

# The Complexity of Some Subclasses of Minimal Unsatisfiable Formulas

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## Abstract

This paper is concerned with the complexity of some natural subclasses of minimal unsatisfiable formulas. We show the  $D^P$ -completeness of the classes of maximal and marginal minimal unsatisfiable formulas. Then we consider the class *Unique-MU* of minimal unsatisfiable formulas which have after removing a clause exactly one satisfying truth assignment. We show that *Unique-MU* has the same complexity as the unique satisfiability problem with respect to polynomial reduction. However, a slight modification of this class leads to the  $D^P$ -completeness. Finally we show that the class of minimal unsatisfiable formulas which can be divided for every variable into two separate minimal unsatisfiable formulas is at least as hard as the unique satisfiability problem.

KEYWORDS: *minimal unsatisfiable formulas, maximal MU, marginal MU, unique satisfiability, disjunctive splitting, complexity*

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## 1. Introduction

A propositional formula  $F$  in conjunctive normal form is called minimal unsatisfiable if and only if  $F$  is unsatisfiable and any proper subformula of  $F$  is satisfiable. The class of minimal unsatisfiable formulas is denoted as  $MU$  and shown to be  $D^P$ -complete [10].  $D^P$  is the class of problems which can be described as the difference of two  $NP$ -problems. It is strongly conjectured that  $D^P$  is different from  $NP$  and from  $coNP$ .

We are interested in subclasses of  $MU$  mainly for two reasons. One reason is that for proof calculi hard formulas are almost all minimal unsatisfiable (see for example [2, 5, 12]) and a deeper understanding of  $MU$ -formulas may help to develop new hard formulas and new satisfiability algorithms. For example, the deficiency property leads to new polynomially solvable classes of formulas, where the deficiency is the difference of the number of clauses and the number of variables [4].  $MU(k)$  is the class of formulas in  $MU$  with defi-

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ciency  $k$  and shown to be decidable in polynomial time [7, 8]. The second reason lies in the close relation between some subclasses of  $MU$ -formulas and the  $Unique-SAT$ -problem.

At first we investigate the complexity of two subclasses of  $MU$ , namely the class of maximal and the class of marginal  $MU$ -formulas. A formula  $F$  in  $MU$  is called *maximal*, if for any clause  $f \in F$  and any literal  $L$  which is not in  $f$ , adding  $L$  to  $f$  yields a satisfiable formula. In a certain sense maximal formulas are maximal extensions of  $MU$ -formulas. Instead of “maximal”, the word “saturated” was used in [9]. A *marginal  $MU$ -formula* is a  $MU$ -formula for which the deletion of an occurrence of a literal leads to an unsatisfiable formula which is not in  $MU$ . That means marginal formulas are the minimal kernel of minimal unsatisfiable formulas. Obviously, both classes can be represented as the intersection of a  $NP$ -problem and a  $coNP$ -problem and lie therefore in  $D^P$ . We will show that both classes are  $D^P$ -complete. The results are not astonishing, but the polynomial-time reductions from the problem  $MU$  may be of interest.

Another class of restrictions is based on a limited number of satisfying truth assignments. Besides the unsatisfiability, minimal unsatisfiable means that for any clause  $f$  the formula  $F - \{f\}$  is satisfiable.  $F$  is called *unique minimal unsatisfiable*, if for any clause  $f$  the formula  $F - \{f\}$  has exactly one satisfying truth assignment, that means  $F - \{f\}$  is in  $Unique-SAT$ . The class of these formulas is denoted as  $Unique-MU$ . At a first glance to demand that for all clauses there is exactly one satisfying truth assignment seems to be very strong.

We will show that the problem  $Unique-MU$  is as hard as the  $Unique-SAT$ -problem. Although it is not known whether  $Unique-SAT$  is  $D^P$ -complete, some evidence has been found that supports the belief that  $Unique-SAT$  is not  $D^P$ -complete (see e.g. [11]). Thus,  $Unique-MU$  is unlikely  $D^P$ -complete. A slight modification of  $Unique-MU$  is the class  $Almost-Unique-MU$  of almost unique minimal unsatisfiable formulas. A formula  $F$  is in  $Almost-Unique-MU$  if for at most one clause  $F - \{f\}$  may have more than one satisfying truth assignment. Under the assumption that  $Unique-SAT$  is not  $D^P$ -complete,  $Almost-Unique-MU$  is stronger than  $Unique-MU$ , because we will show  $D^P$ -completeness of  $Almost-Unique-MU$ .

A more detailed analysis of the class  $Unique-SAT$  leads to class  $Dis-MU$ . A minimal unsatisfiable formula  $F$  is in  $Dis-MU$  if and only if  $F$  has a disjunctive splitting on any variable. That means, for any variable  $x$  of  $F$ ,  $F$  can be split into two disjoint subformula  $H$  and  $G$  such that  $H$  resp.  $G$  contains no occurrence of  $x$  resp.  $\neg x$  and that they are in  $MU$  when setting the variable true resp. false.  $Dis-MU$  is of interest, because  $Dis-MU$  is a proper subclass of  $Unique-MU$  and its close relation to tree-like decision procedures. We establish a polynomial-time reduction from  $Unique-SAT$ , which shows that  $Dis-MU$  is at least as hard as  $Unique-SAT$ . However, we did not succeed in finding a reduction from  $Dis-MU$  to  $Unique-SAT$ .

Summarizing, we shall show  $Unique-SAT \approx_p Unique-MU \leq_p Dis-MU$  and  $MU \approx_p MARG-MU \approx_p MAX-MU \approx_p Almost-Unique-MU$ , where  $\leq_p$  denotes the polynomial-time reducibility and  $A \approx_p B$  is an abbreviation for  $A \leq_p B$  and  $B \leq_p A$ .

## 2. Notation

A literal is a propositional variable or a negated propositional variable.  $var(F)$  is the set of variables of a formula  $F$ . Clauses are sets of literals without multiple occurrences of literals. Since formulas with multiple occurrences of clauses are not minimal unsatisfiable, we consider a formula in *CNF* not as a set of clauses but as a multi-set of clauses. Given a formula  $F$  and a variable  $x$ , we use  $F|_{x=0}$  ( $F|_{x=1}$ , respectively) to denote the formula obtained from  $F$  by deleting the clauses containing  $\neg x$  ( $x$ , respectively) and removing all occurrences of  $x$  ( $\neg x$ , respectively). Suppose  $F \in MU$ , we say  $F$  has a disjunctive splitting on a variable  $x$  if  $F$  can be split into two disjoint subformulas  $G$  and  $H$  such that  $G$  contains no occurrences of  $\neg x$ ,  $H$  contains no occurrences of  $x$ ,  $G|_{x=0} \in MU$ , and  $H|_{x=1} \in MU$ . In this case, we call  $(G, H)$  a disjunctive splitting of  $F$  on  $x$ . It has been proved in [4] that for any formula  $F \in MU$  with deficiency 1 (i.e., the difference between the number of clauses and the number of variables is 1),  $F$  always has a disjunctive splitting on some variable. However, there are minimal unsatisfiable formulas  $F$  such that  $F$  has no disjunctive splitting on any variable.

## 3. Maximal MU-Formulas

In this section we will show the  $D^P$ -completeness of so called maximal minimal unsatisfiable formulas. For a formula  $F \in MU$  and a clause  $f \in F$  we say  $f$  is *maximal* in  $F$  if for any literal  $L$  occurring neither positively nor negatively in  $f$  the formula obtained from  $F$  by adding  $L$  to  $f$  is satisfiable. We say  $F \in MU$  is maximal minimal unsatisfiable,  $F \in MAX-MU$ , if every clause in  $F$  is maximal.

That  $MAX-MU$  is in  $D^P$  is not hard to see. For the  $D^P$ -hardness we establish a reduction from the  $D^P$ -complete problem  $MU$ . At first we introduce an auxiliary function by associating to a formula  $F$ , a clause  $f$ , and a new variable  $z$  a formula  $\xi(F, f, z)$  preserving the minimal unsatisfiability. Later on the formula  $\xi(F, f, z)$  will be used in order to associate in polynomial time to each formula in  $MU$  a maximal formula.

**Definition 1.** For a clause  $f = L_1 \vee \dots \vee L_k$ , we use  $\rho(f)$  to denote the formula consisting of the following clauses:

$$\begin{aligned} &\neg L_1 \vee L_2 \vee L_3 \vee \dots \vee L_k, \\ &\quad \neg L_2 \vee L_3 \vee \dots \vee L_k, \\ &\quad \quad \neg L_3 \vee \dots \vee L_k, \\ &\quad \quad \quad \dots \\ &\quad \quad \quad \quad \neg L_k. \end{aligned}$$

For a formula  $F = \{f\} + H$  let  $z$  be a new variable. Then we define

$$\xi(F, f, z) = z \vee_{cl} H + \{f\} + \neg z \vee_{cl} \rho(f),$$

where  $L \vee_{cl} \{g_1, \dots, g_m\}$  denotes the formula  $\{L \vee g_1, \dots, L \vee g_m\}$

The formula  $\rho(f) + \{f\}$  is a maximal minimal unsatisfiable formula. That means we have  $\rho(f) + \{f\} \in MAX-MU$ .

