mathcal{E}'$ is a complete extension. From the $\Rightarrow$) part of the proof for complete semantics, it follows that $(\mathcal{E}', \mathcal{E}'^d, \mathcal{A} \setminus (\mathcal{E}' \cup \mathcal{E}'^d))$ is a complete labelling. Contradicts the fact that $\mathcal{L}$ is the grounded labelling.

Preferred

$\Rightarrow$) Let $\mathcal{E}$ be a preferred extension and prove that $(\mathcal{E}, \mathcal{E}^d, \mathcal{A} \setminus (\mathcal{E} \cup \mathcal{E}^d))$ is a preferred labelling. The proof is similar to the $\Rightarrow$) part of the proof for grounded semantics except that it is based on $\subseteq$-maximization instead of $\subseteq$-minimization.

$\Leftarrow$) Let $\mathcal{L}$ be a preferred labelling and prove that $\mathcal{E} = \text{in}(\mathcal{L})$ is a preferred extension and $\text{out}(\mathcal{L}) = \mathcal{E}^d$. From the $\Leftarrow$) part of the proof for complete semantics, we deduce that $\mathcal{E} = \text{in}(\mathcal{L})$ is a complete extension and that $\text{out}(\mathcal{L}) = \mathcal{E}^d$. Suppose that there is $\mathcal{E}' \subseteq \mathcal{A}$ s.t. $\mathcal{E} \subset \mathcal{E}'$ and $\mathcal{E}'$ is a complete extension. Then, from the $\Rightarrow$) part of the proof for complete semantics, it follows that $(\mathcal{E}', \mathcal{E}'^d, \mathcal{A} \setminus (\mathcal{E}' \cup \mathcal{E}'^d))$ is a complete labelling. Contradiction with the fact that $\mathcal{L}$ is a preferred labelling.

Stable

$\Rightarrow$) Let $\mathcal{E}$ be an stable extension and prove that $(\mathcal{E}, \mathcal{E}^d, \mathcal{A} \setminus (\mathcal{E} \cup \mathcal{E}^d))$ is a stable labelling. if $\mathcal{E}$ is a stable extension, then from its definition: $\mathcal{A} \setminus (\mathcal{E} \cup \mathcal{E}^d) = \emptyset$. Since it is also complete, form the $\Rightarrow$) part of the proof for complete semantics, the labelling $\mathcal{L} = (\mathcal{E}, \mathcal{E}^d, \mathcal{A} \setminus (\mathcal{E} \cup \mathcal{E}^d) = \emptyset)$ is complete and hence also stable since $\text{undec}(\mathcal{L}) = \emptyset$.

$\Leftarrow$) Let $\mathcal{L}$ be a stable labelling and prove that $\mathcal{E} = \text{in}(\mathcal{L})$ is a stable extension and $\text{out}(\mathcal{L}) = \mathcal{E}^d$. From the $\Leftarrow$) part of the proof for complete semantics, we deduce that $\mathcal{E} = \text{in}(\mathcal{L})$ is a complete extension and $\text{out}(\mathcal{L}) = \mathcal{E}^d$. But since $\text{undec}(\mathcal{L}) = \emptyset$, clearly $\mathcal{E}^d = \mathcal{A} \setminus \mathcal{E}$, i.e. $\mathcal{E}$ is a stable extension.

Semi-stable

$\Rightarrow$) Let $\mathcal{E}$ be an semi-stable extension and prove that $(\mathcal{E}, \mathcal{E}^d, \mathcal{A} \setminus (\mathcal{E} \cup \mathcal{E}^d))$ is a semi-stable labelling.

From the $\Rightarrow$) part of the proof for complete semantics, it follows that $\mathcal{L} = (\mathcal{E}, \mathcal{E}^d, \mathcal{A} \setminus (\mathcal{E} \cup \mathcal{E}^d))$ is a complete labelling. Suppose that $\exists \mathcal{L}'$ s.t. $\text{undec}(\mathcal{L}') \subset \text{undec}(\mathcal{L})$. From the $\Leftarrow$) part of the proof for complete semantics, $\text{in}(\mathcal{L}')$ is a complete extension with $\text{out}(\mathcal{L}) = (\text{in}(\mathcal{L}))^d$. It is easy to check that $\text{undec}(\mathcal{L}') \subset \text{undec}(\mathcal{L})$ is equivalent to $(\text{in}(\mathcal{L}') \cup (\text{in}(\mathcal{L}'))^d) \subset (\mathcal{E} \cup \mathcal{E}^d)$. Contradiction with the fact that $\mathcal{E}$ is a semi-stable extension.

$\Leftarrow$) Let $\mathcal{L}$ be a semi-stable labelling and prove that $\mathcal{E} = \text{in}(\mathcal{L})$ is a semi-stable extension and $\text{out}(\mathcal{L}) = \mathcal{E}^d$. From the $\Leftarrow$) part of the proof for complete semantics, it follows that $\mathcal{E} = \text{in}(\mathcal{L})$ is a complete extension and that $\text{out}(\mathcal{L}) = \mathcal{E}^d$. Suppose that there is complete $\mathcal{E}'$ s.t. $(\mathcal{E} \cup \mathcal{E}^d) \subset (\mathcal{E}' \cup \mathcal{E}'^d)$. From the $\Rightarrow$) part of the proof for complete semantics, it follows that $\mathcal{L}' = (\mathcal{E}', \mathcal{E}'^d, \mathcal{A} \setminus (\mathcal{E}' \cup \mathcal{E}'^d))$ is a complete labelling. Moreover, it is easy to check that $(\mathcal{E} \cup \mathcal{E}^d) \subset (\mathcal{E}' \cup \mathcal{E}'^d)$ is equivalent to $\texttt{undec}(\mathcal{L}') \subset \texttt{undec}(\mathcal{L})$. Contradiction with the fact that $\mathcal{L}$ is a semi-stable labelling. $\square$

Proof of Theorem 4.1. The proof is straightforward since the definition of any kind of extension in an AFN $\langle \mathcal{A}, \mathcal{R}, \mathcal{N} \rangle$ with $\mathcal{N} = \emptyset$ coincides with that of the same kind of extension in the AF $\langle \mathcal{A}, \mathcal{R} \rangle$. $\square$

Proof of Proposition 4.4. The proof follows immediately from Definition 4.2. $\square$

Proof of Theorem 4.5. Admissible

$\Rightarrow$) Let $\mathcal{E} \in \mathcal{A}$ be an admissible set of $\mathcal{G}$ and let $\Gamma_{\mathcal{E}}$ be the set of subsets of arguments defined by: $\Gamma_{\mathcal{E}} = \{\mathcal{C}_a \mid a \in \mathcal{E}, \mathcal{C}_a \subseteq \mathcal{E}$ and $\mathcal{C}_a$ is a meta-argument associated to $a\}$. Thus, each argument of $\mathcal{E}$ is represented in $\Gamma_{\mathcal{E}}$ by at least one meta-argument and it is easy to check that $\bigcup_{\mathcal{C} \in \Gamma_{\mathcal{E}}} \mathcal{C} = \mathcal{E}$. Let us show that $\Gamma_{\mathcal{E}}$ is an admissible set of $\mathcal{H}_{\mathcal{G}}$. $\Gamma_{\mathcal{E}}$ is conflict-free. Indeed, supposing the inverse means that there is $\mathcal{C}_a, \mathcal{C}_b \in \Gamma_{\mathcal{E}}$ s.t. $\mathcal{C}_a\ \mathcal{R}'\ \mathcal{C}_b$, i.e., there is $a' \in \mathcal{C}_a$ and $b' \in \mathcal{C}_b$ s.t. $a'\ \mathcal{R}\ b'$. But since $\mathcal{C}_a, \mathcal{C}_b \subseteq \mathcal{E}$, $a', b' \in \mathcal{E}$. Contradiction with the fact that $\mathcal{E}$ is conflict-free.

Let $\mathcal{C}_a \in \Gamma_{\mathcal{E}}$ be a meta-argument of an argument $a \in \mathcal{E}$. Let $\mathcal{C}_b \notin \Gamma_{\mathcal{E}}$ be a meta-argument of an argument $b \in \mathcal{A} \setminus \mathcal{E}$ and $\mathcal{C}_b\ \mathcal{R}'\ \mathcal{C}_a$. From the definition of $\mathcal{R}'$, it follows that there is b' in $\mathcal{C}_b$, hence in $\mathcal{A} \setminus \mathcal{E}$ and there is a' in $\mathcal{C}_a$, hence in $\mathcal{E}$ s.t. $b'\ \mathcal{R}\ a'$. Since $\mathcal{E}$ defends all its elements. Namely, $\mathcal{E}$ defends a' against b'. So, for every coherent set $\mathcal{C}$ containing b' there is $c \in \mathcal{E}$ s.t. $c\ \mathcal{R}\ \mathcal{C}$. Since $\mathcal{C}_b$ is a coherent set containing b', then there exists a meta-argument of $\Gamma_{\mathcal{E}}$ (we can take any meta-argument $\mathcal{C}_c$ of c) which attacks $\mathcal{C}_b$. $\Gamma_{\mathcal{E}}$ defends all its elements. It is an admissible set of $\mathcal{H}_{\mathcal{G}}$.

