Journal of Intelligent & Fuzzy Systems 29 (2015) 2387–2395
DOI:10.3233/IFS-151938
IOS Press

On open problems based on fuzzy filters of pseudo BCK-algebras

Wang Wei[a,b,d], Wan Hui[c,*], Du Kai[a] and Xu Yang[d]
[a]*State Key Laboratory of Astronautic Dynamics, Xi'an, P.R. China*
[b]*College of Sciences, Xi'an Shiyou University, Xi'an, P.R. China*
[c]*Center for Nonlinear Studies, College of Mathematics, Northwest University, Xi'an, P.R. China*
[d]*College of Electrical Engineering, Southwest Jiaotong University, Chengdu, P.R. China*

Abstract. We study the properties and relations of fuzzy pseudo-filters of pseudo-BCK algebras. After we discuss the equivalent conditions of fuzzy normal pseudo-filter of pseudo-BCK algebra (pP), we propose fuzzy implicative pseudo-filter and its relation with fuzzy Boolean filter of (bounded) pseudo-BCK algebras (pP). Then two open problems: "In pseudo-BCK algebra or bounded pseudo-BCK algebra, Is the notion of implicative pseudo-filter equivalent to the notion of Boolean filter?" and "Prove or negate the following conclusion: A pseudo-BCK algebra is an implicative pseudo-BCK algebra if and only if every pseudo-filter of it is a Boolean filter(or an implicative pseudo-filter)" are partly solved.

Keywords: Pseudo BL-algebras, pseudo BCK-algebras, fuzzy pseudo-filter, fuzzy normal pseudo-filter, fuzzy Boolean filter

1. Introduction

The rapid development of mathematical logic, technology and computing science put forward many new requirements, thus contributing to the non-classical logic and the rapid development of modern logic. The study of fuzzy logic has become a hot topic with scientific information and artificial intelligence, which make fuzzy logic study of algebra and logic inseparable. Historically, the logic of algebra initiated in the early Leibniz period, he made use of symbols proposition to establish a two-valued logic calculus theory. Modern classical mathematical logic has been formed, and characterized as any proposition are "true" and "false" binary judgments in the 20th century.

However, science and technology is constantly in progress and development, so in real life, not everything can be described by absolutely true and absolutely false binary logic to deal with the phenomenon around. The classic two-valued logic can not meet the needs of new types of reasoning. Therefore, the classical two-valued logic must be improved and new promotion and reasoning should occur to meet the needs of real life. One way to improve and extend the classical two-valued logic is to expand the domain of assignment, which formed a multi-valued logic systems and fuzzy logic system. This system consists of a well-known Lukasiewicz logic, Godel logic, Gainse-Resche logic systems, basic logic BL, fuzzy propositional calculus formal deductive system, and so on. Fuzzy logic is both the mathematical basis of the artificial intelligence and fuzzy reasoning. Based on the actual background, different forms of fuzzy logic system is proposed.

The logical algebras are the algebraic counterpart of the the non-classical logic. For example, BCK-algebra, BL-algebras, pseudo MTL-algebras and non-commutative residuated lattice are algebraic counterparts of BCK Logic, Basic Logic, monoidal t-norm-based logic and monoidal logic respectively [2, 18, 19].

*Corresponding author. Wan Hui, Center for Nonlinear Studies, College of Mathematics, Northwest University, Xi'an, P.R. China. E-mail: wanhai1000@163.com.

ISSN 1064-1246/15/$35.00

In 1966, BCK-algebra was introduced by Iséki and Imai from BCK/BCI Logic [11]. From then on, numerous research work have been done on this, including ordered structure, algebraic structure, ideal, filter and the relations between them and logic. Afterwards, pseudo-BCK algebras, an extension of BCK-algebras, was proposed to express the non-commutative reasoning by Georgescu and Iorgulescu [6].

