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Covering-based fuzzy rough sets

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Abstract. Many researchers have combined rough set theory and fuzzy set theory in order to easily approach problems of imprecision and uncertainty. Covering-based rough sets are one of the important generalizations of classical rough sets. Naturally, covering-based fuzzy rough sets can be studied as a combination of covering-based rough set theory and fuzzy set theory. It is clear that Pawlak's rough set model and fuzzy rough set model are special cases of the covering-based fuzzy rough set model. This paper investigates the properties of covering-based fuzzy rough sets. In addition, operations of intersection, union and complement on covering-based fuzzy rough sets are investigated. Finally, the corresponding algebraic properties are discussed in detail.

Keywords: Fuzzy rough sets, monotone covering, approximation operator, algebraic property

1. Introduction

Rough set theory, first proposed by Pawlak [35], is an excellent tool with which to handle vagueness and uncertainty in data analysis. The theory has been applied to the fields of medical diagnosis, conflict analysis, pattern recognition and data mining [7, 12, 17, 19].

Pawlak rough set theory is built on equivalence relations. However, an equivalence relation is restrictive for many real-world applications [8, 14, 22]. To overcome this limitation, there are two primary methods to generalize Pawlak rough set theory. Rough set theory has been generalized from the perspective of extending the equivalence relation to other binary relations, such as dominance relations, tolerance relations and similarity relations [15, 32]. In addition, one of the most important generalizations is to replace a partition obtained by the equivalence relation with a covering [3, 9–11, 24, 26–28, 30, 33, 34]. Zakowski, in 1983, first employed the covering of a universe to establish a covering-based generalized rough set. Since then, the study of covering-based rough set theory has attracted many researchers. Many kinds of lower and upper approximation operators have been proposed [18, 24, 25, 27, 29, 31]. Yao proposed approximation operators based on coverings produced by the predecessor and/or successor neighborhoods of serial or inverse serial binary relations [33]. Zhu, et al. studied six types of approximation operators and investigated the properties and relationships among them [26–28]. Qian, et al. simultaneously investigated five pairs of dual covering-based approximation operators by employing the notion of the neighborhood [13]. In addition, Yun, et al., also discussed covering rough sets and solved an open problem identified by Zhu and Wang [27]. To construct the lower and upper approximations of an arbitrary, Chen, et al. proposed a new covering-based on generalized rough set [3].

Alternatively, rough set theory was generalized by combining with other theories that deal with uncertain knowledge. The fuzzy rough set model which combines fuzzy set theory with rough set theory is one of the most important adaptations. It is well known that fuzzy set theory and rough set theory are complementary in terms of handling different kinds of uncertainty. Rough set theory deals with uncertainty resulting from ambiguity of information [1], while fuzzy set theory is adept at dealing with probabilistic uncertainty, connected to the imprecision of states, perceptions and preferences. The

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two theories can be encountered in many specific problems. Therefore, rough set theory has been generalized by combining it with fuzzy set theory. Many researchers have discussed the fuzzy rough set model from various perspectives [1, 4–6, 21]. Dubois and Prade proposed the concepts of the rough fuzzy set and the fuzzy rough set [2]. Morsi, et al. discussed some axioms of fuzzy rough sets [16]. Wu, et al. studied the (I, T)-fuzzy rough approximation operators [20]. Xu, et al. proposed the multi-granulation fuzzy rough set model and studied the properties of multi-granulation fuzzy rough sets [23]. However, it is still an open problem regarding the research of covering-based fuzzy rough sets.

In this paper, the primary objective is to investigate covering-based rough set theory when combined with fuzzy set theory. The paper is organized as follows. In Section 2, some basic concepts of Pawlak's rough set theory and fuzzy rough set theory are described. Furthermore, the concept of the monotone covering is proposed. In Section 3, the properties of covering-based fuzzy approximation operators are investigated. In Section 4, the operations of intersection, union and complement on covering-based fuzzy rough sets are discussed, as are the algebraic properties of covering-based fuzzy rough sets. Finally, Section 5 concludes this study.

2. Preliminaries

In this section, some basic concepts and notions according to Pawlak's theory rough sets, fuzzy sets, and covering are described. Additional details can be found in various references [23, 25, 35].

$(U,\ R)$ is referred to as an approximation space, in which $U = \{x_1,\ x_2, \cdots,\ x_n\}$ is a non-empty finite set. $\mathbb{R} = \{R_1, R_2, \cdots, R_m\}$ is a set of the equivalence relations. Denote $[x]_R = \{y\,|(x, y) \in R\}$, $U/R = \{[x]_R\,|x \in U\}$; then, $[x]_R$ is called the equivalence class of x and the quotient set U/R is called the equivalence class set of U.