$\Leftarrow$) Let Γ be an admissible set of $\mathcal{H}_{\mathcal{G}}$ and let $\mathcal{E}_{\Gamma} = \bigcup_{\mathcal{C} \in \Gamma} \mathcal{C}$. Since each meta-argument of Γ is a coherent set, it follows that $\mathcal{E}_{\Gamma}$ is coherent and from the conflict-freeness of Γ it follows that $\mathcal{E}_{\Gamma}$ is conflict free. Hence, $\mathcal{E}_{\Gamma}$ is strongly coherent. Let $a \in \mathcal{E}$, $b \notin \mathcal{E}$ and $b\ \mathcal{R}\ a$. Then for every meta argument $\mathcal{C}_b \notin \Gamma$ associated to b and every meta-argument $\mathcal{C}_a \in \Gamma$ associated to a, it holds that: $\mathcal{C}_b\ \mathcal{R}'\ \mathcal{C}_a$. Since Γ is admissible, there is a meta-argument $\mathcal{C} \in \Gamma$ s.t. $\mathcal{C}\ \mathcal{R}'\ \mathcal{C}_b$, i.e. $\mathcal{E}_{\Gamma}\ \mathcal{R}\ \mathcal{C}_b$. But since any coherent set of arguments $\mathcal{C}'$ containing b contains at least one sub-argument associated to b, it follows that for every coherent set of arguments $\mathcal{C}'$ containing b, we have $\mathcal{E}_{\Gamma}\ \mathcal{R}\ \mathcal{C}'$. Thus, $\mathcal{E}_{\Gamma}$ is strongly coherent and defends all its elements, i.e. an admissible set of $\mathcal{G}$.

Complete

$\Rightarrow$) Let $\mathcal{E} \in \mathcal{A}$ be a complete extension of $\mathcal{G}$ and let $\Gamma_{\mathcal{E}}$ be the set of subsets of arguments defined by: $\Gamma_{\mathcal{E}} = \{\mathcal{C}_a \mid a \in \mathcal{E}, \mathcal{C}_a \subseteq \mathcal{E}$ and $\mathcal{C}_a$ is a meta-argument associated to $a\}$. Thus, each argument of $\mathcal{E}$ is represented in $\Gamma_{\mathcal{E}}$ by at least one meta-argument and it is easy to check that $\bigcup_{\mathcal{C} \in \Gamma_{\mathcal{E}}} \mathcal{C} = \mathcal{E}$. Let us show that $\Gamma_{\mathcal{E}}$ is a complete extension of $\mathcal{H}_{\mathcal{G}}$. From the previous proof, it follows that $\Gamma_{\mathcal{E}}$ is an admissible set of $\mathcal{H}_{\mathcal{G}}$. It remains to show that $\Gamma_{\mathcal{E}}$ contains every meta-argument it defends.

For the sake of contradiction, suppose that there is a meta-argument $\mathcal{E}_a \notin \Gamma_{\mathcal{E}}$ associated to $a \notin \mathcal{E}$ defended by $\Gamma_{\mathcal{E}}$, i.e., $\forall \mathcal{C}$ s.t. $\mathcal{C}\ \mathcal{R}'\ \mathcal{C}_a$, $\Gamma_{\mathcal{E}}\ \mathcal{R}'\ \mathcal{C}$.

First, let us precise that a first consequence of our hypotheses is that $\Gamma_{\mathcal{E}} \cup \{\mathcal{C}_a\}$ is conflict-free. Indeed if we suppose that $\Gamma_{\mathcal{E}}\ \mathcal{R}'\ \mathcal{C}_a$ then since $\Gamma_{\mathcal{E}}$ defends $\mathcal{C}_a$ it holds that $\Gamma_{\mathcal{E}}\ \mathcal{R}'\ \Gamma_{\mathcal{E}}$ which contradicts the fact that $\Gamma_{\mathcal{E}}$ is conflict-free. If we suppose that $\mathcal{C}_a\ \mathcal{R}\ \Gamma_{\mathcal{E}}$ then since $\Gamma_{\mathcal{E}}$ is admissible, $\Gamma_{\mathcal{E}}\ \mathcal{R}'\ \mathcal{C}_a$ which coincides with the first case implying conflict in $\Gamma_{\mathcal{E}}$.

Now, let us show that our hypotheses contradict the fact that $\mathcal{E}$ is a complete extension of $\mathcal{G}$. Since $\mathcal{C}_a$ is a coherent set, there is $a_0 \in \mathcal{C}_a$ s.t. there is no $E \subseteq \mathcal{A}$ with $E\ \mathcal{N}\ a_0$. $\mathcal{E} \cup a_0$ is coherent since $\mathcal{E}$ is coherent and $a0$ does not have any necessities to satisfy. if b is any argument s.t. $b\ \mathcal{R}\ a_0$ then any meta-argument $\mathcal{C}_b$ associated to b verifies $\mathcal{C}_b\ \mathcal{R}'\ \mathcal{C}_a$. Since $\Gamma_\mathcal{E}$ defends $\mathcal{C}_a$ it follows that $\Gamma_\mathcal{E}\ \mathcal{R}'\mathcal{C}_b$. Since any coherent set $\mathcal{C}'$ containing b contains at least one meta-argument associated to b. Thus, for every coherent set $\mathcal{C}'$ containing b it holds that $\Gamma_\mathcal{E}\ \mathcal{R}'\ \mathcal{C}'$, i.e. $\mathcal{E}\ \mathcal{R}\ \mathcal{C}'$. This means that $\mathcal{E}$ defends $a_0 \notin \mathcal{E}$. Contradicts the fact that $\mathcal{E}$ is a complete extension of $\mathcal{G}$.

$\Leftarrow$) Let Γ be a complete extension of $\mathcal{H}_\mathcal{G}$ and let $\mathcal{E}_\Gamma = \bigcup_{\mathcal{C}\in\Gamma} \mathcal{C}$. From the $\Leftarrow$) part of the proof for admissible sets, it follows that $\mathcal{E}_\Gamma$ is an admissible set of $\mathcal{G}$. For the sake of contradiction, suppose that there exists $a \notin \mathcal{E}_\Gamma$ and $\mathcal{E}_\Gamma$ defends a, i.e. $\mathcal{E}_\Gamma \cup \{a\}$ is coherent and if $b\ \mathcal{R}\ a$ then for every coherent set $\mathcal{C}$ containing b, $\mathcal{E}_\Gamma\ \mathcal{R}\ \mathcal{C}$. Let $\mathcal{C}_a$ be a meta-argument associated to a, hence $\mathcal{C}_a \notin \Gamma$ because $a \notin \mathcal{E}_\Gamma$ and show that Γ defends $\mathcal{C}_a$. From $\mathcal{E}_\Gamma \cup \{a\}$ is coherent, it follows that all arguments of $\mathcal{C}_a$ except a are in $\mathcal{E}_\Gamma$. If $\mathcal{C}_b$ is a meta-argument s.t. $\mathcal{C}_b\ \mathcal{R}'\ \mathcal{C}_a$ then two cases are possible. The first case is that $\mathcal{C}_b\ \mathcal{R}\ a'$ with $a' \neq a$ hence $a' \in \mathcal{E}_\Gamma$, then there is at least a meta-argument associated to a' in Γ. Consequently, $\mathcal{C}_b\ \mathcal{R}\ \Gamma$. But since Γ is admissible it follows that $\Gamma\ \mathcal{R}'\ \mathcal{C}_b$. The second case is that $\mathcal{C}_b\ \mathcal{R}\ a$,i.e., there is $b' \in \mathcal{C}_b$ s.t. $b'\ \mathcal{R}\ a$. Then, from the hypothesis that $\mathcal{E}_\Gamma$ defends a, it follows that for every coherent set $\mathcal{C}$ containing b', $\mathcal{E}_\Gamma\ \mathcal{R}\ \mathcal{C}$. But $\mathcal{C}_b$ is a coherent set containing b', so $\mathcal{E}_\Gamma\ \mathcal{R}\ \mathcal{C}_b$, i.e. there is a meta-argument $\mathcal{C}' \in \Gamma$ s.t. $\mathcal{C}'\ \mathcal{R}'\ \mathcal{C}_b$. This means that Γ defends $\mathcal{C}_a$ which is outside it. Contradiction with the fact that Γ is a complete extension of $\mathcal{H}_\mathcal{G}$.

Grounded

$\Rightarrow$) Let $\mathcal{E}$ be the grounded extension of $\mathcal{G}$ and $\Gamma_\mathcal{E}$ is the corresponding set of meta-arguments defined as above. It follows from the $\Rightarrow$) part of the proof of the complete semantics, that $\Gamma_\mathcal{E}$ is a complete extension of $\mathcal{H}_\mathcal{G}$. Suppose that $\Gamma_\mathcal{E}$ is not the grounded extension of $\mathcal{H}_\mathcal{G}$, i.e., there is $\Gamma' \subset \Gamma_\mathcal{E}$ s.t. Γ' is a complete extension of $\mathcal{H}_\mathcal{G}$. From the $\Leftarrow$) part of the proof of the complete semantics, it follows that $\mathcal{E}_{\Gamma'} = \bigcup_{\mathcal{C}\in\Gamma'} \mathcal{C}$ is a complete extension of $\mathcal{G}$. But since $\Gamma' \subset \Gamma_\mathcal{E}$ it is obvious that $\mathcal{E}_{\Gamma'} \subset \mathcal{E}_{\Gamma_\mathcal{E}} = \mathcal{E}$. Contradiction with the fact that $\mathcal{E}$ is the grounded extension of $\mathcal{G}$.