Filters theory plays a vital role not only in studying of algebraic structure, but also in non-classical logic and Computer Science [23]. From the aspect of logical point, filters correspond to various provable formulae sets [23]. For example, based on filters and prime filters in BL-algebras, Hájek proved the completeness of Basic Logic BL [18]. Literatures [1, 4, 5, 8, 12, 17, 24, 30], further studied filters of BL-algebras, lattice implication algebras, pseudo BL-algebras, pseudo effect-algebras, residuated lattices, triangle algebras and the corresponding algebraic structures.

Fuzzy sets were introduced in 1965 by Zadeh [14]. At present, fuzzy ideas have been applied to algebraic structures. Furthermore, fuzzy filter ideas function well in studying of algebraic structures. In lattice implication algebras fuzzy positive implicative filters was introduced [30]. [27, 28] characterized fuzzy positive implicative filters of lattice implication algebras. [15, 16] further studied fuzzy filters of BL-algebras. Wang and Xin solved an open problem in pseudo BL-algebras between fuzzy normal and fuzzy Boolean filter [20]. Literatures [10, 26, 29] studied interval valued-fuzzy filters in pseudo BL-algebras and MTL-algebras. Therefore, fuzzy filters are a useful tool to obtain results on algebraic structures of logic algebras.

In [22], there is an open problem: "In pseudo-BCK algebra or bounded pseudo-BCK algebra, Is the notion of implicative pseudo-filter equivalent to the notion of Boolean filter?" In [8], there is an open problem: "Prove or negate that pseudo-BCK algebras is implicative BCK algebras if and only if every pseudo-filters of them is implicative pseudo-filters (or Boolean filter)." The two kinds of filters are important to deep study of the algebraic structure of pseudo-BCK algebra, thus the two open problems are interesting and meaningful topics for deep research in pseudo-BCK algebra and are the motivation of this paper. We found it difficult to directly discuss the relation between the two filters, and we also found that fuzzy idea will be helpful in study of the algebraic structure, so we focused on the fuzzy filters of the two kinds and their relation.

We explore the properties and relations of fuzzy pseudo-filters of pseudo-BCK algebras. After we discuss the equivalent conditions of fuzzy normal pseudo-filter of pseudo-BCK algebra (pP), we propose fuzzy implicative pseudo-filter and its relation with fuzzy Boolean filter of (bounded) pseudo-BCK algebras (pP). Then two open problems are partly solved.

2. Preliminaries

Here we recall some basic results.

Definition 2.1. (Nola, Georgescu and Iorgulescu [2]) An algebraic structure $(A, \vee, \wedge, \odot, \rightharpoonup, \rightsquigarrow, 0, 1)$ is called a pseudo-BL algebra if for all $x, y, z \in A$

(1) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice,

(2) $(A, \odot, 1)$ is a monoid,

(3) $x \odot y \leq z \Leftrightarrow x \leq y \rightharpoonup z \Leftrightarrow y \leq x \rightsquigarrow z$,

(4) $x \wedge y = (x \rightharpoonup y) \odot x = x \odot (x \rightsquigarrow y)$,

(5) $(x \rightharpoonup y) \vee (y \rightharpoonup x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$.

Definition 2.2. (Meng and Jun [9]) An algebraic structure $(A, \rightharpoonup, 1)$ is called an BCK-algebra if for all $x, y, z \in A$

(1) $(z \rightharpoonup x) \rightharpoonup (y \rightharpoonup x)) \geq (y \rightharpoonup z)$,

(2) $(y \rightharpoonup x) \rightharpoonup x \geq y$,

(3) $x \geq x$,

(4) $x \geq y$ and $y \geq x$ imply $x = y$,

(5) $x \rightharpoonup 1 = 1$,

where $x \leq y$ means $x \rightharpoonup y = 1$.