Definition 2.1. Let $(U,\ \mathbb{R})$ be an approximation space and R be an equivalence relation. For any $X \subseteq U$, $\underline{R}(X)=\{x \in U|[x]_R \subseteq X\}$, $\overline{R}(X)=\{x \in U|[x]_R \cap X \neq \phi\}$.

These are the Pawlak lower and upper approximations of X with respect to equivalence relation R, respectively.

Let U represent a non-empty finite set. A fuzzy set X is a mapping from U into the unit interval $[0, 1]$; $X : U \to [0, 1]$, where each $X(x)$ is the membership degree of x in X. The set of all the fuzzy sets defined on U is denoted by $F(U)$.

Definition 2.2. Let $(U,\ \mathbb{R})$ be an approximation space and R be an equivalence relation. For any $X \subseteq U$, denote

$$\underline{R}(X)(x) = \wedge\{X(y)\,|y \in [x]_R\}$$
$$\overline{R}(X)(x) = \vee\{X(y)\,|y \in [x]_R\}.$$

$\underline{R}(X)$ and $\overline{R}(X)$ are the lower and upper approximations of the fuzzy set X with respect to equivalence relation R, where $\wedge$ represents "min" and $\vee$ represents "max".

Definition 2.3. Let U be the universe and $\mathbb{C}$ a family of nonempty subsets of U. If $\cup\mathbb{C} = U$, then $\mathbb{C}$ is a covering of U. The ordered pair $(U, \mathbb{C})$ represents a covering approximation space.

Let $(U, \mathbb{C})$ be a covering approximation space. For any $x \in U$, $\{K_x \in \mathbb{C}\,|x \in K_x\}$ is denoted as $st(x, \mathbb{C})$, i.e., $st(x, \mathbb{C}) = \{K_x \in \mathbb{C}\,|x \in K_x\}$.

Definition 2.4. Let $\mathbb{C}$ be a covering of U. For any $x \in U$ and $st(x, \mathbb{C}) = \{K_{x_1}, K_{x_2}, \cdots, K_{x_n}\}$. If K_{x_1}, $K_{x_2}, \cdots, K_{x_n}$ can be reordered to $K_{x_{i1}},\ K_{x_{i2}}, \cdots, K_{x_{in}}$ such that $K_{x_{i1}} \subseteq K_{x_{i2}} \subseteq \cdots K_{x_{in}}$, then $\mathbb{C}$ is a monotone covering of U, and $(U, \mathbb{C})$ is a monotone covering approximation space.

Furthermore, we denote $K_x^{\min} = K_{x_{i1}}$ and $\mathbb{C}^{\min} = \{K_x^{\min}\,|x \in U\}$. In addition, denote, $\mathbb{C}^{MIN} = \{K_{x_i}^{\min}$, $i = 1, 2, \cdots, m\}$, where $\mathbb{C}^{MIN}$ must satisfy the following three conditions.

(1) For any $K_{x_i}^{\min} \in \mathbb{C}^{MIN}$, $K_{x_i}^{\min} \in \mathbb{C}^{\min}$, $i = 1, 2, \cdots, m$;

(2) For any $K_{x_i}^{\min}, K_{x_j}^{\min} \in \mathbb{C}^{MIN}$, $K_{x_i}^{\min} \cap K_{x_j}^{\min} = \phi$, $i \neq j$, $i, j = 1, 2, \cdots, m$;

(3) For any $K_x^{\min} \in \mathbb{C}^{\min}$, there exists $K_{x_i}^{\min} \in \mathbb{C}^{MIN}$ such that $K_{x_i}^{\min} \subseteq K_x^{\min}$.

Remark 2.1. For any $K \in \mathbb{C}^{MIN}$ and any $x \in K$, by the construction of $\mathbb{C}^{MIN}$, $K = K_x^{\min}$ can be easily obtained.

Example 2.1. Let $U = \{x_1, x_2, \cdots, x_8\}$, $K_1 = \{x_1, x_2, x_3, x_4, x_5\}$, $K_2 = \{x_3, x_4, x_5\}$, $K_3 = \{x_5\}$, $K_4 = \{x_6, x_7, x_8\}$, and $K_5 = \{x_8\}$, $\mathbb{C} = \{K_1, K_2, K_3, K_4, K_5\}$. Then $\mathbb{C}$ is a monotone covering of U. Moreover, $\mathbb{C}^{\min} = \{\{x_1, x_2, x_3, x_4, x_5\}, \{x_3, x_4, x_5\}, \{x_5\}, \{x_6, x_7, x_8\}, \{x_8\}\}$, $\mathbb{C}^{MIN} = \{\{x_5\}, \{x_8\}\}$.