$\Leftarrow$) Let Γ be the grounded extension of $\mathcal{H}_\mathcal{G}$, i.e. the $\subseteq$-minimal complete extension of $\mathcal{H}_\mathcal{G}$. Then, from the $\Leftarrow$) part of the proof of the complete semantics, it follows that the set of arguments $\mathcal{E}_\Gamma = \bigcup_{\mathcal{C}\in\Gamma} \mathcal{C}$ is a complete extension of $\mathcal{G}$. Suppose for the sake of contradiction that $\mathcal{E}_\Gamma$ is not the minimal complete extension of $\mathcal{G}$, i.e., that there is a set $\mathcal{E}' \subset \mathcal{E}_\Gamma$ and $\mathcal{E}'$ is a complete extension of $\mathcal{G}$. From the $\Rightarrow$) part of the proof of the complete semantics, it follows that $\Gamma_{\mathcal{E}'}$ is a complete extension of $\mathcal{H}_\mathcal{G}$. But since $\mathcal{E}' \subset \mathcal{E}_\Gamma$ it is obvious that $\Gamma_{\mathcal{E}'} \subset \Gamma_{\mathcal{E}_\Gamma} = \Gamma$. Contradiction with the fact that Γ is the grounded extension of $\mathcal{H}_\mathcal{G}$.

Preferred

$\Rightarrow$) Let $\mathcal{E}$ be a preferred extension of $\mathcal{G}$ and $\Gamma_\mathcal{E}$ is the corresponding set of meta-arguments defined as above. It follows from the $\Rightarrow$) part of the proof of the complete semantics, that $\Gamma_\mathcal{E}$ is a complete extension of $\mathcal{H}_\mathcal{G}$. Suppose that $\Gamma_\mathcal{E}$ is not a preferred extension of $\mathcal{H}_\mathcal{G}$, i.e., there is $\Gamma' \supset \Gamma_\mathcal{E}$ s.t. Γ' is a complete extension of $\mathcal{H}_\mathcal{G}$. From the $\Leftarrow$) part of the proof of the complete semantics, it follows that $\mathcal{E}_{\Gamma'} = \bigcup_{\mathcal{C}\in\Gamma'} \mathcal{C}$ is a complete extension of $\mathcal{G}$. But since $\Gamma' \supset \Gamma_\mathcal{E}$ it is obvious that $\mathcal{E}_{\Gamma'} \supset \mathcal{E}_{\Gamma_\mathcal{E}} = \mathcal{E}$. Contradiction with the fact that $\mathcal{E}$ is a preferred extension of $\mathcal{G}$.

$\Leftarrow$) Let Γ be a preferred extension of $\mathcal{H}_\mathcal{G}$, i.e. a $\subseteq$-maximal complete extension of $\mathcal{H}_\mathcal{G}$. Then, from the $\Leftarrow$) part of the proof of the complete semantics, it follows that the set of arguments $\mathcal{E}_\Gamma = \bigcup_{\mathcal{C}\in\Gamma} \mathcal{C}$ is a complete extension of $\mathcal{G}$. Suppose for the sake of contradiction that $\mathcal{E}_\Gamma$ is not a maximal complete extension of $\mathcal{G}$, i.e., that there is a set $\mathcal{E}' \supset \mathcal{E}_\Gamma$ and $\mathcal{E}'$ is a complete extension of $\mathcal{G}$. From the $\Rightarrow$) part of the proof of the complete semantics, it follows that $\Gamma_{\mathcal{E}'}$ is a complete extension of $\mathcal{H}_\mathcal{G}$. But since $\mathcal{E}' \supset \mathcal{E}_\Gamma$ it is obvious that $\Gamma_{\mathcal{E}'} \supset \Gamma_{\mathcal{E}_\Gamma} = \Gamma$. Contradiction with the fact that Γ is a preferred extension of $\mathcal{H}_\mathcal{G}$.

Stable

$\Rightarrow$) Let $\mathcal{E}$ be a stable extension of $\mathcal{G}$ and $\Gamma_{\mathcal{E}}$ is the corresponding set of meta-arguments defined as above. It follows from the $\Rightarrow$) part of the proof of the complete semantics, that $\Gamma_{\mathcal{E}}$ is a complete extension of $\mathcal{H}_{\mathcal{G}}$. Suppose that $\Gamma_{\mathcal{E}}$ is not a stable extension of $\mathcal{H}_{\mathcal{G}}$, i.e., there is a meta-argument $\mathcal{C}_a \notin \Gamma_{\mathcal{E}}$ associated to an argument $a \notin \mathcal{E}$ s.t. $\Gamma_{\mathcal{E}} \not\mathcal{R}' \mathcal{C}_a$. Thus, there is an argument $a \notin \mathcal{E}$ and a coherent set $\mathcal{C}_a$ containing a s.t. $\mathcal{E} \not\mathcal{R} \mathcal{C}_a$. Contradiction with the fact that $\mathcal{E}$ is a stable extension of $\mathcal{G}$.

$\Leftarrow$) Let Γ be a stable extension of $\mathcal{H}_{\mathcal{G}}$, i.e. a complete extension of $\mathcal{H}_{\mathcal{G}}$ which attacks any meta-argument outside it. Then, from the $\Leftarrow$) part of the proof of the complete semantics, it follows that the set of arguments $\mathcal{E}_{\Gamma} = \bigcup_{\mathcal{C}\in\Gamma} \mathcal{C}$ is a complete extension of $\mathcal{G}$. Suppose for the sake of contradiction that $\mathcal{E}_{\Gamma}$ is not a stable extension of $\mathcal{G}$, i.e., that there is an argument $a \notin \mathcal{E}_{\Gamma}$ and a coherent set $\mathcal{C}$ containing a s.t. $\mathcal{E} \not\mathcal{R} \mathcal{C}$. This means, there exists a meta-argument $\mathcal{C}_a \subseteq \mathcal{C}$ associated to a (if $\mathcal{C}$ itself is a $\subseteq$-minimal coherent set containing a, then $\mathcal{C}_a = \mathcal{C}$, otherwise it is easy to check that a $\subseteq$-minimal coherent set $\mathcal{C}_a \subset \mathcal{C}$ containing a always exists) s.t. $\mathcal{C}_a \notin \Gamma$ and $\Gamma \not\mathcal{R}' \mathcal{C}_a$. Contradicts the fact that Γ is a stable extension of $\mathcal{H}_{\mathcal{G}}$.

Semi-stable

$\Rightarrow$) Let $\mathcal{E}$ be a semi-stable extension of $\mathcal{G}$ and $\Gamma_{\mathcal{E}}$ is the corresponding set of meta-arguments defined as above. It follows from the $\Rightarrow$) part of the proof of the complete semantics, that $\Gamma_{\mathcal{E}}$ is a complete extension of $\mathcal{H}_{\mathcal{G}}$. Suppose that $\Gamma_{\mathcal{E}}$ is not a semi-stable extension of $\mathcal{H}_{\mathcal{G}}$, i.e., there is $\Gamma' \subseteq \mathcal{A}'$ s.t. Γ' is a complete extension of $\mathcal{H}_{\mathcal{G}}$ and $(\Gamma' \cup \Gamma'^d) \supset (\Gamma_{\mathcal{E}} \cup \Gamma^d_{\mathcal{E}})$. From the $\Leftarrow$) part of the proof of the complete semantics, it follows that $\mathcal{E}_{\Gamma'} = \bigcup_{\mathcal{C}\in\Gamma'} \mathcal{C}$ is a complete extension of $\mathcal{G}$. From $(\Gamma' \cup \Gamma'^d) \supset (\Gamma_{\mathcal{E}} \cup \Gamma^d_{\mathcal{E}})$ it is easy to verify that $(\mathcal{E}_{\Gamma'} \cup \mathcal{E}^d_{\Gamma'}) \subset (\mathcal{E} \cup \mathcal{E}^d)$. Contradiction with the fact that $\mathcal{E}$ is a semi-stable extension of $\mathcal{G}$.

$\Leftarrow$) Let Γ be a semi-stable extension of $\mathcal{H}_{\mathcal{G}}$, i.e. a $\subseteq$-maximal complete extension of $\mathcal{H}_{\mathcal{G}}$. Then, from the $\Leftarrow$) part of the proof of the complete semantics, it follows that the set of arguments $\mathcal{E}_{\Gamma} = \bigcup_{\mathcal{C}\in\Gamma} \mathcal{C}$ is a complete extension of $\mathcal{G}$. Suppose for the sake of contradiction that $\mathcal{E}_{\Gamma}$ is not a semi-stable extension of $\mathcal{G}$, i.e., that there is a set $\mathcal{E}' \subseteq \mathcal{A}$ s.t. $\mathcal{E}'$ is a complete extension of $\mathcal{G}$ and $(\mathcal{E}' \cup \mathcal{E}'^d) \supset (\mathcal{E}_{\Gamma} \cup \mathcal{E}^d_{\Gamma})$. From the $\Rightarrow$) part of the proof of the complete semantics, it follows that $\Gamma_{\mathcal{E}'}$ is a complete extension of $\mathcal{H}_{\mathcal{G}}$. But from $(\mathcal{E}' \cup \mathcal{E}'^d) \supset (\mathcal{E}_{\Gamma} \cup \mathcal{E}^d_{\Gamma})$ it is easy to check that $(\Gamma_{\mathcal{E}'} \cup \Gamma^d_{\mathcal{E}'}) \supset \Gamma \cup \Gamma^d$. Contradiction with the fact that Γ is a semi-stable extension of $\mathcal{H}_{\mathcal{G}}$. □

Proof of Theorem 5.1. To prove the theorem let us first prove the following lemma which says that any atom not appearing as a head in any rule of an LP is interpreted as false in any P-stable model of this LP.