Definition 2.3. (Georgescu and Iorgulescu [6]) An algebraic structure $(A, \geq, \rightharpoonup, \rightsquigarrow, 1)$ is called a (reversed left-) pseudo-BCK algebra, such that for all $x, y, z \in A$, the axioms hold

(1) $(z \rightharpoonup x) \rightsquigarrow (y \rightharpoonup x)) \geq y \rightharpoonup z, (z \rightsquigarrow x) \rightharpoonup (y \rightsquigarrow x)) \geq y \rightsquigarrow z$,

(2) $(y \rightharpoonup x) \rightsquigarrow x \geq y, (y \rightsquigarrow x) \rightharpoonup x \geq y$,

(3) $x \geq x$,

(4) $1 \geq x$,

(5) $x \geq y$ and $y \geq x$ imply $x = y$,

(6) $x \geq y \Leftrightarrow y \rightharpoonup x = 1 \Leftrightarrow y \rightsquigarrow x = 1$.

Example 2.4 (Lee and Park [13]) Let $X = [0, \infty]$ and let $\leq$ be the usual order on X. Define binary operations $\rightharpoonup$ and $\rightsquigarrow$ on X by $x \rightharpoonup y = 0 (if x \leq y)$, $x \rightharpoonup y = \frac{2x}{\pi} arctan(ln(\frac{x}{y}))(if y < x)$; $x \rightsquigarrow y = 0(if x \leq y), x \rightsquigarrow y = xe^{-tan(\frac{\pi y}{2x})}(if y < x)$. for all $x, y \in X$. Then $(X, \leq, \rightharpoonup, \rightsquigarrow, 0)$ is a pseudo-BCK algebra.

Definition 2.5. (Georgescu and Iorgulescu [6]) A pseudo-BCK algebra $(A, \geq, \rightharpoonup, \rightsquigarrow, 1)$ is called bounded if there exists unique element 0 such that $0 \rightharpoonup x = 1$ or $0 \rightsquigarrow x = 1$ for any $x \in A$.

In a bounded pseudo-BCK algebra A we define: $x^{-} = x \rightharpoonup 0, x^{\sim} = x \rightsquigarrow 0$ for any $x \in A$.

Proposition 2.6. *(Georgescu and Iorgulescu [6]) Let $(A, \geq, \rightharpoonup, \rightsquigarrow, 1)$ be a pseudo-BCK algebra. Then the following properties hold for any $x, y, z \in A$*

(1) $x \leq y \rightharpoonup y \rightharpoonup z \leq x \rightharpoonup z$ and $y \rightsquigarrow z \leq x \rightsquigarrow z$,

(2) $x \leq y \rightharpoonup z \rightharpoonup x \leq z \rightharpoonup y$ and $z \rightsquigarrow x \leq z \rightsquigarrow y$,

(3) $z \rightharpoonup x \leq (y \rightharpoonup z) \rightharpoonup (y \rightharpoonup x), z \rightsquigarrow x \leq (y \rightsquigarrow z) \rightsquigarrow (y \rightsquigarrow x)$,

(4) $z \rightsquigarrow (y \rightharpoonup x) = y \rightharpoonup (z \rightsquigarrow x)$.

Definition 2.7. (Georgescu and Iorgulescu [6]) A pseudo-BCK algebra (pP) (i.e., with pseudo-product) is a pseudo-BCK algebra $(A, \geq, \rightharpoonup, \rightsquigarrow, 1)$ satisfying the condition (pP)

(pP) for all $x, y \in A$, there exists $x \odot y = min\{z | x \leq y \rightharpoonup z\} = min\{z | y \leq x \rightsquigarrow z\}$.

Theorem 2.8. *(Zhang [21]) Let $(A, \geq, \rightharpoonup, \rightsquigarrow, 1)$ be a pseudo-BCK algebras (pP), $x \odot y$ is defined as $min\{z|x \leq y \rightharpoonup z\}$ or $min\{z|y \leq x \rightsquigarrow z\}$, then the followings hold in A*

(1) $(x \odot y) \rightharpoonup z = x \rightharpoonup (y \rightharpoonup z)$,

(2) $(y \odot x) \rightsquigarrow z = x \rightsquigarrow (y \rightsquigarrow z)$,

(3) $(x \rightharpoonup y) \odot x \leq x, y$,
$x \odot (x \rightsquigarrow y) \leq x, y$,

(4) $x \odot y \leq x \wedge y \leq x, y$.