**Lemma 1.** For a *CNF* formula  $F$ , a clause  $f \in F$ , and a new variable  $z$ ,  $F \in MU$  if and only if  $\xi(F, f, z) \in MU$ , and  $f$  is maximal in  $\xi(F, f, z)$ .

*Proof.* For short we write  $\xi(F)$  instead of  $\xi(F, f, z)$ .

( $\Rightarrow$ ) Suppose  $F \in MU$ . Then obviously  $\xi(F)$  is unsatisfiable. Next we show for any  $h \in \xi(F)$  that  $\xi(F) - \{h\}$  is satisfiable. We proceed by a case distinction.

*Case 1.*  $h = f$ :  $F - \{f\}$  is satisfiable for some truth assignment  $t$ : We extend  $t$  to  $t'$  by defining  $t'(z) = 0$ . Clearly,  $t'$  is a truth assignment of  $\xi(F) - \{f\}$ .

*Case 2.*  $h = z \vee g$  for some  $g \in H$ : Let  $t$  be a satisfying truth assignment for  $F - \{g\}$ . We extend  $t$  to  $t'$  by defining  $t'(z) = 0$ . Then  $t'$  is a truth assignment of  $\xi(F) - \{h\}$ .

*Case 3.*  $h = \neg z \vee g$  for some  $g \in \rho(f)$ : Since  $\{f\} + (\rho(f) - \{g\})$  is satisfiable, we extend some satisfying truth assignment for  $\{f\} + (\rho(f) - \{g\})$  to  $t'$  by defining  $t'(z) = 1$ . Then  $t'$  is a (partial) satisfying truth assignment for  $\xi(F) - \{h\}$ .

Altogether, we have shown  $\xi(F) \in MU$ .

It remains to show that  $f$  is maximal in  $\xi(F)$ . Let  $L$  be a literal occurring neither positively nor negatively in  $f$ .

*Case 1.*  $L \notin \{z, \neg z\}$ :  $\{L \vee f\} + \rho(f)$  is satisfiable, since  $f + \rho(f) \in MAX-MU$ . Therefore,  $(\xi(F) - \{f\}) + \{L \vee f\}$  is satisfiable (set  $z = 1$ ).

*Case 2.*  $L = \neg z$ :  $(\xi(F) - \{f\}) + \{L \vee f\}$  is satisfiable, since  $H$  is satisfiable.

*Case 3.*  $L = z$ : Similar to the case  $L = \neg z$ .

Altogether we have shown  $f$  is maximal in  $\xi(F)$ .

( $\Leftarrow$ ) Suppose  $\xi(F) \in MU$  and  $f$  is maximal in  $\xi(F)$ . Then  $F = H + \{f\}$  is unsatisfiable. Suppose  $F \notin MU$ .  $H$  must be minimal unsatisfiable, since  $\xi(F) \in MU$ . Then we obtain  $(\xi(F) - \{f\}) \models z$ . Hence,  $(\xi(F) - \{f\}) + \{\neg z \vee f\}$  is minimal unsatisfiable, in contradiction to the maximality of  $f$ . Thus  $F$  is minimal unsatisfiable.  $\square$

**Lemma 2.** *Suppose  $F \in MU$ .*

1) *Any clause of  $\neg z \vee_{cl} \rho(f)$  is maximal in  $\xi(F)$ .*

2)  *$g \in H$  is maximal in  $F$  if and only if  $z \vee g$  is maximal in  $\xi(F)$ .*

*Proof.* *Ad 1:* For a clause  $h = \neg z \vee g$  with  $g \in \rho(f)$  let  $L$  be a literal with  $L, \neg L \notin h$ .  $\{f\} + (\rho(f) - \{g\}) + \{L \vee g\}$  is satisfiable, since  $\{f\} + \rho(f)$  is in  $MAX-MU$ . Thus for  $z = 1$  the formula  $(\xi(F) - \{h\}) + \{L \vee h\}$  is satisfiable. That means  $\neg z \vee \rho(f)$  is maximal in  $\xi(F)$ .

*Ad 2:* ( $\Rightarrow$ ) Suppose  $g$  is maximal in  $F$ , and  $L$  is a literal with  $L, \neg L \notin (z \vee g)$ .  $(F - \{g\}) + \{L \vee g\}$  is satisfiable, because  $g$  is maximal in  $F$ . Then for  $z = 0$  the formula  $(\xi(F) - \{z \vee g\}) + \{L \vee z \vee g\}$  is satisfiable. Therefore,  $z \vee g$  is maximal in  $\xi(F)$ .

( $\Leftarrow$ ) Suppose  $z \vee g$  is maximal in  $\xi(F)$ , but  $g$  is not maximal in  $F$ . Then there is a literal  $L$ , such that  $(F - \{g\}) + \{L \vee g\}$ , denoted as  $F'$ , is in  $MU$ . It is easy to see that  $\xi(F')$  equals the formula  $(\xi(F) - \{z \vee g\}) + \{L \vee z \vee g\}$ . By the previous lemma we obtain  $\xi(F') \in MU$  in contradiction to the maximum of  $z \vee g$ .  $\square$

**Theorem 1.**  *$MAX-MU$  is  $D^P$ -complete.*

*Proof.* As mentioned above the class  $MAX-MU$  lies in  $D^P$ . We establish a polynomial-time transformation  $\delta(F)$  for which we show  $F \in MU$  if and only if  $\delta(F) \in MAX-MU$ . Then the  $D^P$ -hardness of  $MAX-MU$  follows from the  $D^P$ -completeness of  $MU$  [10].

**Procedure MU-MAX**

**Input:** A formula  $F$  in  $CNF$

**Output:** A formula  $\delta(F)$  in  $CNF$

```

begin
 $\mathcal{C} :=$ the set of clauses in  $F$ 
while  $\mathcal{C}$  is non-empty
    for a clause  $f$  in  $\mathcal{C}$ ; for a new variable  $z$ 
         $F := \xi(F, f, z)$ 
         $\mathcal{C} := z \vee_{cl} (\mathcal{C} - \{f\})$ 
end while
 $\delta(F) := F$ 
end
    
```

We define  $\mathcal{C}_0 = \mathcal{C}$ , and for  $i \geq 1$ , let  $\mathcal{C}_i$  be the set of clauses we obtain after the  $i$ -th run of the **while**-loop. Please note that  $|\mathcal{C}_0| = m$  and that  $|\mathcal{C}_{i+1}| = |\mathcal{C}_i| - 1$ , where  $m$  is the number of clauses in  $F$ . There are at most  $m$  runs of the **while**-loop. It is easy to see that the running time within the **while**-loop is bound by  $O((mn)^2)$ , where  $n$  is the number of variables of  $F$ . Therefore, the procedure requires not more than  $O((mn)^3)$  steps.

Now it remains it to prove  $F \in MU$  if and only if  $\delta(F) \in MAX-MU$ .

We define  $F_0 = F$ , and for  $i \geq 1$ , let  $F_i$  be the formula we obtain after the  $i$ -th run of the **while**-loop of the procedure MU-MAX.

( $\Rightarrow$ ) Suppose  $F \in MU$ . Then by Lemma 1, every  $F_i$  is minimal unsatisfiable. By Lemma 1 and 2, any clause in  $F_i - \mathcal{C}_i$  is maximal in  $F_i$ . Suppose  $\delta(F) = F_k$ . Then  $\mathcal{C}_k$  is empty. That implies,  $\delta(F)$  is maximal minimal unsatisfiable.

( $\Leftarrow$ ) Suppose  $\delta(F) = F_k$  is maximal. Then by Lemma 1 and 2,  $F_{k-1} \in MU$  and any clause in  $F_{k-1} - \mathcal{C}_{k-1}$  is maximal in  $F_{k-1}$ . By an iterative application of Lemma 1 and 2 finally we obtain  $F \in MU$ .  $\square$

#### 4. Marginal MU-Formulas

A  $MU$ -formula  $F$  is called *marginal* if, and only if removing an arbitrary occurrence of a literal from  $F$  leads to an unsatisfiable formula which is not in  $MU$ . The class of marginal formulas is denoted as  $MARG-MU$ .

**Theorem 2.** *MARG-MU is  $D^P$ -complete.*

*Proof.* Obviously, the class  $MARG-MU$  is in  $D^P$ . We will show the  $D^P$ -hardness by a reduction from the  $D^P$ -complete problem  $MU$  [10]. We establish a procedure running in polynomial time generating a formula  $\sigma(F)$  from a formula  $F$ , such that  $F \in MU$  if and only if  $\sigma(F) \in MARG-MU$ . The procedure is based on an iterative application of the following function  $\zeta$ .