Lemma 2. *Let Π be an LP and $I = \langle T, F\rangle$ be a P-stable model of Π. If a is an atom of* $(\mathtt{HB}_{\Pi} \setminus \mathtt{Head}(\Pi))$ *then $I(a) = f$.*

Proof of Lemma 2. By definition, I is the least model of the generalized reduct Π^I of Π by I. Hence, I is the fixpoint obtained by repeating the application of the operator Ψ on Π^I starting from the interpretation $\langle \emptyset, \mathtt{HB}_{\Pi}\rangle$. Let $a \in (\mathtt{HB}_{\Pi} \setminus \mathtt{Head}(\Pi))$. From the definition of Ψ it is clear that the only way to assign the values **t** or **u** to a is to have a rule r in π^I s.t. $a = \mathtt{Head}(r)$ which is obviously not possible for a. Then, a receives necessarily the value **f**. Notice also that from the previous lemma, it follows that I and I' agree on atoms whose truth value is undefined (**u**), i.e. $\mathtt{HB}_{\Pi} \setminus (T \cup F) = \mathtt{HB}_{\Pi'} \setminus (T \cup F')$. □

Now let us turn to our theorem. Let Π be an LP and Π' be the LP resulting from its simplification. We consider the partition of the rules of Π constituted from the following sets of rules. $\Pi_1 = \{r \in$

$\Pi \mid (\texttt{Body}^+(r) \cup \texttt{Body}^-(r)) \subseteq \texttt{Head}(\Pi)\}$ (rules whose all atoms of the body appear in the set of heads of Π rules); $\Pi_2 = \{r \in \Pi \mid a \notin \texttt{Head}(\Pi) \text{ for some } a \in \texttt{Body}^+(r)\}$ and Π_3 contains the possible remaining rules, i.e., rules whose positive body is included in the set of heads of Π rules, but at least an atom of the negative body does not appear in the set of heads of Π rules, i.e., $\Pi_3 = \{r \in \Pi | \texttt{Body}^+(\Pi) \subseteq \texttt{Head}(\Pi) \text{ and } a \notin \texttt{Head}(\Pi) \text{ for some } a \in \texttt{Body}^-(r)\}$. Then, clearly $\Pi' = \Pi_1 \cup \Pi_3'$ where Π_3' is obtained from Π_3 by removing any expression not a s.t. $a \notin \texttt{Head}(\Pi)$. Let Π^I (resp. $\Pi'^{I'}$) be the generalized reduct of Π (resp. of Π') by I (resp. by I'). Then, $\Pi^I = \Pi_1^I \cup \Pi_2^I \cup \Pi_3^I$ and $\Pi'^{I'} = \Pi_1^{I'} \cup \Pi_3'^{I'}$ where Π_1^I (resp. Π_2^I, Π_3^I, $\Pi_1^{I'}$, $\Pi_3'^{I'}$) is obtained from Π_1 (resp. from Π_2, Π_3, Π_1, Π_3') by replacing in every rule of Π_1 (resp. of Π_2, Π_3, Π_1, Π_3') any expression not a by $\mathbf{f}$ if $I(a) = \mathbf{t}$ (resp. if $I(a) = \mathbf{t}$, $I(a) = \mathbf{t}$, $I'(a) = \mathbf{t}$, $I'(a) = \mathbf{t}$), by $\mathbf{t}$ if $I(a) = \mathbf{f}$ (resp. if $I(a) = \mathbf{f}$, $I(a) = \mathbf{f}$, $I'(a) = \mathbf{f}$, $I'(a) = \mathbf{f}$) and by $\mathbf{u}$ if $I(a) = \mathbf{u}$ (resp. if $I(a) = \mathbf{u}$, $I(a) = \mathbf{u}$, $I'(a) = \mathbf{u}$, $I'(a) = \mathbf{u}$). It is easy to verify that $\Pi_1^I = \Pi_1^{I'}$ since I and I' agree on each atom appearing in Π_1. Each rule in Π_3^I corresponds exactly to a rule in $\Pi_3'^{I'}$ to which we add to the body at least one $\mathbf{t}$ corresponding to expressions not a where a does not appear in $\texttt{Head}(\Pi)$ (Lemma 2 shows that their value in I is $\mathbf{f}$).

Now let $I_0 = \langle \emptyset, \texttt{HB}_\Pi \rangle$ and $I_0' = \langle \emptyset, \texttt{HB}_{\Pi'} \rangle$. Let $\Psi^n(I_0) = I_n = \langle T_n, F_n \rangle$ (resp. $\Psi^n(I_0') = I_n' = \langle T_n', F_n' \rangle$) be the result of n successive applications of Ψ on Π (resp. Π') starting from I_0 (resp. from I_0'). The proof of our theorem turns out to prove that for every $n \geqslant 0$, $T_n' = T_n$ and $F_n' = F_n \cap \texttt{HB}_{\Pi'}$. It is easy to check that this is equivalent to the fact that $T_n' = T_n$ and $U_n' = U_n$ where U_n (resp. U_n') are the atoms having the truth value $\mathbf{u}$ in the interpretation $\Psi^n(I_0)$ (resp. $\Psi^n(I_0')$). Let us prove this fact by induction on n.

- For $n = 0$, $\Psi^0(I_0) = I_0$ and $\Psi^0(I_0') = I_0'$. The fact is true since $T_0 = T_0' = \emptyset$ and $U_0 = U_0' = \emptyset$.
- Suppose $T_n = T_n'$ and $U_n = U_n'$ and prove that $T_{n+1} = T_{n+1}'$ and $U_{n+1} = U_{n+1}'$. Let $a \in \texttt{HB}_\Pi$ s.t. $I_{n+1}(a) = \mathbf{t}$, then there is a rule $r : a \leftarrow a_1, \ldots, a_k$ in Π^I s.t. for all $i \leqslant k$, $I_n(a) = \mathbf{t}$. It is clear that $r \notin \Pi_2^I$. Indeed, every rule $r' \notin \Pi_2^I$ has in its body at least an atom from $(\texttt{HB}_\Pi \setminus \texttt{Head}(\Pi))$ which has necessarily the value $\mathbf{f}$ since from the definition of Ψ, such an atom can never receive the value $\mathbf{t}$ or $\mathbf{u}$. If $r \in \Pi_1^I$ then $I_{n+1}'(a) = I_{n+1}(a) = \mathbf{t}$ since $\Pi_1^I = \Pi'^{I'}$ and $T_n = T_n'$. If $r \in \Pi_3^I$ then there is a rule $r' : a \leftarrow a_1', \ldots, a_m'$ in $\Pi_3'^{I'}$ s.t. $m < k$, $\{a_1', \ldots, a_m'\} \subseteq \{a_1, \ldots, a_k\}$ and for all $b \in (\{a_1, \ldots, a_k\} \setminus \{a_1', \ldots, a_m'\})$, $b = \mathbf{t}$. Since $T_n = T_n'$ it follows that $I_{n+1}'(a) = I_{n+1}(a) = \mathbf{t}$. Inversely, let $a \in \texttt{HB}_{\Pi'}$ s.t. $I_{n+1}'(a) = \mathbf{t}$, then there is a rule $r : a \leftarrow a_1, \ldots, a_k$ in $\Pi'^{I'}$ s.t. for all $i \leqslant k$, $I_n'(a) = \mathbf{t}$. If $r \in \Pi_1'^{I'}$ then $I_{n+1}(a) = I_{n+1}'(a) = \mathbf{t}$ since $\Pi_1^I = \Pi'^{I'}$ and $T_n = T_n'$. If $r \in \Pi_3'^{I'}$ then there is a rule $r' : a \leftarrow a_1', \ldots, a_m'$ in $\Pi_3^{I'}$ s.t. $m > k$, $\{a_1, \ldots, a_k\} \subseteq \{a_1', \ldots, a_k'\}$ and for all $b \in (\{a_1', \ldots, a_m'\} \setminus \{a_1, \ldots, a_k\})$, $b = \mathbf{t}$. Since $T_n = T_n'$ it follows that $I_{n+1}(a) = I_{n+1}'(a) = \mathbf{t}$.
  Let $a \in \texttt{HB}_\Pi$ s.t. $I_{n+1}(a) = \mathbf{u}$, then $I_{n+1}(a) \neq \mathbf{t}$ and there is a rule $r : a \leftarrow a_1, \ldots, a_k$ in Π^I s.t. for all $i \leqslant k$, $I_n(a) \neq \mathbf{f}$. It is clear that $r \notin \Pi_2^I$. Indeed, every rule $r' \notin \Pi_2^I$ has in its body at least an atom from $(\texttt{HB}_\Pi \setminus \texttt{Head}(\Pi))$ which has necessarily the value $\mathbf{f}$ since from the definition of Ψ, such an atom can never receive the value $\mathbf{t}$ or $\mathbf{u}$. If $r \in \Pi_1^I$ then $I_{n+1}'(a) = I_{n+1}(a) = \mathbf{u}$ since $\Pi_1^I = \Pi'^{I'}$, $T_n = T_n'$ and $U_n = U_n'$. If $r \in \Pi_3^I$ then there is a rule $r' : a \leftarrow a_1', \ldots, a_m'$ in $\Pi_3'^{I'}$ s.t. $m < k$, $\{a_1', \ldots, a_m'\} \subseteq \{a_1, \ldots, a_k\}$ and for all $b \in (\{a_1, \ldots, a_k\} \setminus \{a_1', \ldots, a_m'\})$, $b = \mathbf{t}$. Since $T_n = T_n'$ and $U_n = U_n'$ it follows that $I_{n+1}'(a) = I_{n+1}(a) = \mathbf{u}$. Inversely, let $a \in \texttt{HB}_{\Pi'}$ s.t. $I_{n+1}'(a) = \mathbf{u}$, then $I_{n+1}'(a) \neq \mathbf{t}$ and there is a rule $r : a \leftarrow a_1, \ldots, a_k$ in $\Pi'^{I'}$ s.t. for all $i \leqslant k$, $I_n'(a) \neq \mathbf{f}$. If $r \in \Pi_1'^{I'}$ then $I_{n+1}(a) = I_{n+1}'(a) = \mathbf{u}$ since $\Pi_1^I = \Pi'^{I'}$, $T_n = T_n'$ and $U_n = U_n'$. If $r \in \Pi_3'^{I'}$ then there is a rule $r' : a \leftarrow a_1', \ldots, a_m'$ in $\Pi_3'^{I'}$ s.t. $m > k$, $\{a_1, \ldots, a_k\} \subseteq \{a_1', \ldots, a_k'\}$

and for all $b \in (\{a'_1, \ldots, a'_m\} \setminus \{a_1, \ldots, a_k\})$, $b = \mathbf{t}$. Since $T_n = T'_n$ and $U_n = U'_n$ it follows that $I_{n+1}(a) = I'_{n+1}(a) = \mathbf{u}$. $\square$