Proposition 2.9. *(Iorgulescu [3]) Any pseudo-BL algebra is a pseudo-BCK algebra (pP).*

Lemma 2.10. *(Zhang [21]) pseudo-BCK algebra (pP) is categorically equivalent to partially ordered residuated integral monoid.*

Proposition 2.11. *(Georgescu [7]) In a pseudo-BL algebra A, for all $x, y, z \in A$ the followings hold*

(1) $x \vee y = ((x \rightharpoonup y) \rightsquigarrow y) \wedge ((y \rightharpoonup x) \rightsquigarrow x)$,

(2) $x \vee y = ((x \rightsquigarrow y) \rightharpoonup y) \wedge ((y \rightsquigarrow x) \rightharpoonup x)$,

(3) $x \rightharpoonup y = x \rightharpoonup x \wedge y, x \rightsquigarrow y = x \rightsquigarrow x \wedge y$.

Definition 2.12. (Meng and Jun [9]) A BCK algebra A is called an implicative BCK-algebra if for any $x, y \in A$, $(x \rightharpoonup y) \rightharpoonup x = x$.

Definition 2.13. (Zhang [22]) A pseudo-BCK algebra $(A, \geq, \rightharpoonup, \rightsquigarrow, 1)$ is called a 1-type implicative pseudo-BCK algebra if for any $x, y \in A$, $(x \rightharpoonup y) \rightharpoonup x = (x \rightsquigarrow y) \rightsquigarrow x = x$.

Theorem 2.14. *(Zhang and Gong [22]) Let $(A, \geq, \rightharpoonup, \rightsquigarrow, 1)$ be a pseudo-BCK algebras. Then A is a 1-type implicative pseudo-BCK algebra if and only if A is an implicative BCK-algebra.*

Definition 2.15. (Zhang and Gong [22]) A pseudo-BCK algebra $(A, \geq, \rightharpoonup, \rightsquigarrow, 1)$ is called a 2-type implicative pseudo-BCK algebra if for any $x, y \in A$, $(x \rightsquigarrow y) \rightharpoonup x = (x \rightharpoonup y) \rightsquigarrow x = x$.

Theorem 2.16. *(Zhang and Gong [22]) Let $(A, \geq, \rightharpoonup, \rightsquigarrow, 1)$ be a pseudo-BCK algebras. Then A is a 2-type implicative pseudo-BCK algebra if and only if A is an implicative BCK-algebra.*

3. Some kinds of pseudo-filters and fuzzy pseudo-filters in pseudo *BCK*-algebras

First, we recall the definitions of pseudo-filter, Boolean filter, normal pseudo-filter and implicative pseudo-filter of pseudo-BCK algebras.

Definition 3.1. (Zhang and Gong [22]) A nonempty subset F of a pseudo-BCK algebra A is called a pseudo-filter if

(F1) $x \in F, y \in A, x \leq y \Rightarrow y \in F$,

(F2) $x \in F, x \rightharpoonup y \in F$ or $x \rightsquigarrow y \in F \Rightarrow y \in F$.

We could get the equivalent conditions of pseudo-filter in pseudo-BCK algebra as follows.

Theorem 3.2. *A nonempty subset F of a pseudo-BCK algebra A is a pseudo-filter if and only if it satisfies*

(F3) $1 \in F$,

(F4) $x \in F$, $x \rightharpoonup y \in F$ or $x \rightsquigarrow y \in F$ implies $y \in F$.

Proof. Let F be a nonempty subset of A. It is easy to see that $1 \in F$, i.e., (F3) and (F4) hold if F is a pseudo-filter of A. Conversely, suppose F satisfies (F3) and (F4). Let $x \in F, x \leq y \in A$, then $x \rightharpoonup y = x \rightsquigarrow y = 1 \in F$, thus $y \in F$, so F is a pseudo-filter of A.