Definition 2.5. Let $(U, \mathbb{C})$ be a covering approximation space. For any fuzzy set $X \in F(U)$, denote

$$\underline{C}(X)(x) = \vee_{K_x \in \mathbb{C}}\{\wedge\{X(y) \,|\, y \in K_x\}\}$$

$$\overline{C}(X)(x) = \wedge_{K_x \in \mathbb{C}}\{\vee\{X(y) \,|\, y \in K_x\}\}.$$

$\underline{C}(X)$ and $\overline{C}(X)$ are the lower and upper covering fuzzy approximations of X, respectively. The pair $(\underline{C}(X), \overline{C}(X))$ is the covering fuzzy rough set of X and $\tilde{C} = \{(\underline{C}(X), \overline{C}(X)) \,|\, X \in F(U)\}$ represent all of the covering fuzzy rough sets.

Remark 2.2. In a special case, when $\mathbb{C}$ is a partition of the universe, then $\underline{C}(X)$ and $\overline{C}(X)$ of Definition 2.5 will degenerate into $\underline{R}(X)$ and $\overline{R}(X)$ of Definition 2.2.

Example 2.2. Let $U = \{x_1, x_2, x_3, x_4\}$, $K_1 = \{x_1, x_2\}$, $K_2 = \{x_2\}$, $K_3 = \{x_1, x_3\}$, $K_4 = \{x_3, x_4\}$, and $\mathbb{C} = \{K_1, K_2, K_3, K_4\}$. $\mathbb{C}$ is a covering of U. For fuzzy set $X = (0.3, 0.4, 0.2, 0.5)$, $\underline{C}(X) = (0.3, 0.4, 0.2, 0.5)$, $\overline{C}(X) = (0.3, 0.4, 0.3, 0.5)$.

3. Covering-based fuzzy approximation operators

In the section, the properties of the lower and upper covering fuzzy approximation operators in a covering approximation space are considered.

Proposition 3.1. *Let $(U, \mathbb{C})$ be a covering approximation space and $X, Y \in F(U)$. Then, the following properties hold:*

(1) $\underline{C}(U) = \overline{C}(U) = U$, $\underline{C}(\phi) = \overline{C}(\phi) = \phi$;

(2) $\underline{C}(X) \subseteq X \subseteq \overline{C}(X)$;

(3) $X \subseteq Y \Rightarrow \underline{C}(X) \subseteq \underline{C}(Y), \overline{C}(X) \subseteq \overline{C}(Y)$;

(4) $\underline{C}(\sim X) = \sim \overline{C}(X), \overline{C}(\sim X) = \sim \underline{C}(X)$.

Proof: The properties can be easily proved by Definition 2.5. □

Example 3.1. (Continued from Example 2.2) For another fuzzy set $Y = (0.3, 0.2, 0.4, 0.1)$, $\underline{C}(X) \cap \underline{C}(Y) = (0.3, 0.2, 0.2, 0.1)$. $\underline{C}(X \cap Y) = (0.2, 0.2, 0.2, 0.1)$. $\underline{C}(X) \cap \underline{C}(Y) \neq \underline{C}(X \cap Y)$. Similarly, $\overline{C}(X) \cup \overline{C}(Y) \neq \overline{C}(X \cup Y)$.

If $\mathbb{C}$ is a monotone covering of U, then the following properties are observed.

Proposition 3.2. *Let $(U, \mathbb{C})$ be a monotone covering approximation space and $X, Y \in F(U)$. Then the following properties hold:*

(1) $\underline{C}(X \cap Y) = \underline{C}(X) \cap \underline{C}(Y)$;

(2) $\overline{C}(X \cup Y) = \overline{C}(X) \cup \overline{C}(Y)$.

Proof: (1) ($\Rightarrow$:) can be proved easily by Proposition 3.1.