Let  $F = \{L \vee f, L \vee g\} + H$  be a formula with at least two occurrences of the literal  $L$ . For new variables  $y$  and  $z$  we define

$$\zeta(F, L \vee f, L \vee g, y, z) = \{y \vee f, z \vee g, \neg y \vee z, y \vee \neg z, \neg y \vee \neg z \vee L\} + H.$$

The formula describes the equivalence of  $y$  and  $z$ , the two occurrences of  $L$  are replaced by one occurrence, and  $\zeta(F, L \vee f, L \vee g, y, z) \models F$ . For short we write  $\zeta(F)$ .

**Claim 1.**  $F \in MU$  if and only if  $\zeta(F) \in MU$

*Proof.* ( $\Rightarrow$ ) Let  $F$  be a formula in  $MU$ . At first we show  $\zeta(F)$  is unsatisfiable. Suppose, by contrary, that  $\zeta(F)$  is satisfiable for some satisfying truth assignment  $t$ . Then we have either  $t(y) = 1$  or  $t(z) = 1$ , since  $\{f, g\} + H$  is unsatisfiable. Because of the clauses  $(\neg y \vee z)$  and  $(y \vee \neg z)$ , we have  $t(L) = 1$ . Thus,  $t$  is a satisfying truth assignment for  $F$  in contradiction to  $F \in MU$ . Hence,  $\zeta(F)$  is unsatisfiable. Next we show for any clause  $h \in \zeta(F)$  the satisfiability of  $\zeta(F) - \{h\}$  by a case distinction.

*Case 1.*  $h \in H$ : Let  $t$  be a satisfying truth assignment for  $F - \{h\}$ . If  $t(L) = 1$  then we extend  $t$  to  $t'$  by defining  $t'(y) = t'(z) = 1$ . Clearly,  $t'$  is a satisfying truth assignment for  $\zeta(F) - \{h\}$ . If  $t(L) = 0$  then we extend  $t$  to  $t'$  by defining  $t'(y) = t'(z) = 0$ . Then  $t'$  is a satisfying truth assignment for  $\zeta(F) - \{h\}$ .

*Case 2.*  $h = (\neg y \vee z)$ : Since  $F \in MU$ , the formula  $H + \{L \vee g\}$  is satisfiable. Because  $L \vee f$  is in  $F$ ,  $H + \{g\}$  is satisfiable. Let  $t$  be a satisfying truth assignment for  $H + \{g\}$ . We extend  $t$  to  $t'$  by adding  $t'(y) = 1$  and  $t'(z) = 0$ . Obviously,  $t'$  is a satisfying truth assignment for  $\zeta(F) - \{h\}$ .

*Case 3.*  $h = (y \vee \neg z)$  or  $h = (\neg y \vee \neg z \vee L)$ : Similar to case 2.

*Case 4.*  $h = (y \vee f)$ :  $H + \{g\}$  is satisfiable, because  $H + \{L \vee g\}$  is satisfiable and the clause  $L \vee f$  is in  $F$ . Let  $t$  be a satisfying truth assignment for  $H + \{g\}$ . We extend  $t$  to  $t'$  by adding  $t'(y) = t'(z) = 0$ . Obviously,  $t'$  satisfies  $\zeta(F) - \{h\}$ .

*Case 5.*  $h = (z \vee g)$ : Similar to case 4.

Altogether, we have shown  $\zeta(F) \in MU$ .

( $\Leftarrow$ ) For  $\zeta(F) \in MU$  at first we show the unsatisfiability of  $F$ . Suppose  $F$  is satisfiable for some satisfying truth assignment  $t$ . If  $t(L) = 1$  resp.  $t(L) = 0$  then we extend  $t$  to  $t'$  by defining  $t'(y) = t'(z) = 1$  resp.  $t'(y) = t'(z) = 0$ . Then  $t'$  is a satisfying truth assignment for  $\zeta(F)$  in contradiction to  $\zeta(F) \in MU$ . Thus  $F$  is unsatisfiable. Next we show for any clause  $h \in F$  the satisfiability of  $F - \{h\}$ .

*Case 1.*  $h \in H$ : If  $\{f, g\} + (H - \{h\})$  is satisfiable then  $F - \{h\}$  is satisfiable. Therefore we assume  $\{f, g\} + (H - \{h\})$  is unsatisfiable. Since  $\zeta(F) \in MU$ ,  $\zeta(F) - \{h\}$  is satisfiable for some satisfying truth assignment  $t$ . Then we obtain  $t(y) = t(z) = 1$ , and therefore  $t(L) = 1$ . Hence,  $t$  is a satisfying truth assignment for  $F - \{h\}$ .

*Case 2.*  $h = (L \vee f)$ : Let  $t$  be a satisfying truth assignment for  $\zeta(F) - \{y \vee f\}$ . Since  $\zeta(F) \in MU$ , we obtain  $t(y) = 0$ ,  $t(z) = 0$ , and therefore  $t(L) = 0$ . Hence,  $\{g\} + H$  is satisfiable. Thus  $F - \{h\}$  is satisfiable.

*Case 3.*  $h = (L \vee g)$ : Similar to case 2.

Altogether, we have shown  $F \in MU$ . This completes the proof of Claim 1.

For a formula  $F \in MU$  and a literal  $L$  we say  $F$  is *marginal w.r.t. the literal  $L$*  if removing any occurrence of  $L$  from  $F$  results in an unsatisfiable formula which is not in  $MU$ .

**Claim 2.** For  $F \in MU$ ,  $\zeta(F)$  is marginal w.r.t. the new literals  $y, \neg y, z, \neg z$ .

*Proof.* For sake of symmetry it suffices to show the marginality of  $F$  w.r.t  $y$  and  $\neg y$ . At first we prove  $\zeta(F)$  is marginal w.r.t.  $y$ . Clearly, removing  $y$  from the clause  $y \vee \neg z$  violates the minimal unsatisfiability, because  $\neg z$  would be a subclause of  $\neg y \vee \neg z \vee L$ . Removing  $y$  from  $y \vee f$  also violates the minimal unsatisfiability. That can be seen by showing the

unsatisfiability of the formula

$$G := \{f, z \vee g, y \vee \neg z, \neg y \vee \neg z \vee L\} + H.$$

Suppose  $G$  is satisfiable for some satisfying truth assignment  $t$ . Since  $F$  is minimal unsatisfiable, we obtain the unsatisfiability of  $\{f, g\} + H$ , and therefore  $t(z) = 1$ . Then we obtain  $t(y) = 1$ , because of  $y \vee \neg z \in G$ , and therefore  $t(L) = 1$ . That would be a satisfying truth assignment for  $F$  in contradiction to  $F \in MU$ . Thus  $G$  is unsatisfiable. Therefore, removing  $y$  from  $y \vee f$  would result in a non-minimal unsatisfiable formula. That is to say,  $\zeta(F)$  is marginal w.r.t.  $y$ .

Next we show  $F$  is marginal w.r.t.  $\neg y$ . It is easy to see that removing  $\neg y$  from the clause  $\neg y \vee z$  violates the minimal unsatisfiability. Now we prove that  $\neg y$  cannot be removed from  $\neg y \vee \neg z \vee L$  preserving the minimal unsatisfiability. We only need to prove that the following formula

$$K := \{y \vee f, z \vee g, \neg y \vee z, \neg z \vee L\} + H$$

is unsatisfiable. Suppose, by contrary, that  $t$  is a satisfying truth assignments for  $K$ . For  $t(L) = 0$  we obtain  $t(y) = t(z) = 0$ . That means  $\{f, g\} + H$  is satisfiable in contradiction to  $F \in MU$ . For  $t(L) = 1$  the formula  $H$  is unsatisfiable, because  $F\{L \vee f, L \vee g\} + H$  is unsatisfiable. Thus  $K$  is unsatisfiable. Consequently,  $\zeta(F)$  is marginal w.r.t.  $\neg y$ , and we finish the proof of Claim 2.

**Claim 3.** For  $F \in MU$ , if  $F$  is marginal w.r.t. a literal  $B$  different from  $L$ , then  $\zeta(F)$  is marginal w.r.t the literal  $B$ .

*Proof.* Suppose, by contrary, that there is  $h \in \zeta(F)$  such that  $(\zeta(F) - \{h\}) + \{h_B\}$  is also in  $MU$ , where  $h_B$  is obtained from  $h$  by removing  $B$ . We proceed by a case distinction:

*Case 1.*  $h \in H$ : Let  $F' = (F - \{h\}) + \{h_B\}$ . It is easy to see that  $(\zeta(F) - \{h\}) + \{h_B\}$  equals  $\zeta(F')$ . By Proposition 1, we know  $F'$  is in  $MU$  in contradiction to the marginality of  $F$  w.r.t.  $B$ .

*Case 2.*  $h = y \vee f$ : We define  $F' = \{L \vee f_B, L \vee g\} + H$ , where  $f_B$  is obtained from  $f$  by removing  $B$ . It is easy to see that  $(\zeta(F) - \{h\}) + \{h_B\}$  equals the formula  $\zeta(F')$ . By Proposition 1, we obtain  $F' \in MU$  in contradiction to the marginality w.r.t.  $B$ .