Proof of Corollary 5.2. The proof follows easily from Definition 2.7. $\square$

Proof of Theorem 5.6. Complete labelling – P-stable model

- **$\mathcal{L}$ is a complete labelling $\Rightarrow$ $\mathtt{Int}(\mathcal{L})$ is a P-stable model**
  Let $\mathcal{L} = (\mathrm{IN}, \mathrm{OUT}, \mathrm{UND})$ be a complete labelling of $\mathcal{G}_\Pi$ and $I = \mathtt{Int}(\mathcal{L}) = \langle T, F\rangle$ be the associated 3-valued interpretation of Π. To prove that I is a P-stable model of Π we have to prove that I is the minimal fixpoint of the operator Ψ applied on the generalized reduct Π^I, i.e. the fixpoint obtained by the successive application of the operator Ψ on Π^I starting from the interpretation $I_0 = \langle \emptyset, \mathtt{HB}_\Pi\rangle$. Let us denote by Ψ_n the operator corresponding to the application of Ψ for n successive times.
  First, let us show that I is a fixpoint of Ψ, i.e. $\Psi(I) = I$. For that purpose, it suffices to show that: $\forall a \in \mathtt{HB}_\Pi$, $\Psi(I)(a) = \mathbf{t}$ iff $I(a) = \mathbf{t}$ and $\Psi(I)(a) = \mathbf{f}$ iff $I(a) = \mathbf{f}$ (it follows then that $\Psi(I)(a) = \mathbf{u}$ iff $I(a) = \mathbf{u}$). Let $a \in \mathtt{HB}_\Pi$:
  $\Psi(I)(a) = \mathbf{t}$ iff $\exists r \in \Pi$ s.t. $r = a \leftarrow a_1, \ldots, a_m, \mathrm{not}\ a_{m+1}, \ldots, a_n.$ with
  $\forall i \in \{1, \ldots, m\} : I(a_i) = \mathbf{t}$ and $\forall j \in \{m+1, \ldots, n\} : I(a_j) = \mathbf{f}$.
  iff $\exists r \in \Pi$ s.t. $r = a \leftarrow a_1, \ldots, a_m, \mathrm{not}\ a_{m+1}, \ldots, a_n.$ with
  $\forall i \in \{1, \ldots, m\}, \exists r_i \in \Pi : \mathtt{Head}(r_i) = a_i$ and $\mathcal{L}(r_i) = \mathtt{in}$ and
  $\forall j \in \{m+1, \ldots, n\}, \forall r_j \in \Pi$ s.t. $\mathtt{Head}(r_j) = a_j$, $\mathcal{L}(r_i) = \mathtt{out}$. iff $\exists r \in \Pi$ s.t.
  $r = a \leftarrow a_1, \ldots, a_m, \mathrm{not}\ a_{m+1}, \ldots, a_n.$ and $\mathcal{L}(r) = \mathtt{in}$.
  iff $I(a) = \mathbf{t}$.

  $\Psi(I)(a) = \mathbf{f}$ iff $\forall r \in \Pi$ s.t. $r = a \leftarrow a_1, \ldots, a_m, \mathrm{not}\ a_{m+1}, \ldots, a_n.$,
  $\exists i \in \{1, \ldots, m\}$: $I(a_i) = \mathbf{f}$ or $\exists j \in \{m+1, \ldots, n\}$: $I(a_j) = \mathbf{t}$.
  iff $\forall r \in \Pi$ s.t. $r = a \leftarrow a_1, \ldots, a_m, \mathrm{not}\ a_{m+1}, \ldots, a_n.$,
  either $\exists i \in \{1, \ldots, m\}, \forall r_i \in \Pi$ s.t. $\mathtt{Head}(r_i) = a_i$, $\mathcal{L}(r_i) = \mathtt{out}$
  or $\exists j \in \{m+1, \ldots, n\}, \exists r_j \in \Pi$ s.t. $\mathtt{Head}(r_j) = a_j$, $\mathcal{L}(r_i) = \mathtt{in}$. iff $\forall r \in \Pi$ s.t.
  $r = a \leftarrow a_1, \ldots, a_m, \mathrm{not}\ a_{m+1}, \ldots, a_n.$, $\mathcal{L}(r) = \mathtt{out}$.
  iff $I(a) = \mathbf{f}$.

  Now it remains to show minimality of I. To do so, it suffices to show that the successive applications of Ψ on Π^I starting from I_0 will mark as true any atom a s.t. $I(a) = \mathbf{t}$ ($I(a) = \mathbf{t} \Rightarrow \exists n \geqslant 1$ s.t. $\Psi_n(I_0)(a) = \mathbf{t}$) and as false any atom a s.t. $I(a) = \mathbf{f}$ ($I(a) = \mathbf{f} \Rightarrow \forall n \geqslant 1$, $\Psi_n(I_0)(a) = \mathbf{f}$). Indeed, since I has been shown to be a fixpont of Ψ and since the successive application of Ψ on Π^I starting from I_0 yields the minimal fixpoint of Ψ then it is clear that further applications of Ψ cannot add to I neither true nor false atoms.

  * ($I(a) = \mathbf{t} \Rightarrow \exists n \geqslant 1$ s.t. $\Psi_n(I_0)(a) = \mathbf{t}$)
    Let $a \in \mathtt{HB}_\Pi$ s.t. $I(a) = \mathbf{t}$. From safety of $\mathcal{L}$ it follows that there is a sequence $r_1, \ldots, r_s \in \mathrm{IN}(\mathcal{L})$ s.t. $\mathtt{Head}(r_s) = a$ and $\{r_1, \ldots, r_s\}$ is a meta-argument for r_s in the sense of Definition 4.2. We can construct a partition $R_1, \ldots, R_k$ of k sets of $\{r_1, \ldots, r_s\}$ ($k \leqslant s$) with $R_k = \{r_s\}$, if $r \in R_1$ then $\mathtt{Body}^+ r = \emptyset$ ($R_1 \neq \emptyset$ since at least $r_1 \in R_1$), if $r \in R_i$ ($1 < i \leqslant k$) then $\mathtt{Body}^+ r \subseteq \bigcup_{j<i} \mathtt{Head}(r_j)$ and $\forall r \in R_i$ ($1 \leqslant i \leqslant k$), if $r' \in \Pi$ is s.t. $\mathtt{Head}(r') \in \mathtt{Body}^-(r)$

then $\mathcal{L}(r') = \texttt{out}$. Now, we prove by induction on i that: (Prop) : $\forall i \in \{1, \dots, k\}$, if $r \in R_k$ and $\texttt{Head}(r) = a$ then $\Psi_i(I_0)(a) = \mathbf{t}$ (our result follows then by taking $i = k$).
For $i = 0$, let $r \in R_0$ and $\texttt{Head}(r) = a$. Since we have: if $r' \in \Pi$ is s.t. $\texttt{Head}(r') \in \texttt{Body}^-(r)$ then $I(a') = \texttt{out}$, it follows that $I(a') = \mathbf{f}$ for every $a' \in \texttt{Body}^-(r)$. Thus $\Psi_1(I_0)(a) = \Psi(I_0)(a) = \mathbf{t}$.
Suppose that (Prop) holds for any $j \leqslant i < k$ and show that it holds for $i + 1$. Let $r \in R_{i+1}$ and $\texttt{Head}(r) = a$. If $a' \in \texttt{Body}^+(r)$ then it holds that $a' = \texttt{Head}(r')$ for some $r' \in R_j$ with $j \leqslant i$. From the induction hypothesis it follows that $\Psi_j(I_0)(a') = \mathbf{t}$, hence $\Psi_i(I_0)(a') = \mathbf{t}$. Moreover, if $a' \in \texttt{Body}^-(r)$ then $I(a') = \mathbf{f}$. It follows that $\Psi_{i+1}(I_0)(a) = \mathbf{t}$.