($\Leftarrow$:) For $\forall x \in U$, we have $\underline{C}(X \cap Y)(x)$

$$\begin{aligned}
&= \vee_{K_x \in \mathbb{C}}\{\wedge\{(X \cap Y)(y) \,|\, y \in K_x\}\} \\
&= \wedge\{(X \cap Y)(y) \,\Big|\, y \in K_x^{\min}\} \\
&\geq \{\wedge\{X(y) \,\Big|\, y \in K_x^{\min}\}\} \wedge \{\wedge\{Y(y) \,\Big|\, y \in K_x^{\min}\}\} \\
&= \underline{C}(X)(x) \wedge \underline{C}(Y)(x) \\
&= (\underline{C}(X) \cap \underline{C}(Y))(x)
\end{aligned}$$

(2) The item can be proved similarly to (1). □

Proposition 3.3. *Let $(U, \mathbb{C})$ be a monotone covering approximation space and $X \in F(U)$. Then the following properties hold:*

(1) $\underline{C}(\underline{C}(X)) = \underline{C}(X)$;

(2) $\overline{C}(\overline{C}(X)) = \overline{C}(X)$.

Proof: (1) ($\Rightarrow$:) can be proved easily by Proposition 3.1.

($\Leftarrow$:) For any $x \in U$, we have $\underline{C}(\underline{C}(X))(x)$

$$\begin{aligned}
&= \vee_{K_x \in \mathbb{C}}\{\wedge\{\underline{C}(X)(y) \,|\, y \in K_x\}\} \\
&= \wedge\{\underline{C}(X)(y) \,\Big|\, y \in K_x^{\min}\} \\
&= \underline{C}(X)(y_0)(\text{where}\underline{C}(X)(y_0) = \min_{y \in K_x^{\min}} \underline{C}(X)(y)) \\
&= \vee_{K_{y_0} \in \mathbb{C}}\{\wedge\{X(z) \,\Big|\, z \in K_{y_0}\}\} \\
&= \wedge\{X(z) \,\Big|\, z \in K_{y_0}^{\min}\} \\
&\geq \wedge\{X(z) \,\Big|\, z \in K_x^{\min}\} \\
&= \vee_{K_x \in \mathbb{C}}\{\wedge\{X(z) \,|\, z \in K_x\}\} \\
&= \underline{C}(X)(x)
\end{aligned}$$

(2) The item can be proved similarly to (1). □

Proposition 3.4. *Let* $(U, \mathbb{C})$ *be a monotone covering approximation space and* $X, Y \in F(U)$. *Then the following properties hold:*

(1) $\underline{C}(\underline{C}(X) \cap \underline{C}(Y)) = \underline{C}(X) \cap \underline{C}(Y)$;

(2) $\underline{C}(\underline{C}(X) \cup \underline{C}(Y)) = \underline{C}(X) \cup \underline{C}(Y)$;

(3) $\overline{C}(\overline{C}(X) \cap \overline{C}(Y)) = \overline{C}(X) \cap \overline{C}(Y)$;

(4) $\overline{C}(\overline{C}(X) \cup \overline{C}(Y)) = \overline{C}(X) \cup \overline{C}(Y)$.

Proof: (1) It is clear according to Proposition 3.1 and Proposition 3.2.

(2) ($\Rightarrow$:) is clear according to Proposition 3.1.

($\Leftarrow$:) For $\forall x \in U$, $\underline{C}(\underline{C}(X) \cup \underline{C}(Y))(x)$

$$= \vee_{K_x \in \mathbb{C}}\{\wedge\{(\underline{C}(X) \cup \underline{C}(Y))(y) \,|\, y \in K_x\}\}$$

$$= \wedge\{(\underline{C}(X) \cup \underline{C}(Y))(y) \,\Big|\, y \in K_x^{\min}\}$$

$$\geq \{\wedge\{\underline{C}(X)(y) \,\Big|\, y \in K_x^{\min}\}\} \vee \{\wedge\{\underline{C}(Y)(y) \,\Big|\, y \in K_x^{\min}\}\}$$

$$= \underline{C}(\underline{C}(X))(x) \vee \underline{C}(\underline{C}(Y))(x)$$

$$= \underline{C}(X) \cup \underline{C}(Y)(x).$$

(3) This item can be proved similarly to (2).

(4) This item can be proved similarly to (1). □

Proposition 3.5. *Let* $(U, \mathbb{C})$ *be a monotone covering approximation space. For any* $x \in U$ *and* $X \in F(U)$, *if* $K_x^{\min} = \{x\}$, *then* $\underline{C}(X)(x) = \overline{C}(X)(x)$.

Proof: Clear according to Definition 2.5. □

Proposition 3.6. *Let* $(U, \mathbb{C})$ *be a monotone covering approximation space. For* $\forall x, y \in U$ *and* $X \in F(U)$, *if* $K_x^{\min} = K_y^{\min}$, *the following properties hold:*

(1) $\underline{C}(X)(x) = \underline{C}(X)(y)$;

(2) $\overline{C}(X)(x) = \overline{C}(X)(y)$.