*Case 3.*  $H = z \vee g$ : Similar to case 2.

Now we introduce the above mentioned procedure.

**Procedure** MU-MARG

**Input:** A formula  $F$  in *CNF*

**Output:** A formula  $\sigma(F)$  in *CNF*.

**begin**

$\mathcal{L}$  := the set of literals occurring at least twice in  $F$

**while**  $\mathcal{L}$  is non-empty

for some  $L \in \mathcal{L}$

for two clauses  $L \vee f, L \vee g \in F$ ; for new variables  $y, z$

$F := \zeta(F, L \vee f, L \vee g, y, z)$

remove from  $\mathcal{L}$  literals occurring in  $F$  exactly once

**end while**

$\sigma(F) := F$   
**end**

The running time of the procedure MU–MARG is bound by a polynomial depending on the length of  $F$ , because within the **while**–loop a double occurrence of a literal  $L$  is replaced by one occurrence. Please note, that any literal of the input formula occurs exactly once in  $\sigma(F)$ . Now it suffices to prove  $F \in MU \Leftrightarrow \sigma(F) \in MARG\text{--}MU$ .

By an iterative application of Claim 1, we see that  $F \in MU$  if and only if  $\sigma(F) \in MU$ . Now it remains to show that for a formula  $F \in MU$  the formula  $\sigma(F)$  is marginal, that means marginal w.r.t any literal. Since a literal  $L \in lit(F)$  occurs in  $\sigma(F)$  exactly once,  $\sigma(F)$  is marginal w.r.t  $L$ . By means of Claim 2 and Claim 3, we see that  $\sigma(F)$  is marginal w.r.t. the introduced literals. Thus,  $\sigma(F)$  is marginal w.r.t. any literal and therefore marginal.  $\square$

## 5. Unique–MU and Almost–Unique–MU

In this section we investigate  $MU$ –formulas  $F$  for which  $F - \{f\}$  is in  $Unique\text{--}SAT$  for all clauses resp. for all but one clause  $f \in F$ . The classes are defined as

$Unique\text{--}MU = \{F \in MU \mid \forall f \in F : F - \{f\} \text{ has exactly one satisfying truth assignment}\}$   
 $Almost\text{--}Unique\text{--}MU = \{F \in MU \mid \text{there is at most one clause } f \in F, \text{ such that } F - \{f\} \text{ has more than one satisfying truth assignment}\}$

In the first part of this section we prove  $Unique\text{--}SAT \approx_p Unique\text{--}MU$ , whereas in the second part the  $D^P$ –completeness of  $Almost\text{--}Unique\text{--}MU$  is shown.

**Theorem 3.**  $Unique\text{--}MU \leq_p Unique\text{--}SAT$

*Proof.* We introduce a pol–time computable function  $\theta$  for which we show  $F \in Unique\text{--}MU$  if and only if  $\theta(F)$  is in  $Unique\text{--}SAT$ . In order to simplify the construction we demand that any literal occurs negatively and positively in the formula. If this is not the case then obviously the formula  $F$  is not in  $MU$  and therefore not in  $Unique\text{--}MU$ . For  $F = \{f_1, \dots, f_m\}$  we define

$$\theta(F) := ((F - \{f_1\}) + \overline{\{f_1\}}) \wedge \bigwedge_{1 \leq i \leq m} (F - \{f_i\})^{i+1}.$$

$(F - \{f_i\})^{i+1}$  is the formula we obtain by renaming the variables of the formulas  $(F - \{f_i\})$ , such that the formulas  $(F - \{f_j\})^{j+1}$  ( $1 \leq j \leq m$ ) and  $((F - \{f_1\}) + \overline{\{f_1\}})$  have pairwise different variables.  $\overline{f_1}$  is the conjunction of the negated literals of  $f_1$ .

If  $F$  is in  $Unique\text{--}MU$  then obviously  $\theta(F)$  belongs to  $Unique\text{--}SAT$ . For the other direction the only non–trivial part is the unsatisfiability of  $F$ . Since  $\theta(F)$  belongs to  $Unique\text{--}SAT$ , the formulas  $(F - \{f_1\})^2$  and  $((F - \{f_1\}) + \overline{\{f_1\}})$  have unique satisfying truth assignments. That means  $(F - \{f_1\})$  and  $((F - \{f_1\}) + \overline{\{f_1\}})$  have the same unique satisfying truth assignment. Hence, we have  $(F - \{f_1\}) \models \overline{\{f_1\}}$ . That means  $(F - \{f_1\}) + \{f_1\} = F$  is unsatisfiable.  $\square$

That w.r.t. polynomial reducibility  $Unique\text{--}SAT$  is not harder than  $Unique\text{--}MU$  will be shown by establishing an appropriate reduction. At first we introduce the transformation  $\omega(F)$ , which will be used later on as a basis for our desired transformation. Let  $F =$



$\{f_1, f_2, \dots, f_m\}$  be a 3-CNF formula over variables  $\{x_1, x_2, \dots, x_n\}$  with clauses  $f_i = L_{i1} \vee L_{i2} \vee L_{i3}$ . We introduce new variables  $\{y_1, y_2, \dots, y_m\}$ .  $\pi_i$  ( $1 \leq i \leq m$ ) denotes the clause

$$y_1 \vee \dots \vee y_{i-1} \vee y_{i+1} \vee \dots \vee y_m.$$

$\omega(F)$  is the conjunction of the following groups of clauses:

- (A) The clauses  $f_1 \vee \pi_1, f_2 \vee \pi_2, \dots, f_m \vee \pi_m$   
 (B) The clauses

$$\begin{aligned} &\neg L_{11} \vee \pi_1 \vee \neg y_1, \quad \neg L_{21} \vee \pi_2 \vee \neg y_2, \quad \dots, \quad \neg L_{m1} \vee \pi_m \vee \neg y_m \\ &\neg L_{12} \vee \pi_1 \vee \neg y_1, \quad \neg L_{22} \vee \pi_2 \vee \neg y_2, \quad \dots, \quad \neg L_{m2} \vee \pi_m \vee \neg y_m \\ &\neg L_{13} \vee \pi_1 \vee \neg y_1, \quad \neg L_{23} \vee \pi_2 \vee \neg y_2, \quad \dots, \quad \neg L_{m3} \vee \pi_m \vee \neg y_m \end{aligned}$$

- (C) The clauses  $\neg y_i \vee \neg y_j$  ( $1 \leq i < j \leq m$ )  
 (D) The clause  $y_1 \vee y_2 \vee \dots \vee y_m$

Next we show some Lemmas for the function  $\omega(F)$ .

**Lemma 3.**  $\omega(F)$  is unsatisfiable.

*Proof.* Suppose, by contrary,  $\omega(F)$  is satisfiable for a satisfying truth assignment  $t$ . From the clauses in (C) and (D), we know that there is exactly one  $y_i$  such that  $t(y_i) = 1$ . W.o.l.g. we assume  $t(y_1) = 1$ . Then  $t(y_2) = \dots = t(y_m) = 0$  and from the clauses  $\neg L_{11} \vee \pi_1 \vee \neg y_1, \neg L_{12} \vee \pi_1 \vee \neg y_1, \neg L_{13} \vee \pi_1 \vee \neg y_1$ , we get  $t(L_{11}) = t(L_{12}) = t(L_{13}) = 0$ . But then we obtain  $t(f_1 \vee \pi_1) = 0$ , a contradiction. Thus,  $F$  is unsatisfiable.  $\square$

**Lemma 4.**  $\forall h \in \omega(F) : h \neq (y_1 \vee y_2 \vee \dots \vee y_m) \Rightarrow \omega(F) - \{h\}$  is satisfiable.

*Proof.* The satisfying truth assignments are given below:  $s_i$  is a satisfying (partial) truth assignment for  $\omega(F) - \{f_i \vee \pi_i\}$ , where  $s_i(y_i) = 1, s_i(y_j) = 0$  ( $1 \leq j \leq m, j \neq i$ ),  $s_i(L_{i1}) = s_i(L_{i2}) = s_i(L_{i3}) = 0$ .

$t_{ik}$  is a satisfying (partial) truth assignment for  $\omega(F) - \{\neg L_{ik} \vee \pi_i \vee \neg y_i\}$ , where  $t_{ik}(y_i) = 1, t_{ik}(y_j) = 0$  ( $1 \leq j \leq m, j \neq i$ ),  $t_{ik}(L_{ik}) = 1, t_{ik}(L_{i(k \oplus 1)}) = t_{ik}(L_{i(k \oplus 2)}) = 0$ . The symbol  $\oplus$  denotes the addition module 3.