$*$ $I(a) = \mathbf{f} \Rightarrow \forall n \geqslant 1, \Psi_n(I_0)(a) = \mathbf{f}$ Let $a \in \texttt{HB}_\Pi$ s.t. $I(a) = \mathbf{f}$. Then, for every rule r whose head is a i.e., of the form $r : a \to a_0, \dots, a_m, \text{not } a_{m+1}, \dots, \text{not } a_n$ we have that $\mathcal{L}(r) = \texttt{out}$. Since $\mathcal{L}$ is completely, r is legally out and two cases are possible. The first case is that for some $j \in \{m + 1, \dots, n\}$ there is a rule r_j s.t. $\texttt{Head}(r_j) = a_j$ and $\mathcal{L}(r_j) = \texttt{in}$ which implies that $I(a_j) = \mathbf{t}$. In this case, it is clear that for every $n \geqslant 1$, $\Psi_n(I_0)(a) = \mathbf{f}$. The second case is that for some $i \in \{0, \dots, m\}$, for all rule r_i s.t. $\texttt{Head}(r_i) = a_i$ it holds that $\mathcal{L}(r_i) = \text{out}$ which implies that $I(a_i) = \mathbf{f}$. In this case we repeat for a_j the same reasoning for r. since the number of rules is finite, either every starting rule r is involved in a cycle or for any (minimal) sequence of rules $r_1, \dots r_k$ verifying: $\texttt{Head}(r_k) = a$, $\texttt{Body}^+(r_1) = \emptyset$ and for $i > 1$, $\texttt{Body}^+(r_i) = \bigcup_{j<i} \texttt{Head}(r_j)$, there is at least one rule r of the sequence s.t. $I(b) = \mathbf{t}$ for some $b \in \texttt{Body}^-(r)$. In both cases, it is clear that there is no mean to assign to a the value $\mathbf{t}$ or $\mathbf{u}$.

- **$\texttt{Label}(I)$ is a complete labelling $\Leftarrow$ I is a P-stable model**
Let I be a P-stable model of Π, i.e. a minimal fixpoint of the generalized reduct Π^I. Then, $\Psi(I) = I$ (The application of Ψ is on Π^I) and I is a minimal interpretation satisfying the previous equality. Let $\mathcal{L} = \texttt{Label}(I)$, we prove that for every $r \in \mathcal{G}_\Pi$, if r is labelled $\texttt{in}$ (resp. out, undec) in $\mathcal{L}$ then it is legally $\in$ (resp. legally $\texttt{out}$, legally $\texttt{undec}$) and that $\mathcal{L}$ is safe.
Let $r :\in \mathcal{G}_\Pi$ of the form $a \to a_0, \dots, a_m, \text{not } a_{m+1}, \dots, \text{not } a_n$ s.t. $\mathcal{L}(r) = \text{in}$. Since $\mathcal{L}(r) = \text{in}$ and from the definition of the function $\texttt{Label}$, it follows that for all $i \in \{0, \dots, m\}$, $I(a_i) = \mathbf{t}$ and for all $j \in \{m+1, \dots, n\}$, $I(a_j) = \mathbf{f}$. Now, suppose for the sake of contradiction that r is illegally $\texttt{in}$. Two cases are possible.
Case 1: there is a rule r' s.t. $\mathcal{L}(r') \neq \texttt{out}$ and $r'\mathcal{R}_\Pi r$, i.e., there is a rule r' s.t. $\mathcal{L}(r') \neq \texttt{out}$ and $\texttt{Head}(r') = a_j$ for some $j \in \{m + 1, \dots, n\}$. Suppose that r' has the form: $r' : a_j \to a'_0, \dots, a'_{m'}, \text{not } a'_{m'+1}, \dots, a'_{n'}$. If $\mathcal{L}(r') = \texttt{in}$ then from the definition of $\texttt{Label}$, for all $i \in \{0, \dots, m'\}$, $I(a'_i) = \mathbf{t}$ and for all $j \in \{m' + 1, \dots, n'\}$, $I(a'_j) = \mathbf{f}$. But from $\Psi(I) = I$ it follows that $I(a_j) = \mathbf{t}$ which contradicts the fact that for all $j \in \{m + 1, \dots, n\}$, $I(a_j) = \mathbf{f}$. If $\mathcal{L}(r') = \texttt{undec}$, a similar reasoning yields $I(a_j) = \mathbf{u}$ which contradicts also the same fact.
Case 2: there is a set of rules R_i s.t. for every $r' \in R_i$, $\mathcal{L}(r') \neq\in$ and $R_i\mathcal{N}_\Pi r$, i.e., for some $i \in \{0, \dots, m\}$, for every rule r' s.t. $\texttt{Head}(r') = a_i$ (the set R_i contains all such rules and only them), we have that $\mathcal{L}(r') \neq \texttt{in}$. Let $r' \in R_i$ and consider that r' has the form $a_j \to a'_0, \dots, a'_{m'}, \text{not } a'_{m'+1}, \dots, a'_{n'}$. If $\mathcal{L}(r') = \texttt{out}$ then from the definition of $\texttt{Label}$ it follows that: there is either $i \in \{0, \dots, m'\}$ s.t. $I(a'_i) = \mathbf{f}$ or $j \in \{m'+1, \dots, n'\}$ s.t. $I(a'_j) = \mathbf{t}$. If $\mathcal{L}(r') = \texttt{undec}$ then from the definition of $\texttt{Label}$ it follows that: (1) there is either $i \in \{0, \dots, m'\}$ s.t. $I(a'_i) \neq \mathbf{t}$ or $j \in \{m' + 1, \dots, n'\}$ s.t. $I(a'_j) \neq \mathbf{f}$ and (2) for all $i \in \{0, \dots, m'\}$, $I(a'_i) \neq \mathbf{f}$ and for all $j \in \{m' + 1, \dots, n'\}$, $I(a'_j) \neq \mathbf{t}$. But then from $\Psi(I) = I$ it follows that either $I(a_i) = \mathbf{f}$ or $I(a_i) = \mathbf{f}$ i.e., $I(a_i) \neq \mathbf{t}$ which contradicts the fact that for all $i \in \{0, \dots, m\}$, $I(a_i) = \mathbf{t}$.

By a similar reasoning one can prove that if r is labelled out (resp. undec) in $\mathcal{L}$ then it is legally $\texttt{out}$ (resp. $\texttt{undec}$).

For safety of $\mathcal{L}$, let r be a rule $r \in \text{IN}(\mathcal{L})$ with $\texttt{Head}(r) = a$. From the definition of the function $\texttt{Label}$ and the fact that r is legally $\texttt{in}$ it follows that $I(a) = \mathbf{t}$. Now, for the sake of contradiction, suppose that r is not powerful in $\text{IN}(\mathcal{L})$. Thus, if $r_1 \dots r_n$ is a minimal sequence of rules s.t. $r_n = r$, $\texttt{Body}^+(r_1) = \emptyset$ and for all $i \in \{2, \dots, n\}$, $\texttt{Body}^+(r_i) \subseteq \bigcup_{j<i} \texttt{Head}(r_j)$, there is a rule r' of the sequence s.t. $\mathcal{L}(r') \neq \texttt{in}$, i.e. $\mathcal{L}(r') = \texttt{out}$ or $\mathcal{L}(r') = \texttt{undec}$. By considering the structure of r' and the definition of the function label it follows that the possible values of I for the atoms appearing in the body of r' do never allow one to use r' in a process ending by assigning the value $\mathbf{t}$ to a. Contradiction.

Grounded labelling – Well-founded model

- **$\mathcal{L}$ is a grounded labelling $\Rightarrow$ $\texttt{Int}(\mathcal{L})$ is a well-founded model**
  Let $\mathcal{L} = (\text{IN}, \text{OUT}, \text{UND})$ be the grounded labelling of $\mathcal{G}_\Pi$. Then $\mathcal{L}$ is also complete and hence $I = \texttt{Int}(\mathcal{L}) = \langle T, F\rangle$ is a P-stable model of Π. Suppose that I is not the P-stable model of Π having the minimal set T. Thus, it exists a P-stable model $I' = \langle T', F'\rangle$ of Π s.t. $T' \subset T$. It follows that $\mathcal{L}' = \texttt{Label}(I') = (\text{IN}', \text{OUIT}', \text{UND}')$ is a complete labelling of $\mathcal{G}_\Pi$. But, from the definition of the function $\texttt{Label}$ it is easy to check that $\text{IN}' \subset \text{IN}$ which contradicts the fact that $\mathcal{L}$ is the grounded labelling of $\mathcal{G}_\Pi$.
- **$\texttt{Label}(\mathcal{L})$ is a grounded labelling $\Leftarrow$ I is a well-founded model**
  Let $I = \langle T, F\rangle$ be the Well-founded model of Π. Then I is also a P-stable model of Π and hence $\mathcal{L} = \texttt{Label}(I) = (\text{IN}, \text{OUT}, \text{UND})$ is a complete labelling of $\mathcal{G}_\Pi$. Suppose that $\mathcal{L}$ is not the complete labelling $\mathcal{G}_\Pi$ having the minimal set IN. Thus, it exists a complete labelling $\mathcal{L}' = (\text{IN}', \text{OUT}', \text{UND}')$ of $\mathcal{G}_\Pi$ s.t. $\text{IN}' \subset \text{IN}$. It follows that $I' = \texttt{Int}(\mathcal{L}') = (T', F')$ is a P-stable model Π. But, from the definition of the function $\texttt{Int}$ it is easy to check that $T' \subset T$ which contradicts the fact that I is the well-founded model of Π.

Preferred labelling – M-stable model

We obtain that if $\mathcal{L}$ is a preferred labelling of $\mathcal{G}_\Pi$ (resp. if I is an M-stable model of Π) then $\text{Int}(\mathcal{L})$ is an M-stable model of Π (resp. then $\texttt{Label}(I)$ is a preferred labelling of $\mathcal{G}_\Pi$) by a proof similar to the previous one.