Proof: Clear according to Definition 2.5. □

4. Covering-based fuzzy rough sets

4.1. Operations on covering-based fuzzy rough sets

In this section, the operations of intersection, union and complement on covering-based fuzzy rough sets are investigated. We first propose the concepts of intersection, union and complement of covering-based fuzzy rough sets.

Definition 4.1. Let $(U, \mathbb{C})$ be a covering approximation space. For any $(\underline{C}(X), \overline{C}(X))$, $(\underline{C}(Y), \overline{C}(Y)) \in \tilde{C}$, the intersection and union are defined as follows:

(1) $(\underline{C}(X), \overline{C}(X)) \cap (\underline{C}(Y), \overline{C}(Y))$
$= (\underline{C}(X) \cap \underline{C}(Y), \overline{C}(X) \cap \overline{C}(Y))$;

(2) $(\underline{C}(X), \overline{C}(X)) \cup (\underline{C}(Y), \overline{C}(Y))$
$= (\underline{C}(X) \cup \underline{C}(Y), \overline{C}(X) \cup \overline{C}(Y))$.

Definition 4.2. Let $(U, \mathbb{C})$ be a covering approximation space. For any $(\underline{C}(X), \overline{C}(X)) \in \tilde{C}$, the complement is defined as follows:

$$\sim (\underline{C}(X), \overline{C}(X)) = (\sim \overline{C}(X), \sim \underline{C}(X))$$

Here, a question is raised: do all the covering-based fuzzy rough sets satisfy the operations of intersection, union and complement as defined above? The following will employ an example to illustrate the question.

Example 4.1. Let $U = \{x_1, x_2, x_3\}$, $K_1 = \{x_1, x_2\}$, $K_2 = \{x_2, x_3\}$, $\mathbb{C} = \{K_1, K_2\}$. Clearly, $\mathbb{C}$ is a covering of U. For fuzzy sets $X = (0.1, 0.2, 0.3)$ and $Y = (0.3, 0.2, 0.1)$, we have $\underline{C}(X) \cap \underline{C}(X) = (0.1, 0.2, 0.1)$ and $\overline{C}(X) \cap \overline{C}(Y) = (0.2, 0.2, 0.2)$. Clearly, there does not exist a fuzzy set $V \in F(U)$ such that $(\underline{C}(V), \overline{C}(V)) = (\underline{C}(X), \overline{C}(X)) \cap (\underline{C}(Y), \overline{C}(Y))$.

Example 4.1 indicates that all the covering-based fuzzy rough sets do not meet the operation of intersection.

Let $(U, \mathbb{C})$ be a monotone covering approximation space. If $\mathbb{C}$ satisfies the following condition (*).

For any $x \in U$, either $K_x^{\min} = \{x\}$ or there exists $y \in U$ such that $K_x^{\min} = K_y^{\min}$. $(*)$

Thus, we can present the following proposition.

Proposition 4.1. *Let* $(U, \mathbb{C})$ *be a monotone covering approximation space and* $\mathbb{C}$ *satisfy the condition* (*). *For any* $X, Y \in F(U)$, *the following properties hold:*

(1) $\underline{C}(V) = \underline{C}(X) \cap \underline{C}(Y)$, $\overline{C}(V) = \overline{C}(X) \cap \overline{C}(Y)$;

(2) $\underline{C}(W) = \underline{C}(X) \cup \underline{C}(Y)$, $\overline{C}(W) = \overline{C}(X) \cup \overline{C}(Y)$.

Proof: (1) Suppose that $\mathbb{C}^{MIN} = \{K_1, K_2, \cdots, K_m\}$.

For $K_1 \in \mathbb{C}^{MIN}$, choose $x_{11} \in K_1$ randomly. According to Remark 2.1, $K_1 = K_{x_{11}}^{\min}$ can be obtained. Let $st(x_{11}, \mathbb{C}) = \{K_{x_{11}}^1, K_{x_{11}}^2, \cdots, K_{x_{11}}^n\}$. Since $\mathbb{C}$ is a monotone covering, without loss of generality, suppose that $x_{11} \in K_{x_{11}}^{\min} = K_{x_{11}}^1 \subset K_{x_{11}}^2 \subset \cdots \subset K_{x_{11}}^n$.