$v_{ij}$  is a satisfying (partial) truth assignment for  $\omega(F) - \{\neg y_i \vee \neg y_j\}$ , where  $v_{ij}(y_i) = v_{ij}(y_j) = 1, v_{ij}(y_k) = 0$  ( $1 \leq k \leq m, k \neq i, k \neq j$ ).  $\square$

**Lemma 5.**  $F \in SAT$  if and only if  $\omega(F) \in MU$ .

*Proof.* Because of the Lemma 3 and 4 it suffices to show  $F$  is satisfiable if and only if  $\omega(F) - \{y_1 \vee y_2 \vee \dots \vee y_m\}$  is satisfiable.

Suppose  $F$  is satisfiable with a truth assignment  $t$ . Now we extend  $t$  to  $t'$  by defining  $t'(y_i) = 0$  for all  $1 \leq i \leq m$ . Clearly,  $t'$  is a satisfying truth assignment for  $\omega(F) - \{y_1 \vee y_2 \vee \dots \vee y_m\}$ . Suppose  $\omega(F)$  is minimal unsatisfiable. Then  $\omega(F) - \{y_1 \vee y_2 \vee \dots \vee y_m\}$  is satisfiable with some truth assignments  $t$ . Since  $\omega(F)$  is unsatisfiable, we obtain  $t(y_1) = t(y_2) = \dots = t(y_m) = 0$  and therefore  $t(f_1) = t(f_2) = \dots = t(f_m) = 1$ . That implies the satisfiability of  $F$ .  $\square$

After these preparations we introduce the function  $\Omega$  for which  $F \in \text{Unique-SAT}$  if and only  $\Omega(F) \in \text{Unique-MU}$  will be shown.

$\chi_i$  is the disjunction of all literals  $\neg x$ , where  $x \in \text{var}(F) - \text{var}(f_i)$ , and  $\chi$  denotes the disjunction of all literals  $\neg x$ , where  $x \in \text{var}(F)$ .  $\Omega(F)$  is the formula consisting of the following groups of clauses:

(A') For each clause  $(f_i \vee \pi_i) \in \omega(F)$ :

$$f_i \vee \pi_i \vee \chi_i, \quad f_i \vee \pi_i \vee x \quad \text{for all } x \in \text{var}(F) - \text{var}(f_i)$$

(B') For each clause  $(\neg L_{ik} \vee \pi_i \vee \neg y_i) \in \omega(F)$ :

$$\neg L_{ik} \vee \pi_i \vee \neg y_i \vee \chi_i, \quad \neg L_{ik} \vee \pi_i \vee \neg y_i \vee x \quad \text{for all } x \in \text{var}(F) - \text{var}(f_i)$$

(C') For each clause  $(\neg y_i \vee \neg y_j) \in \omega(F)$ :

$$\neg y_i \vee \neg y_j \vee \chi, \quad \neg y_i \vee \neg y_j \vee x \quad \text{for all } x \in \text{var}(F)$$

(D') The clause  $y_1 \vee y_2 \vee \dots \vee y_m$ .

Next we show some Lemmas for the function  $\Omega$ .

**Lemma 6.**  $\Omega(F)$  is unsatisfiable.

*Proof.* Suppose  $\Omega(F)$  is satisfiable with a satisfying truth assignment  $t$ . Since  $\omega(F)$  is unsatisfiable, see Lemma 1, some clauses in  $\omega(F)$  are false for  $t$ . Since the clause  $y_1 \vee y_2 \vee \dots \vee y_m$  belongs to  $\Omega(F)$  and to  $\omega(F)$ , this clause is true for  $t$ . If  $t(f_i \vee \pi_i) = 0$ , then we obtain the contradiction  $t(\chi_i) = 1$  and  $t(x) = 1$  for all  $x \in \text{var}(F) - \text{var}(f_i)$ . By the same argument we obtain a contradiction for the clauses  $\neg L_{ij} \vee \pi_i \vee \neg y_i$  and  $\neg y_i \vee \neg y_j$ . Hence,  $\Omega(F)$  is unsatisfiable.  $\square$

The next lemma states that almost all  $(\Omega(F) - \{h\})$  are uniquely satisfiable independent on the satisfiability of  $F$ .

Let  $s, t$  be two (partial) truth assignments, we say  $s$  is a proper segment of  $t$  if  $s(x)$  is defined implies that  $t(x)$  is also defined and  $s(x) = t(x)$  for each variable  $x$ . Clearly, whenever  $s$  is a satisfying (partial) truth assignment of a formula and  $s$  is a proper segment of  $t$  then  $t$  is a satisfying truth assignment of the formula, too.

**Lemma 7.**  $\forall h \in \Omega(F) : h \neq (y_1 \vee \dots \vee y_m) \Rightarrow \Omega(F) - \{h\} \in \text{Unique-SAT}$ .

*Proof.* We claim that  $s'_i$  is the unique satisfying truth assignment of  $\Omega(F) - \{f_i \vee \pi_i \vee \chi_i\}$ , where

$$\begin{aligned} s'_i(y_i) &= 1, \\ s'_i(y_j) &= 0, \quad 1 \leq j \leq m, j \neq i, \\ s'_i(L_{i1}) &= s'_i(L_{i2}) = s'_i(L_{i3}) = 0, \\ s'_i(x) &= 1, \quad \text{for all } x \in \text{var}(F) - \text{var}(f_i) \end{aligned}$$

We first show that  $s'_i$  is a satisfying truth assignment of  $\Omega(F) - \{f_i \vee \pi_i \vee \chi_i\}$ . Clearly,  $s'_i$  makes all the clauses  $f_i \vee \pi_i \vee x$  true,  $x \in \text{var}(F) - \text{var}(f_i)$ . Please note that  $s_i$  (defined in

the proof of Lemma 4) is a proper segment of  $s'_i$ . Thus  $s'_i$  is a satisfying truth assignment of  $\omega(F) - \{f_i \vee \pi_i\}$ . Since except for clauses  $f_i \vee \pi_i \vee x$ ,  $x \in \text{var}(F) - \text{var}(f_i)$  every clause in  $\Omega(F) - \{f_i \vee \pi_i \vee \chi_i\}$  is a super clause of some clause in  $\omega(F) - \{f_i \vee \pi_i\}$ , it follows that  $s'_i$  is a satisfying truth assignment of  $\Omega(F) - \{f_i \vee \pi_i \vee \chi_i\}$ .

Next we shall show the uniqueness of  $s'_i$ . W.o.l.g., we assume  $i = 1$ . Now let  $t$  be any satisfying truth assignment of  $\Omega(F) - \{f_1 \vee \pi_1 \vee \chi_1\}$ . Since  $\Omega(F)$  is unsatisfiable, it follows that

$$\begin{aligned} t(y_2) &= \dots = t(y_m) = 0, \\ t(L_{11}) &= t(L_{12}) = t(L_{13}) = 0, \\ t(x) &= 1, \text{ for all } x \in \text{var}(F) - \text{var}(f_1). \end{aligned}$$

Now from the clause  $y_1 \vee y_2 \vee \dots \vee y_m$  we get  $t(y_1) = 1$ . Hence we obtain  $t = s'_1$ . Consequently,  $s'_1$  is the unique satisfying truth assignment of  $\Omega(F) - \{f_1 \vee \pi_1 \vee \chi_1\}$ .

The proofs of the following statements below are analogical.

$s''_i$  is the unique satisfying truth assignment of  $\Omega(F) - \{f_i \vee \pi_i \vee z\}$ , where

$$\begin{aligned} s''_i(y_i) &= 1, \\ s''_i(y_j) &= 0, \quad 1 \leq j \leq m, j \neq i, \\ s''_i(L_{i1}) &= s''_i(L_{i2}) = s''_i(L_{i3}) = 0, \\ s''_i(z) &= 0, \\ s''_i(x) &= 1, \text{ for all } x \in \text{var}(F) - \text{var}(f_i), \quad x \neq z \end{aligned}$$

$t'_i$  is the unique satisfying truth assignment of  $\Omega(F) - \{\neg L_{ik} \vee \pi_i \vee \neg y_i \vee \chi_i\}$ , where

$$\begin{aligned} t'_{ik}(y_i) &= 1, \\ t'_{ik}(y_j) &= 0, \quad 1 \leq j \leq m, j \neq i, \\ t'_{ik}(L_{ik}) &= 1, \\ t'_{ik}(L_{i(k \oplus 1)}) &= t'_{ik}(L_{i(k \oplus 2)}) = 0, \\ t'_{ik}(x) &= 1, \text{ for all } x \in \text{var}(F) - \text{var}(f_i). \end{aligned}$$

where  $\oplus$  is the addition module 3.

$t''_i$  is the unique satisfying truth assignment of  $\Omega(F) - \{\neg L_{ik} \vee \pi_i \vee \neg y_i \vee z\}$ , where

$$\begin{aligned} t''_{ik}(y_i) &= 1, \\ t''_{ik}(y_j) &= 0, \quad 1 \leq j \leq m, j \neq i, \\ t''_{ik}(L_{ik}) &= 1, \\ t''_{ik}(L_{i(k \oplus 1)}) &= t''_{ik}(L_{i(k \oplus 2)}) = 0, \\ t''_{ik}(z) &= 0, \\ t''_{ik}(x) &= 1, \text{ for all } x \in \text{var}(F) - \text{var}(f_i), \quad x \neq z. \end{aligned}$$

where  $\oplus$  is the addition module 3.