Stable labelling – Stable model

- **$\mathcal{L}$ is a stable labelling $\Rightarrow$ $\texttt{Int}(\mathcal{L})$ is a stable model**
  Let $\mathcal{L} = (\text{IN}, \text{OUT}, \text{UND})$ be a stable labelling of $\mathcal{G}_\Pi$. Then $\mathcal{L}$ is a complete labelling of $\mathcal{G}_\Pi$ s.t. $\text{OUT} = \emptyset$. Hence, $I = \texttt{Int}(\mathcal{L}) = \langle T, F\rangle$ is a P-stable model of Π. But, from the definition of $\texttt{Label}$ it is easy to check that no atom of Π can receive the value $\mathbf{u}$, i.e., $\text{IN} \cup \text{OUT} = \texttt{HB}(\Pi)$. Thus, $I = \texttt{Int}(\mathcal{L})$ is a stable model of Π.
- **$\texttt{Label}(\mathcal{L})$ is a stable labelling $\Leftarrow$ I is a stable model**
  Let $I = \langle T, F\rangle$ be a stable model of Π. Then I is P-stable model of Π s.t. $T \cup F = \emptyset$. Hence, $\mathcal{L} = \texttt{Label}(I) = (\text{IN}, \text{OUT}, \text{UND})$ is a complete labelling of $\mathcal{G}_\Pi$. But, from the definition of $\texttt{Int}$ it is easy to check that $\text{UND} = \emptyset$ Thus, $\mathcal{L} = \texttt{Label}(I)$ is a stable labelling of $\mathcal{G}_\Pi$.

Semi-stable labelling vs L-stable model

- **$\mathcal{L}$ is a semi-stable labelling $\not\Rightarrow$ $\texttt{Int}(\mathcal{L})$ is an L-stable model**
  Consider a counter-example. We take the LP Π with the rules: $\{(x) : c \rightarrow \text{not}\, c; (y) : a \rightarrow \text{not}\, b; (z) : c \rightarrow \text{not}\, c; (u) : c \rightarrow \text{not}\, c; (v) : c \rightarrow \text{not}\, c\}$. The corresponding AFN using Definition 5.3 is $\mathcal{G}_\Pi = \langle \mathcal{A}_\Pi, \mathcal{R}_\Pi, \mathcal{N}_\Pi\rangle$ with $\mathcal{A}_\Pi = \{x, y, z, u, v\}$, $\mathcal{R}_\Pi =$

$\{(x, x), (x, u), (y, z), (z, y), (u, u), (u, x), (v, v)\}$ and $\mathcal{N} = \{(z, u), (y, v)\}$. $\mathcal{G}_\Pi$ has two semi-stable labellings: $\mathcal{L}_1 = (\{y\}, \{z, u\}, \{x, v\})$ and $\mathcal{L}_2 = (\{z\}, \{y, v\}, \{x, u\})$. We can verify that $\mathtt{Int}(\mathcal{L}_1) = \langle\{a\}, \{b\}\rangle$ and $\mathtt{Int}(\mathcal{L}_2) = \langle\{b\}, \{a, g\}\rangle$ but $\mathtt{Int}(\mathcal{L}_2)$ is an L-stable model of Π but $\mathtt{Int}(\mathcal{L}_1)$ is not.

- **$\mathtt{Label}(I)$ is a semi-stable labelling $\not\Leftarrow$ I is an L-stable model**
Consider a counter-example. We take the LP Π with the rules: $\{(x) : c \rightarrow e$ not c; $(y) : a \rightarrow$ not b; $(z) : b \rightarrow$ not a; $(u) : c \rightarrow b$, not c; $(v) : g \rightarrow a$ not g; $(w) : d \rightarrow$ not e; $(k) : e \rightarrow$ not $d\}$. The corresponding AFN using Definition 5.3 is $\mathcal{G}_\Pi = \langle\mathcal{A}_\Pi, \mathcal{R}_\Pi, \mathcal{N}_\Pi\rangle$ with $\mathcal{A}_\Pi = \{x, y, z, u, v, w, k\}$, $\mathcal{R}_\Pi = \{(x, x), (x, u), (y, z), (z, y), (u, u), (u, x), (v, v), (w, k), (k, w)\}$ and $\mathcal{N} = \{(z, u), (y, v), (k, x)\}$. Π has three L-stable models: $I_1 = \langle\{a, d\}, \{b, c, e\}\rangle$, $I_2 = \langle\{b, d\}, \{a, e, g\}\rangle$ and $I_3 = \langle\{b, e\}, \{a, d, g\}\rangle$. We can verify that $\mathtt{Label}(I_1) = (\{y, w\}, \{x, z, u, k\}, \{v\})$, $\mathtt{Label}(I_2) = (\{z, w\}, \{x, y, v, k\}, \{u\})$ and $\mathtt{Label}(I_3) = (\{z, k\}, \{y, v, w\}, \{x, u\})$. Only $\mathtt{Label}(I_1)$ and $\mathtt{Label}(I_2)$ are semi-stable labellings. $\mathtt{Label}(I_3)$ is not a semi-stable labelling. $\square$

Proof of Theorem 5.7. Follows from Theorem 5.6 and Definitions 5.4 and 5.5. $\square$

Proof of Theorem 5.11. Complete labelling – P-stable model
$\Rightarrow$) Let $\mathcal{L}$ be a complete labelling of $\mathcal{G}$ and $I_\mathcal{L} = \mathtt{Int}'(\mathcal{L})$. We prove that $I_\mathcal{L}$ is a least fix point of Ψ.

(1) Case 1. $a \in \mathcal{A}$

- prove that $\Psi(I_\mathcal{L})(a) = \mathbf{t}$ iff $I_\mathcal{L}(a) = \mathbf{t}$
$\Psi(I_\mathcal{L})(a) = \mathbf{t}$ iff there is a rule $r : a \leftarrow e_1, \dots, e_m$, not a_1, not a_n s.t. $\forall i \in \{1, \dots, m\}$, $I_\mathcal{L}(e_i) = \mathbf{t}$ and $\forall j \in \{1, \dots, n\}$, $I_\mathcal{L}(a_j) = \mathbf{f}$.
iff there is a rule $r : a \leftarrow e_1, \dots, e_m$, not a_1, not a_n s.t. $\forall i \in \{1, \dots, m\}$ there is a rule: $e_i \leftarrow a_i$ with $\mathcal{L}(a_i) = \mathtt{in}$ and $\forall j \in \{1, \dots, n\}$, $I_\mathcal{L}(a_j) = \mathbf{f}$.
iff there is a rule $r : a \leftarrow e_1, \dots, e_m$, not a_1, not a_n s.t. $\forall E_i \subseteq \mathcal{A}$ with $E_i\ \mathcal{N}\ r$, $\exists a_i \in E_i$ s.t. $\mathcal{L}(a_i) = \mathtt{in}$ and $\forall j \in \{1, \dots, n\}$, $\mathcal{L}(a_j) = \mathtt{out}$.
iff $\mathcal{L}(a) = \mathtt{in}$.
iff $I_\mathcal{L}(a) = \mathbf{t}$.
- prove that $\Psi(I_\mathcal{L})(a) = \mathbf{f}$ iff $I_\mathcal{L}(a) = \mathbf{f}$
$\Psi(I_\mathcal{L})(a) = \mathbf{f}$ iff for all rule $r : a \leftarrow e_1, \dots, e_m$, not a_1, not a_n, $\exists i \in \{1, \dots, m\} s.t. I_\mathcal{L}(e_i) = \mathbf{f}$ or $\exists j \in \{1, \dots, n\}$ s.t. $I_\mathcal{L}(a_j) = \mathbf{t}$.
iff for all rule $r : a \leftarrow e_1, \dots, e_m$, not a_1, not a_n, $\exists i \in \{1, \dots, m\}$ s.t; for all rule: $e_i \leftarrow a_i$, $\mathcal{L}(a_i) = \mathtt{out}$ or $\exists j \in \{1, \dots, n\}$ s.t. $I_\mathcal{L}(a_j) = \mathbf{t}$.
iff for all rule $r : a \leftarrow e_1, \dots, e_m$, not a_1, not a_n, $\exists E_i \subseteq \mathcal{A}$ with $E_i\ \mathcal{N}\ r$, $\forall a_i \in E_i$, $\mathcal{L}(a_i) = \mathtt{out}$ or $\exists j \in \{1, \dots, n\}$ s.t. $a_j\ \mathcal{R}\ r$ and $\mathcal{L}(a_j) = \mathtt{in}$.
iff $\mathcal{L}(a) = \mathtt{out}$.
iff $I_\mathcal{L}(a) = \mathbf{f}$.
- The proof that $\Psi(I_\mathcal{L})(a) = \mathbf{u}$ iff $I_\mathcal{L}(a) = \mathbf{u}$ follows immediately from the previous two proofs.

(2) Case 2. $e \in \mathcal{A}'$

- prove that $\Psi(I_\mathcal{L})(e) = \mathbf{t}$ iff $I_\mathcal{L}(e) = \mathbf{t}$
$\Psi(I_\mathcal{L})(e) = \mathbf{t}$ iff there is a rule $r : e \leftarrow a$ s.t. $I_\mathcal{L}(a) = \mathbf{t}$.
iff there is a rule $r : e \leftarrow a$ s.t. $\mathcal{L}(a) = \mathtt{in}$.
iff $I_\mathcal{L}(e) = \mathbf{t}$.