(1_{K_1}) If $K_{x_{11}}^1$ has only one element, i.e., $K_{x_{11}}^{\ 1} = \{x_{11}\}$. Denote $V(x_{11}) = (\underline{C}(X) \cap \underline{C}(Y))(x_{11}) = (\overline{C}(X) \cap \overline{C}(Y))(x_{11})$ according to Proposition 3.5. If $K_{x_{11}}^1$ has at least two elements, i.e., $\{x_{11}\} \subset K_{x_{11}}^{\ 1}$. According to Proposition 3.6, we denote $V(x_{11}) = (\underline{C}(X) \cap \underline{C}(Y))(x_{11})$. For $\forall\ x \in K_{x_{11}}^{\ 1}/\{x_{11}\}$, let $V(x) = (\overline{C}(X) \cap \overline{C}(Y))(x)$.

(2_{K_1}) If $K_{x_{11}}^2/K_{x_{11}}^1$ has only one element, suppose that $K_{x_{11}}^2/K_{x_{11}}^1 = \{x_{12}\}$. According to proposition 3.5, we denote $V(x_{12}) = (\underline{C}(X) \cap \underline{C}(Y))(x_{12}) = \overline{C}(X) \cap \overline{C}(Y)(x_{12})$. If $K_{x_{11}}^2/K_{x_{11}}^1$ has at least two elements, according to Proposition 3.6, denote $V(x_{12}) = (\underline{C}(X) \cap \underline{C}(Y))(x_{12})$. For $\forall\ x \in K_{x_{11}}^{\ 2}/\ (K_{x_{11}}^{\ 1} \cup \{x_{12}\})$, let $V(x) = (\overline{C}(X) \cap \overline{C}(Y))(x)$.

(3_{K_1}) If $K_{x_{11}}^3/K_{x_{11}}^2$ has only one element, suppose that $K_{x_{11}}^{\ 3}/K_{x_{11}}^{\ 2} = \{x_{13}\}$. According to Proposition 3.5, denote $V(x_{13}) = (\underline{C}(X) \cap \underline{C}(Y))(x_{13}) = \overline{C}(X) \cap \overline{C}(Y)(x_{13})$. If $K_{x_{11}}^3/K_{x_{11}}^2$ has at least two elements, according to Proposition 3.6, we denote $V(x_{13}) = (\underline{C}(X) \cap \underline{C}(Y))(x_{13})$. For $\forall\ x \in K_{x_{11}}^{\ 3}/\ (K_{x_{11}}^{\ 2} \cup \{x_{13}\})$, let $V(x) = (\overline{C}(X) \cap \overline{C}(Y))(x)$.

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(n_{K_1}) If $K_{x_{11}}^n/K_{x_{11}}^{n-1}$ has only one element, suppose that $K_{x_{11}}^{\ n}/K_{x_{11}}^{\ n-1} = \{x_{1n}\}$. According to Proposition 3.5, denote $V(x_{1n}) = (\underline{C}(X) \cap \underline{C}(Y))(x_{1n}) = \overline{C}(X) \cap \overline{C}(Y)(x_{1n})$. If $K_{x_{11}}^n/K_{x_{11}}^{n-1}$ has at least two elements, according to Proposition 3.6, denote $V(x_{1n}) = (\underline{C}(X) \cap \underline{C}(Y))(x_{1n})$. For $\forall\ x \in K_{x_{11}}^{\ n}/\ (K_{x_{11}}^{\ n-1} \cup \{x_{1n}\})$, let $V(x) = (\overline{C}(X) \cap \overline{C}(Y))(x)$.

For $K_i \in \mathbb{C}^{MIN}, i = 2, 3, \cdots, m$, repeat the above steps from 1_{K_1} to n_{K_1}, respectively. Finally, a fuzzy set $V \in F(U)$ is obtained such that $\underline{C}(V) = \underline{C}(X) \cap \underline{C}(Y)$ and $\overline{C}(V) = \overline{C}(X) \cap \overline{C}(Y)$.

(2) The property can be proved according to (1). □

Remark 4.1. For any two covering-based fuzzy rough sets $(\underline{C}(X), \overline{C}(X))$, $(\underline{C}(Y), \overline{C}(Y)) \in \tilde{C}$, two fuzzy sets $V, W \in F(U)$ can be obtained such that $(\underline{C}(V), \overline{C}(V)) = (\underline{C}(X), \overline{C}(X)) \cap (\underline{C}(Y), \overline{C}(Y))$ and $(\underline{C}(W), \overline{C}(W)) = (\underline{C}(X), \overline{C}(X)) \cup (\underline{C}(Y), \overline{C}(Y))$.