$v'_{ij}$  is the unique satisfying truth assignment of  $\Omega(F) - \{\neg y_i \vee \neg y_j \vee \chi\}$ , where

$$\begin{aligned} v'_{ij}(y_i) &= v'_{ij}(y_j) = 1, \\ v'_{ij}(y_k) &= 0, \quad 1 \leq k \leq m, k \neq i, k \neq j, \\ v'_{ij}(x) &= 1, \text{ for all } x \in \text{var}(F). \end{aligned}$$

$v''_{ij}$  is the unique satisfying truth assignment of  $\Omega(F) - \{\neg y_i \vee \neg y_j \vee z\}$ , where

$$\begin{aligned} v''_{ij}(y_i) &= v''_{ij}(y_j) = 1, \\ v''_{ij}(y_k) &= 0, \quad 1 \leq k \leq m, k \neq i, k \neq j, \\ v''_{ij}(z) &= 0, \\ v''_{ij}(x) &= 1, \quad \text{for all } x \in \text{var}(F), x \neq z. \end{aligned}$$

□

**Lemma 8.**  $F \in SAT \Leftrightarrow \omega(F) \in MU \Leftrightarrow \Omega(F) \in MU$  .

*Proof.* Because of Lemma 5 we have only to show the second equivalence.

( $\Rightarrow$ ) Suppose  $\omega(F) \in MU$ . Note that every clause in  $\Omega(F) - \{y_1 \vee y_2 \vee \dots \vee y_m\}$  is a super clause of some clause in  $\omega(F) - \{y_1 \vee y_2 \vee \dots \vee y_m\}$  which is satisfiable. Thus,  $\Omega(F) - \{y_1 \vee y_2 \vee \dots \vee y_m\}$  is satisfiable. This fact and Lemma 6–7 imply the minimal unsatisfiability of  $\Omega(F)$ .

( $\Leftarrow$ ) Suppose  $\Omega(F) \in MU$ . Please note that each clause in  $\Omega(F)$  is a super clause of some clause in  $\omega(F)$ . Thus, if  $\omega(F)$  is satisfiable then  $\Omega(F)$  is satisfiable, too. Hence  $\omega(F)$  is unsatisfiable. It is not difficult to see that any satisfying truth assignment of  $\Omega(F) - \{y_1 \vee y_2 \vee \dots \vee y_m\}$  is also a satisfying truth assignment of  $\omega(F) - \{y_1 \vee y_2 \vee \dots \vee y_m\}$ . This fact and Lemma 3–4 imply the minimal unsatisfiability of  $\omega(F)$ . □

A simple consequence of Lemma 7 and 8 is the following Corollary.

**Corollary 1.**  $F \in SAT \Leftrightarrow \Omega(F) \in \text{Almost-Unique-MU}$  .

**Lemma 9.**  $F \in \text{Unique-SAT} \Leftrightarrow \Omega(F) \in \text{Unique-MU}$  .

*Proof.* ( $\Rightarrow$ ) Suppose,  $F$  is uniquely satisfiable. Because of Lemma 8,  $\Omega(F)$  is minimal unsatisfiable. Because of Lemma 7, it suffices to prove the unique satisfiability of  $\Omega(F) - \{y_1 \vee y_2 \vee \dots \vee y_m\}$ . Suppose  $t$  is a satisfying truth assignment. Since  $\Omega(F)$  is unsatisfiable,  $t$  assigns the truth value 0 to every variable  $y_i$ ,  $1 \leq i \leq m$ . Then it is not difficult to see that  $t$  restricted to the variables in  $\text{var}(F)$  is a satisfying truth assignments for  $F$ . Since  $F$  is uniquely satisfiable,  $t$  is the only satisfying truth assignment for  $\Omega(F) - \{y_1 \vee y_2 \vee \dots \vee y_m\}$ . ( $\Leftarrow$ ) Suppose  $\Omega(F) \in \text{Unique-MU}$ , and  $t_1, t_2$  are satisfying truth assignments for  $F$ . We extend  $t_1, t_2$  to  $t'_1, t'_2$  by defining  $t'_1(y_i) = t'_2(y_i) = 0$ ,  $1 \leq i \leq m$ . Clearly,  $t'_1$  and  $t'_2$  are satisfying truth assignments for  $\Omega(F) - \{y_1 \vee y_2 \vee \dots \vee y_m\}$ . Since  $\Omega(F) - \{y_1 \vee y_2 \vee \dots \vee y_m\} \in \text{Unique-SAT}$ , it follows that  $t_1 t_2$ . Thus,  $F$  is in  $\text{Unique-SAT}$ . □

We have shown  $\text{Unique-SAT} \leq_p \text{Unique-MU}$ , because  $\Omega(F)$  can be computed in polynomial time. That shows together with Theorem 3 the equivalence of both problems with respect to polynomial reducibility.

**Theorem 4.**  $\text{Unique-SAT} \approx_p \text{Unique-MU}$  .

In the remainder of this section we investigate minimal unsatisfiable formulas  $F$  for which up to at most one clause  $f$  the formula  $F - \{f\}$  is in  $\text{Unique-SAT}$ . We will see that in this case the decision problem is  $D^P$ -complete. The proof of the  $D^P$ -completeness of the problem  $\text{Almost-Unique-MU}$  is based on a reduction from the  $D^P$ -complete problem

*SAT*–*UNSAT* of determining for a given pair  $(F, G)$  of propositional formulas whether  $F$  is satisfiable and  $G$  is unsatisfiable (see [10]). For the reduction we make use of the previously used function  $\Omega(F)$ . Additionally we define  $\Lambda(F) = \Omega(F) - \{y_1 \vee y_2 \vee \dots \vee y_m\}$ .

**Lemma 10.** *Suppose,  $F$  contains at least six negative clauses with pairwise distinct variables.*

- (1)  $F \in \text{UNSAT} \Rightarrow \Lambda(F) \in \text{Unique-MU}$  .
- (2)  $\Lambda(F) \in \text{MU} \Rightarrow F \in \text{UNSAT}$  .

*Proof. Ad 1:* Suppose  $F$  is unsatisfiable. At first we show  $\lambda(F) := \omega(F) - \{y_1 \vee \dots \vee y_m\}$  is in *MU* . By Lemma 4, it is sufficient to show  $\lambda(F)$  is unsatisfiable. Suppose, by contrary, that  $\lambda(F)$  is satisfiable. Then let  $t$  be a satisfying truth assignment of  $\lambda(F)$ . From the clauses of group (C), see definition of  $\omega(F)$ , we know that there is at most one  $i$  such that  $t(y_i) = 1$ .

*Case 1.*  $t(y_i) = 1$  for some  $i$ : W.o.l.g., we assume  $t(y_1) = 1$ . Then we obtain  $t(y_2) = \dots = t(y_m) = 0$ . Therefore, we get the contradiction

$$t(f_1) = 1, t(L_{11}) = t(L_{12}) = t(L_{13}) = 0.$$

*Case 2.*  $t(y_i) = 0$  for all  $i$ ,  $1 \leq i \leq m$ : Then we get  $t(f_1) = t(f_2) = \dots = t(f_m) = 1$ , in contradiction to the unsatisfiability of  $F$ .

Altogether, we have shown  $\lambda(F)$  is minimal unsatisfiable.

From the proof of Lemma 7 we see that for all  $h \in \Lambda(F)$  :  $(\Lambda(F) - \{h\}) \in \text{SAT}$  . Thus, to prove  $\Lambda(F) \in \text{MU}$  it is sufficient to show the unsatisfiability of  $\Lambda(F)$ . Suppose  $\Lambda(F)$  is satisfiable with a satisfying truth assignment  $t$ . Since  $\lambda(F)$  is unsatisfiable, some clauses in  $\lambda(F)$  are false for  $t$ . If  $t(f_i \vee \pi_i) = 0$ , then we obtain the contradiction  $t(\chi_i) = 1$  and  $t(x) = 1$  for all  $x \in \text{var}(F) - \text{var}(f_i)$ . By the same argument we obtain a contradiction for the clauses  $\neg L_{ij} \vee \pi_i \vee \neg y_i$  and  $\neg y_i \vee \neg y_j$ . Hence,  $\Lambda(F)$  is unsatisfiable. Moreover,  $\Lambda(F) \in \text{MU}$  .