- prove that $\Psi(I_{\mathcal{L}})(e) = \mathbf{f}$ iff $I_{\mathcal{L}}(e) = \mathbf{f}$
  $\Psi(I_{\mathcal{L}})(e) = \mathbf{f}$ iff for all rule $r : e \leftarrow a$, $I_{\mathcal{L}}(a) = \mathbf{f}$.
  iff for all rule $r : e \leftarrow a$, $\mathcal{L}(a) = \texttt{out}$.
  iff $I_{\mathcal{L}}(e) = \mathbf{f}$.
- The proof that $\Psi(I_{\mathcal{L}})(e) = \mathbf{u}$ iff $I_{\mathcal{L}}(e) = \mathbf{u}$ follows immediately from the previous two proofs.

To prove minimality of $I_{\mathcal{L}}$ it suffices to show (in a similar way as in theorem 5.6) that the successive application of the operator Ψ on $\Pi_{\mathcal{G}}$ starting from I_0 yields to an interpretation in which all true (resp. false) atoms of $I_{\mathcal{L}}$ are true (resp. false).

$\Leftarrow$) Let $I = \langle T, F\rangle$ be a P-stable model of $\Pi_{\mathcal{G}}$ and prove that $\mathcal{L}_I = (I)(T \cap \mathcal{A}, F \cap \mathcal{A}, \mathcal{A} \setminus (T \cup F))$ is a complete labelling of $\mathcal{G}$, i.e., all arguments of $\mathcal{G}$ are labelled legally in $\mathcal{L}_I$ and $\mathcal{L}_I$ is safe.

- Let a be an argument of $\mathcal{G}$ labelled $\texttt{in}$. Then, there is in $\Pi_{\mathcal{G}}$ a unique rule $r : a \leftarrow e_1, \ldots, e_m, \text{not } a_1, \ldots, \text{not } a_n$ with $I(a) = \mathbf{t}$. It follows that $\forall i \in \{1, \ldots, m\}$, $I(e_i) = \mathbf{t}$ and $\forall j \in \{1, \ldots, m\}$, $I(a_j) = \mathbf{f}$. Thus, for all $E \subseteq \mathcal{A}$ s.t. $E\ \mathcal{N}\ a$, there exists $a_i \in E$ s.t. $\mathcal{L}_I(a_i) = \texttt{in}$ and for all $a_j \in \mathcal{A}$ s.t. $a_j\ \mathcal{R}\ a$, $\mathcal{L}(a_j) = \texttt{out}$. It follows that a is legally $\texttt{in}$.
- Let a be an argument of $\mathcal{G}$ labelled $\texttt{out}$. Then, there is in $\Pi_{\mathcal{G}}$ a unique rule $r : a \leftarrow e_1, \ldots, e_m, \text{not } a_1, \ldots, \text{not } a_n$ with $I(a) = \mathbf{f}$. It follows that either $\exists i \in \{1, \ldots, m\}$, $I(e_i) = \mathbf{f}$ or $\exists j \in \{1, \ldots, m\}$, $I(a_j) = \mathbf{t}$. Thus, either there exists $E \subseteq \mathcal{A}$ s.t. $E\ \mathcal{N}\ a$ and for all $a_i \in E$, $\mathcal{L}_I(a_i) = \texttt{out}$ or there exists $a_j \in \mathcal{A}$ s.t. $a_j\ \mathcal{R}\ a$ and $\mathcal{L}(a_j) = \texttt{in}$. It follows that a is legally $\texttt{out}$.
- Let a be an argument of $\mathcal{G}$ labelled $\texttt{undec}$. Then, there is in $\Pi_{\mathcal{G}}$ a unique rule $r : a \leftarrow e_1, \ldots, e_m, \text{not } a_1, \ldots, \text{not } a_n$ with $I(a) = \mathbf{u}$. It follows that $\forall i \in \{1, \ldots, m\}$, $I(e_i) \neq \mathbf{f}$ and $\forall j \in \{1, \ldots, m\}$, $I(a_j) \neq \mathbf{t}$ and either $\exists i \in \{1, \ldots, m\}$ s.t. $I(e_i) \neq \mathbf{t}$ or $\exists j \in \{1, \ldots, n\}$ s.t. $I(a_j) \neq \mathbf{f}$. Thus, for all $E \subseteq \mathcal{A}$ s.t. $E\ \mathcal{N}\ a$, $E \nsubseteq \text{out}(\mathcal{L}_I)$ and for all $a_j \in \mathcal{A}$ s.t. $a_j\ \mathcal{R}\ a$, $\mathcal{L}(a_j) \neq \texttt{in}$ and either there is $E \subseteq \mathcal{A}$ s.t. $E\ \mathcal{N}\ a$ and $E \cap \text{in}(\mathcal{L}_I) \neq \emptyset$ or there is $a_j \in \mathcal{A}$ s.t. $a_j\ \mathcal{R}\ a$ and $\mathcal{L}(a_j) \neq \texttt{out}$. It follows that a is legally $\texttt{undec}$.

To show safety of $\mathcal{L}_I$, observe that if we suppose the contrary, then by assigning $\mathbf{f}$ to each atom corresponding to an argument which is not powerful in $\text{in}(\mathcal{L}_I)$ and to the corresponding atoms e_i, the obtained interpretation is still a fixpoint of $\Pi_{\mathcal{G}}$ which contradicts the minimality of I.

Grounded labelling – Well-founded model

$\Rightarrow$) If $\mathcal{L}$ is the grounded labelling of $\mathcal{G}$ then it is also complete, thus $I_{\mathcal{L}} = \texttt{Int}'(\mathcal{L})$ is a P-stable model of $\Pi_{\mathcal{G}}$. Suppose that $I_{\mathcal{L}}$ is not well-founded, i.e., that there is a P-stable model $I' = \langle T', F'\rangle$ of $\Pi_{\mathcal{G}}$ with $T' \subset T$. It follows that the labelling $\mathcal{L}_{I'} = (T' \cap \mathcal{A}, F' \cap \mathcal{A}, \mathcal{A} \setminus (T' \cup F'))$ is a complete labelling of $\mathcal{G}$. It is easy to check that $\text{in}(L_{I'}) \subset \text{in}(\mathcal{L})$ which contradicts the fact that $\mathcal{L}$ is the grounded labelling of $\mathcal{G}$.

$\Leftarrow$) If $I = \langle T, F\rangle$ is the well-founded model of $\Pi_{\mathcal{G}}$ then it is also P-stable, thus $\mathcal{L}_I = \texttt{Label}'(I)$ is a complete labelling of $\mathcal{G}$. Suppose that $\mathcal{L}_I$ is not grounded, i.e., that there is a complete labelling $\mathcal{L}'$ s.t; $\text{in}(\mathcal{L}') \subset \text{in}(\mathcal{L}_I)$. It follows that the labelling $\texttt{Int}(\mathcal{L}') = \langle T', F'\rangle$ is a P-stable model of $\Pi_{\mathcal{G}}$. It is easy to check that $T' \subset T$ which contradicts the fact that I is the well-founded model of $\Pi_{\mathcal{G}}$.

Preferred labelling – M-stable model

The proof is similar to the previous one.

Stable labelling – Stable model

$\Rightarrow$) If $\mathcal{L}$ is a stable labelling of $\mathcal{G}$ then it is complete and $\text{undec}(\mathcal{L}) = \emptyset$. Thus, $I_{\mathcal{L}} = \texttt{Int}'(\mathcal{L})$ is a P-stable model of $\Pi_{\mathcal{G}}$. Moreover, for each atom of $\mathcal{A}$ in $\Pi_{\mathcal{G}}$ there is one and only one rule of $\Pi_{\mathcal{G}}$ which labelled in $\mathcal{L}$ either $\texttt{in}$ or $\texttt{out}$ but never $\texttt{undec}$. From the definition of the function $\texttt{Int}'$, it is easy to check that for each atom $a \in \mathcal{A} \cup \mathcal{A}'$, we have that $I_(\mathcal{L})(a)$ equals either $\mathbf{t}$ or $\mathbf{f}$ but never $\mathbf{u}$. Thus $I_(\mathcal{L})$ is a P-stable model of $\Pi_{\mathcal{G}}$ without undefined atoms, i.e., a stable model of $\Pi_{\mathcal{G}}$.

$\Leftarrow$) If $I\langle T, F\rangle$ is a stable model of $\Pi_{\mathcal{G}}$ then it is a P-stable model and $T \cup F = \mathtt{HB}_{\Pi_{\mathcal{G}}}$. Thus, $\mathcal{L}_I = \mathtt{Label}'(I)$ is a complete labelling of $\mathcal{G}$. Moreover, for each argument of $\mathcal{G}$ there is one and only one rule of $\Pi_{\mathcal{G}}$ whose any body's atom has either the truth value **t** or **f** by I but never the value **u**. From the definition of the function $\mathtt{Label}'$, it is easy to check that for each argument a in $\mathcal{G}$, we have that $\mathcal{L}_(I)(a)$ is either `in` or `out` but never `undec`. Thus $\mathcal{L}_(I)$ is a complete labelling of $\mathcal{G}$ without arguments labelled `undec`, i.e., a stable labelling of $\mathcal{G}$.

Semi-stable labelling – L-stable model

The proof is similar to that for the link between grounded (resp. preferred) labelling of $\mathcal{G}$ and well-founded (resp. M-stable) models of $\Pi_{\mathcal{G}}$. $\square$

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