Example 4.2. (Continued from Example 2.1) It is clear that $\mathbb{C}$ satisfies condition (*). For two fuzzy sets $X = (0.3, 0.1, 0.2, 0.4, 0.5, 0.4, 0.3, 0.4)$ and $Y = (0.2, 0.3, 0.4, 0.2, 0.6, 0.3, 0.2, 0.1)$, fuzzy sets $V = (0.1, 0.5, 0.2, 0.5, 0.5, 0.1, 0.3, 0.1)$ and $W = (0.2, 0.6, 0.2, 0.6, 0.6, 0.3, 0.4, 0.4)$ can be obtained such that $(\underline{C}(V), \overline{C}(V)) = (\underline{C}(X), \overline{C}(X)) \cap (\underline{C}(Y), \overline{C}(Y))$ and $(\underline{C}(W), \overline{C}(W)) = (\underline{C}(X), \overline{C}(X)) \cup (\underline{C}(Y), \overline{C}(Y))$.

Moreover, for $Z = (0.2, 0.3, 0.4, 0.5, 0.4, 0.6, 0.1, 0.3)$, Let $C = (0.2, 0.5, 0.4, 0.5, 0.5, 0.1, 0.6, 0.3)$, $D = (0.2, 0.5, 0.2, 0.5, 0.4, 0.1, 0.4, 0.3)$, then $(\underline{C}(C), \overline{C}(C)) = ((\underline{C}(X), \overline{C}(X)) \cap (\underline{C}(Y), \overline{C}(Y))) \cup (\underline{C}(Z), \overline{C}(Z))$ and $(\underline{C}(D), \overline{C}(D)) = ((\underline{C}(X), \overline{C}(X)) \cup (\underline{C}(Y), \overline{C}(Y))) \cap (\underline{C}(Z), \overline{C}(Z))$.

4.2. Algebraic properties of covering-based fuzzy rough sets

In this section, the algebraic properties of covering-based fuzzy rough sets are investigated. Suppose that $(U, \mathbb{C})$ is a monotone covering approximation space and that $\mathbb{C}$ satisfies condition (*). Then, we obtain the following conclusions.

Proposition 4.3. $(\tilde{C}, \cup, \cap, \sim)$ is an assignment lattice.

Proof: For any $(\underline{C}(X), \overline{C}(X))$, $(\underline{C}(Y), \overline{C}(Y))$ and $(\underline{C}(Z), \overline{C}(Z)) \in \tilde{C}$, we have

(1) $(\underline{C}(X), \overline{C}(X)) \cap (\underline{C}(X), \overline{C}(X))$

$= (\underline{C}(X), \overline{C}(X))\ (\underline{C}(X), \overline{C}(X)) \cup (\underline{C}(X), \overline{C}(X))$

$= (\underline{C}(X), \overline{C}(X))$

(2) $(\underline{C}(X), \overline{C}(X)) \cap (\underline{C}(Y), \overline{C}(Y))$

$= (\underline{C}(Y), \overline{C}(Y)) \cap (\underline{C}(X), \overline{C}(X))$

$(\underline{C}(X), \overline{C}(X)) \cup (\underline{C}(Y), \overline{C}(Y))$

$= (\underline{C}(Y), \overline{C}(Y)) \cup (\underline{C}(X), \overline{C}(X))$

(3) $((\underline{C}(X), \overline{C}(X)) \cap (\underline{C}(Y), \overline{C}(Y))) \cap (\underline{C}(Z), \overline{C}(Z))$

$= ((\underline{C}(X), \overline{C}(X)) \cap ((\underline{C}(Y), \overline{C}(Y)) \cap (\underline{C}(Z), \overline{C}(Z)))$

$((\underline{C}(X), \overline{C}(X)) \cup (\underline{C}(Y), \overline{C}(Y))) \cup (\underline{C}(Z), \overline{C}(Z))$

$= ((\underline{C}(X), \overline{C}(X)) \cup ((\underline{C}(Y), \overline{C}(Y)) \cup (\underline{C}(Z), \overline{C}(Z)))$

(4) $((\underline{C}(X), \overline{C}(X)) \cap ((\underline{C}(X), \overline{C}(X)) \cup (\underline{C}(Y), \overline{C}(Y)))$

$= (\underline{C}(X), \overline{C}(X))$

$((\underline{C}(X), \overline{C}(X)) \cup ((\underline{C}(X), \overline{C}(X)) \cap (\underline{C}(Y), \overline{C}(Y)))$