Next we show  $\Lambda(F) \in \text{Unique-MU}$  . Consider any clause  $h \in \Lambda(F)$ . We shall show  $\Lambda(F) - \{h\} \in \text{Unique-SAT}$  . The only two cases in which the proofs are different from that in Lemma 7 are  $h = f_i \vee \pi_i \vee \chi_i$  and  $h = f_i \vee \pi_i \vee z$ . We only show the case  $h = f_i \vee \pi_i \vee \chi_i$ . W.o.l.g., we assume  $i = 1$ . Let  $t$  be a satisfying truth assignment of  $\Lambda(F) - \{h\}$ . Clearly,  $t$  and  $s'_1$  are equal on every variable except for  $y_1$ . Thus to prove  $t = s'_1$  we only need to show  $t(y_1) = 1$ . Now  $t$  assigns the truth value 0 to at most three variables (at most four variables in case  $h = f_1 \vee \pi_1 \vee z$ ) in  $\text{var}(F)$ . Please note that there are in  $F$  six negative clauses whose variables are pairwise disjoint. By the pigeon hole principle, there must be a negative clause, say  $f_i$ ,  $i \neq 1$ , such that  $t(f_i) = 0$ . Suppose  $t(y_1) = 0$ . Then  $t$  makes the clause  $f_i \vee \pi_i$  false. Thus, we obtain the contradiction  $t(\chi_i) = 1$  and  $t(x) = 1$  for all  $x \in \text{var}(F) - \text{var}(f_i)$ . Hence we get  $t(y_1) = 1$ . Consequently,  $s'_1$  is the unique satisfying truth assignment of  $\Lambda(F) - \{h\}$ . For cases when  $h$  is one of the clauses  $\neg L_{ik} \vee \pi_i \vee \neg y_i \vee \xi_i$ ,  $\neg L_{ik} \vee \pi_i \vee \neg y_i \vee z$ ,  $\neg y_i \vee \neg y_j \vee \xi$ ,  $\neg y_i \vee y_j \vee z$ , it is easy to see that the truth assignments  $t'_i, t''_i, v'_{ij}, v''_{ij}$  defined in the proof of Lemma 7 are respectively the unique satisfying truth assignment of  $\Lambda(F) - \{h\}$ .

*Ad 2:* Suppose  $\Lambda(F) \in \text{MU}$ . Since every clause in  $\Lambda(F)$  is a super clause of some clause in  $\lambda(F)$ , it follows that  $\lambda(F)$  is unsatisfiable. Note that if  $F$  is satisfiable then  $\omega(F)$  is minimal unsatisfiable, and hence  $\lambda(F)$  is satisfiable. Thus,  $F$  must be unsatisfiable.  $\square$

**Theorem 5.** *The problem Almost-Unique-MU is  $D^P$ -complete.*

*Proof.* We define a reduction from *SAT-UNSAT* to *Almost-Unique-MU*. For a pair of formulas  $F_1, F_2$ , w.o.l.g. we assume  $F_2$  contains at least six negative clauses whose variables are distinct. Otherwise, we extend  $F_2$  for new variables  $x_1, \dots, x_{18}$  to the formula

$$F_2 + \{\neg x_1 \vee \neg x_2 \vee \neg x_3, \dots, \neg x_{16} \vee \neg x_{17} \vee \neg x_{18}\}.$$

$F_2$  and the generated formula are equivalent with respect to satisfiability. We can also assume that  $\Omega(F_1)$  and  $\Lambda(F_2)$  have different variables. Let  $h_1$  be a clause in  $\Omega(F_1)$  such that  $\Omega(F_1) - \{h_1\} \in \textit{Unique-SAT}$  (from our construction we can easily find such a clause). For a fixed clause  $h_2 \in \Lambda(F_2)$  we define

$$G := (\Omega(F_1) - \{h_1\}) + \{h_1 \vee h_2\} + (\Lambda(F_2) - \{h_2\}).$$

The theorem follows from the claim.

**Claim.**  $F_1 \in \textit{SAT}$  and  $F_2 \in \textit{UNSAT}$  if and only if  $G \in \textit{Almost-Unique-MU}$ .

*Proof of the claim.*

( $\Rightarrow$ ) Suppose  $F_1$  is satisfiable while  $F_2$  is unsatisfiable. Then by Corollary 1 and Lemma 10,  $\Omega(F_1), \Lambda(F_2) \in \textit{MU}$ . We first show  $G$  is unsatisfiable. Suppose, by contrary,  $t$  is a satisfying truth assignment of  $G$ . Then  $t(h_1 \vee h_2) = 1$ . W.o.l.g., we assume that  $t(h_1) = 1$ . Then  $\Omega(F_1)$  is satisfiable, a contradiction. Since  $\Omega(F_1)$  and  $\Lambda(F_2)$  have different variables, it is easy to see that  $G - \{h_1 \vee h_2\}$  is satisfiable. For any clause  $h \in \Omega(F_1) - \{h_1\}$ , since  $\Omega(F_1) - \{h\} \in \textit{SAT}$  and  $\Lambda(F_2) - \{h_2\} \in \textit{SAT}$ , it follows that  $G - \{h\}$  is satisfiable. Similarly, we can show that  $G - \{h\}$  is also satisfiable for each  $h \in \Lambda(F_2) - \{h_2\}$ . Thus  $G \in \textit{MU}$ . The almost unique minimal unsatisfiability of  $G$  follows from the almost unique minimal unsatisfiability of  $\Omega(F_1)$  and the unique minimal satisfiability of  $\Lambda(F_2)$  and the choice of  $h_1$ .

( $\Leftarrow$ ) Suppose  $G \in \textit{MU}$ . First we show  $\Omega(F_1)$  is unsatisfiable. Suppose  $\Omega(F_1)$  is satisfiable. Since  $G \in \textit{MU}$ , we get that  $\Lambda(F_2) - \{h_2\}$  is satisfiable. Then  $G$  is satisfiable (note that the two formulas have different variables). Thus  $\Omega(F_1) \in \textit{UNSAT}$ . By the same argument,  $\Lambda(F_2) \in \textit{UNSAT}$ . Now the minimal unsatisfiability of  $\Omega(F_1)$  and  $\Lambda(F_2)$  follows from the minimal unsatisfiability of  $G$ . Then by Lemma 8 and Lemma 10 we have  $F_1 \in \textit{SAT}$  and  $F_2 \in \textit{UNSAT}$ .  $\square$

## 6. Dis-MU Formulas

A subset of *Unique-MU*-formulas are *MU*-formulas which have for each variable a so called disjunctive splitting. We say a *MU*-formula  $F$  has a disjunctive splitting on a variable  $x$ , if  $F$  can be partitioned into two subformulas  $G$  and  $H$ , where  $G$  resp.  $H$  contains no occurrence of  $\neg x$  resp.  $x$ , and  $G|_{x=0}, H|_{x=1} \in \textit{MU}$ . We define

$\textit{Dis-MU} = \{F \in \textit{MU} \mid \forall x \in \textit{var}(F) \exists G, H : F = G \wedge H, G \cap H = \textit{empty}, G \textit{ resp. } H \textit{ contains no literal } \neg x \textit{ resp. } x, G|_{x=0}, H|_{x=1} \in \textit{MU}\}$ .

We shall show that the problem whether a formula is in *Dis-MU* is at least as hard as the unique satisfiability problem. Whether *Dis-MU* is equivalent to *Unique-SAT* w.r.t. polynomial reducibility or  $D^P$ -complete is still open. Clearly, it is sufficient to show that  $F$

is unique satisfiable if and only if  $\Omega(F) \in \text{Dis-MU}$ . The direction from right to left follows from Lemma 9 and the fact that  $\text{Dis-MU}$  is a subclass of  $\text{Unique-MU}$ .

Suppose there is some  $F \in \text{Dis-MU}$ , but  $F \notin \text{Unique-MU}$ . That means there is some  $f \in F$  such that  $F - \{f\}$  is not in  $\text{Unique-SAT}$ . Let  $x$  be a variable for which we have neither  $F - \{f\} \models x$  nor  $F - \{f\} \models \neg x$ . Now we split  $F$  on  $x$ . Let  $G, H$  be two disjoint subformulas such that  $G|_{x=0} \in \text{MU}$  and  $H|_{x=1} \in \text{MU}$ . Suppose w.l.o.g.  $f \in G$ . Then  $f$  occurs not  $H$  and hence  $H \subseteq F - \{f\}$ . Since  $H|_{x=1} \in \text{MU}$ , we obtain  $H \models \neg x$  and therefore  $F - \{f\} \models \neg x$  in contradiction to our assumption  $F - \{f\} \not\models \neg x$ .

Our next task is devoted to the proof of the direction from left to right,  $F \in \text{Unique-SAT}$  implies  $\Omega(F) \in \text{Dis-MU}$ .