$= (\underline{C}(X), \overline{C}(X))$

$$
\begin{aligned}
(5)\ &((\underline{C}(X), \overline{C}(X)) \cap (\underline{C}(Y), \overline{C}(Y))) \cup (\underline{C}(Z), \overline{C}(Z)) \\
&= ((\underline{C}(X), \overline{C}(X)) \cup (\underline{C}(Z), \overline{C}(Z))) \\
&\quad \cap((\underline{C}(Y), \overline{C}(Y)) \cup (\underline{C}(Z), \overline{C}(Z))) \\
&\quad ((\underline{C}(X), \overline{C}(X)) \cup (\underline{C}(Y), \overline{C}(Y))) \cap (\underline{C}(Z), \overline{C}(Z)) \\
&= ((\underline{C}(X), \overline{C}(X)) \cap (\underline{C}(Z), \overline{C}(Z))) \\
&\quad \cup((\underline{C}(Y), \overline{C}(Y)) \cap (\underline{C}(Z), \overline{C}(Z))).
\end{aligned}
$$

Hence, $(\tilde{C}, \cup, \cap, \sim)$ is an assignment lattice. □

Let $(\phi, \phi) = 0$, $(U, U) = 1$, then the following conclusion is obtained.

Proposition 4.4. $(\tilde{C}, \cup, \cap, \sim)$ *is a soft algebra.*

Proof: (1) For any $(\underline{C}(X), \overline{C}(X)), (\underline{C}(Y), \overline{C}(Y)) \in \tilde{C}$,

$$
\begin{aligned}
&(\underline{C}(X), \overline{C}(X)) \cup 0 = (\underline{C}(X), \overline{C}(X)) \\
&(\underline{C}(X), \overline{C}(X)) \cap 0 = 0 \\
&(\underline{C}(X), \overline{C}(X)) \cup 1 = 1 \\
&(\underline{C}(X), \overline{C}(X)) \cap 1 = (\underline{C}(X), \overline{C}(X)).
\end{aligned}
$$

Thus, 0 and 1 are the minimal and maximal element of $(\tilde{C}, \cup, \cap, \sim)$, respectively.

$$
\begin{aligned}
(2)\ &\sim (\sim (\underline{C}(X), \overline{C}(X))) = \sim (\sim \overline{C}(X), \sim \underline{C}(X)) \\
&= (\underline{C}(X), \overline{C}(X))
\end{aligned}
$$

$$
\begin{aligned}
(3)\ &\sim ((\underline{C}(X), \overline{C}(X)) \cap (\underline{C}(X), \overline{C}(X))) \\
&= \sim ((\underline{C}(X) \cap \underline{C}(Y), \overline{C}(X) \cap \overline{C}(Y))) \\
&= (\sim (\overline{C}(X) \cap \overline{C}(Y)), \sim (\underline{C}(X) \cap \underline{C}(Y))) \\
&= (\sim \overline{C}(X) \cup \sim \overline{C}(Y), \sim \underline{C}(X) \cup \sim \underline{C}(Y)) \\
&= (\sim \overline{C}(X), \sim \underline{C}(X)) \cup (\sim \overline{C}(Y), \sim \underline{C}(Y)) \\
&= \sim (\underline{C}(X), \overline{C}(X)) \cup \sim (\underline{C}(Y), \overline{C}(Y)).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\sim ((\underline{C}(X), \overline{C}(X)) \cup (\underline{C}(X), \overline{C}(X))) \\
&\quad = \sim (\underline{C}(X), \overline{C}(X)) \cap \sim (\underline{C}(Y), \overline{C}(Y)).
\end{aligned}
$$

Hence, $(\tilde{C}, \cup, \cap, \sim)$ is a soft algebra. □

5. Conclusion

To easily deal with problems of uncertainty and imprecision, Xu, et al. proposed the multi-granulation fuzzy rough set model based on equivalence relations [23]. The model is a meaningful contribution toward the generalization of the classical rough set model.

It is well known that a multi-granulation rough set is a generalization of a Pawlak rough set. Covering-based rough sets are also an important generalization of classical rough sets. In this paper, we proposed the covering-based fuzzy rough set model and discussed its corresponding properties. Although many researchers have studied many properties of rough sets, the operations of intersection, union and complement on rough sets have yet to be investigated. In this paper, we proposed the concept of monotone covering and researched the operations of intersection, union and complement on covering-based fuzzy rough sets. Thus, the construction of the covering-based fuzzy rough set model is a meaningful generalization of rough set theory.

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