Let  $F = \{f_1, f_2, \dots, f_m\}$  be a 3-CNF formula,  $x \in \text{var}(F)$ . We define  $\omega(F, x)$  to be the formula obtained from  $\omega(F)$  as follows:

(1) If  $x \in f_i$  and  $L_{ik}$  is  $x$ , then add  $x$  to clauses

$$\neg L_{i(k \oplus 1)} \vee \pi_i \vee \neg y_i, \quad \neg L_{i(k \oplus 2)} \vee \pi_i \vee \neg y_i$$

(2) If  $\neg x \in f_i$  and  $\neg x$  is  $L_{ik}$ , then add  $\neg x$  to clauses

$$\neg L_{i(k \oplus 1)} \vee \pi_i \vee \neg y_i, \quad \neg L_{i(k \oplus 2)} \vee \pi_i \vee \neg y_i$$

(3) Add  $x$  to the clause  $y_1 \vee y_2 \vee \dots \vee y_m$

**Lemma 11.** *Suppose  $F \in \text{SAT}$  but  $F|_{x=1} \in \text{UNSAT}$ . Then  $\omega(F, x) \in \text{MU}$ .*

*Proof.* By means of Lemma 5, we obtain  $\omega(F) \in \text{MU}$ . By the construction of  $\omega(F, x)$ , it is sufficient to show the unsatisfiability of  $\omega(F, x)$ . Suppose  $\omega(F, x)$  is satisfiable. Let  $t$  be a satisfying truth assignment of  $\omega(F, x)$ . We shall derive a contradiction. From the clauses in group (C), we know that there are only two cases.

*Case 1.*  $t(y_1) = t(y_2) = \dots = t(y_m) = 0$ : Then  $t(x) = 1$ , and  $t(f_1) = \dots = t(f_m) = 1$  contradicts the assumption that  $F|_{x=1} \in \text{UNSAT}$ .

*Case 2.* There is exactly one  $i$  such that  $t(y_i) = 1$ : Then  $t(f_i) = 1$ . On the other hand, it is not hard to show  $t(L_{i1}) = t(L_{i2}) = t(L_{i3}) = 0$  (there are two subcases: one is the case in which  $f_i$  contains  $x$  or  $\neg x$ , the other is that  $f_i$  contains neither  $x$  nor  $\neg x$ ). A contradiction.

Therefore,  $\omega(F, x)$  is unsatisfiable, and the proof completes.  $\square$

Now we define  $\Omega(F, x)$  to be the formula consisting of the following groups of clauses.

(A'1) For each clause  $f_i \vee \pi_i$  such that neither  $x \in f_i$  nor  $\neg x \in f_i$ , the clauses

$$f_i \vee \pi_i \vee \chi_i, \quad f_i \vee \pi_i \vee x, \quad f_i \vee \pi_i \vee \neg x \vee z, \quad \text{for all } z \in \text{var}(F) - \text{var}(f_i), z \neq x.$$

(A'2) For each  $f_i \vee \pi_i$  such that  $x \in f_i$  or  $\neg x \in f_i$ , the clauses

$$f_i \vee \pi_i \vee \chi_i, \quad f_i \vee \pi_i \vee z, \quad \text{for all } z \in \text{var}(F) - \text{var}(f_i).$$

(B'1) For each clause  $\neg L \vee \pi_i \vee \neg y_i$ , where  $L \in \{x, \neg x\}$ , the clauses

$$\neg L \vee \pi_i \vee \neg y_i \vee \chi_i, \quad \neg L \vee \pi_i \vee \neg y_i \vee z, \quad \text{for all } z \in \text{var}(F) - \text{var}(f_i).$$

(B'2) for each clauses  $\neg L_{ik} \vee L \vee \pi_i \vee \neg y_i$ , where  $L \in \{x, \neg x\}$ , the clauses

$$\neg L_{ik} \vee L \vee \pi_i \vee \neg y_i \vee \chi_i, \neg L_{ik} \vee L \vee \pi_i \vee \neg y_i \vee z, \text{ for all } z \in \text{var}(F) - \text{var}(f_i).$$

(B'3) For each clause  $\neg L_{ik} \vee \pi_i \vee \neg y_i$  such that  $L_{ik} \notin \{x, \neg x\}$  (then  $f_i$  contains neither  $x$  nor  $\neg x$ ), the clauses

$$\begin{aligned} &\neg L_{ik} \vee \pi_i \vee \neg y_i \vee \chi_i, \neg L_{ik} \vee \pi_i \vee \neg y_i \vee x, \\ &\neg L_{ik} \vee \pi_i \vee \neg y_i \vee \neg x \vee z, \text{ for all } z \in \text{var}(F) - \text{var}(f_i), z \neq x. \end{aligned}$$

(C'1) For each clause  $\neg y_i \vee \neg y_j$ , the clauses

$$\neg y_i \vee \neg y_j \vee \chi, \neg y_i \vee \neg y_j \vee x, \neg y_i \vee \neg y_j \vee \neg x \vee z, \text{ for all } z \in \text{var}(F) - \{x\}.$$

(D'1) The clause  $y_1 \vee y_2 \vee \dots \vee y_m \vee x$ .

**Lemma 12.**  $\omega(F, x) \in MU \Rightarrow \Omega(F, x) \in MU$ .

*Proof.* For  $\omega(F, x) \in MU$ , it is easy to see that  $F$  is satisfiable. Then we have  $\Omega(F) \in MU$ . Please notice that  $\Omega(F, x)$  can be obtained from  $\Omega(F)$  by appropriately adding  $x$  or  $\neg x$  to clauses containing neither  $x$  nor  $\neg x$ . Thus the Lemma follows from the unsatisfiability of  $\Omega(F, x)$  which is clearly implied by the unsatisfiability of  $\omega(F, x)$ .  $\square$

**Theorem 6.**  $Unique\text{-}SAT \leq_p Dis\text{-}MU$ .

*Proof.* It is sufficient to show

$$F \in Unique\text{-}SAT \text{ if and only if } \Omega(F) \in Dis\text{-}MU.$$

The direction from right to left follows from the fact  $Dis\text{-}MU \subseteq Unique\text{-}MU$  and Lemma 9. Now we only need to show the inverse direction. Suppose  $F$  is unique satisfiable. Then we obtain  $\Omega(F) \in Unique\text{-}MU$ . We shall show,  $\Omega(F)$  has a disjunctive splitting on any variable. First we can show that  $\omega(F)$  has a disjunctive splitting on each  $y_i$ . For the sake of symmetry, it is sufficient to show the assertion for  $y_m$ . Let  $G$  be the following formula

$$\{f_m \vee \pi_m, \neg L_{m1} \vee \pi_i \vee \neg y_m, \neg L_{m2} \vee \pi_i \vee \neg y_m, \neg L_{m3} \vee \pi_i \vee \neg y_m, \\ \neg y_1 \vee \neg y_m, \neg y_2 \vee \neg y_m, \dots, \neg y_{m-1} \vee \neg y_m\}.$$

Clearly,  $G$  is a subformula of  $\omega(F)$  and  $G|_{y_m=1} \in MU$ . For  $H = F - G$ , the formula  $H|_{y_m=0}$  is the formula  $\omega(F - \{f_m\})$ .  $F - \{f_m\}$  is satisfiable, since  $F$  is satisfiable. By Lemma 5, we have  $H|_{y_m=0} \in MU$ . Thus,  $(H, G)$  forms a disjunctive splitting of  $\omega(F)$  on  $y_m$ .

Now it is not hard to see that  $\Omega(F)$  has a disjunctive splitting on each  $y_i$ . Next we show  $\Omega(F)$  has a disjunctive splitting on each variable  $x \in \text{var}(F)$ . Consider any variable  $x \in \text{var}(F)$ . Since  $F$  is uniquely satisfiable, either  $F|_{x=0} \in UNSAT$  or  $F|_{x=1} \in UNSAT$ . W.o.l.g., we assume  $F|_{x=1}$  is unsatisfiable. Then by Lemma 11-12,  $\Omega(F, x) \in MU$ . Please notice that  $x$  is a complete variable in  $\Omega(F, x)$ . That means, any clause in  $\Omega(F, x)$  contains either  $x$  or  $\neg x$ .

Thus  $(G^+, G^-)$  is a disjunctive splitting of  $\Omega(F, x)$  on  $x$ , where  $G^+$  resp.  $G^-$  is the formula consisting of all clauses containing  $x$  resp.  $\neg x$ . Please note again that  $\Omega(F, x)$  can be obtained from  $\Omega(F)$  by appropriately adding  $x$  or  $\neg x$  to clauses containing neither  $x$  nor  $\neg x$ . Thus, appropriately removing some occurrences of  $x$  and  $\neg x$  from  $G^+$  and  $G^-$  leads to a disjunctive splitting of  $\Omega(F)$  on  $x$ .  $\square$



## 7. Conclusion

We have shown the hierarchy  $Unique-SAT \approx_p Unique-MU \leq_p Dis-MU$  and  $MU \approx_p MARG-MU \approx_p MAX-MU \approx_p Almost-Unique-MU$ . Although we did not find a reduction from  $Dis-MU$  to  $Unique-SAT$  or to  $Unique-MU$ , we still believe that  $Dis-MU$  is as hard as  $Unique-SAT$ , because of the close relationship between  $Dis-MU$  and  $Unique-MU$ .

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