Theorem 3.3. *A nonempty subset F of a pseudo-BCK algebra(pP) A is a pseudo-filter if and only if*

(F5) $x, y \in F$ implies $x \odot y \in F$,

(F6) $x \in F, x \leq y \in A$ implies $y \in F$.

Proof. Let F be a pseudo-filter of A and $x, y \in F$, we know $x \rightharpoonup (y \rightharpoonup (x \odot y)) = (x \odot y) \rightharpoonup (x \odot y) = 1 \in F$, then $y \rightharpoonup (x \odot y) \in F$, so $x \odot y \in F$.

Conversely, suppose F satisfies (F5) and (F6). If $x, x \rightharpoonup y (or x \rightsquigarrow y) \in F$, then $(x \rightharpoonup y) \odot x, x \odot (x \rightsquigarrow y) \leq y \in F$.

Definition 3.4. (Zhang and Gong [22]) A pseudo-filter F of a pseudo-BCK algebra A is normal, if

(NF) $x \rightharpoonup y \in F$ iff $x \rightsquigarrow y \in F$.

It's obvious that any filter is normal in a BCK-algebra.

Definition 3.5. (Zhang and Gong [22]) Let $(A, \geq, \rightharpoonup, \rightsquigarrow, 1)$ be a pseudo-BCK algebra. A pseudo-filter F of A is called an implicative pseudo-filter if for any $x, y \in A$

(1) $(x \rightharpoonup y) \rightharpoonup x \in F$ implies $x \in F$,

(2) $(x \rightsquigarrow y) \rightsquigarrow x \in F$ implies $x \in F$.

Definition 3.6. (Zhang and Gong [22]) Let $(A, \geq, \rightharpoonup, \rightsquigarrow, 1)$ be a pseudo-BCK algebra. A pseudo-filter F is called Boolean if for any $x, y \in A$

(1) $(x \rightharpoonup y) \rightsquigarrow x \in F$ implies $x \in F$,

(2) $(x \rightsquigarrow y) \rightharpoonup x \in F$ implies $x \in F$.

Theorem 3.7. *(Zhang and Gong [22]) Let $(A, \geq, \rightharpoonup, \rightsquigarrow, 1)$ be a pseudo-BCK algebra and F a normal pseudo-filter of A. Then F is implicative if and only if F is Boolean.*

Theorem 3.8. *(Zhang and Gong [22]) Let $(A; \leq, \rightharpoonup, \rightsquigarrow, 0, 1)$ be a bounded pseudo-BCK algebra and F implicative pseudo-filter. The following properties hold for $\forall x, y \in A$*

(1) $((x \rightharpoonup 0) \rightharpoonup x) \rightsquigarrow x \in F$, i.e., $(x^{-} \rightharpoonup x) \rightsquigarrow x \in F$,

(2) $((x \rightsquigarrow 0) \rightsquigarrow x) \rightharpoonup x \in F$, i.e., $(x^{\sim} \rightsquigarrow x) \rightharpoonup x \in F$,

(3) $((x \rightharpoonup y) \rightharpoonup x) \rightsquigarrow x \in F$,

(4) $((x \rightsquigarrow y) \rightsquigarrow x) \rightharpoonup x \in F$,

(5) if $(x \rightharpoonup y) \rightharpoonup y \in F$, then $(y \rightharpoonup x) \rightsquigarrow x \in F$,

(6) if $(x \rightsquigarrow y) \rightsquigarrow y \in F$, then $(y \rightsquigarrow x) \rightharpoonup x \in F$,

(7) if $x \rightsquigarrow y \in F$, then $((y \rightsquigarrow x) \rightharpoonup x) \rightsquigarrow y \in F$,

(8) if $x \rightharpoonup y \in F$, then $((y \rightharpoonup x) \rightsquigarrow x) \rightharpoonup y \in F$.

Definition 3.9. (Zhang and Jun [25]) A nonempty subset F of a pseudo-BCK algebra A is a positive implicative pseudo-filter of A if for all $x, y \in A$, (F1) and the followings hold

(F7) $x \rightsquigarrow (y \rightharpoonup z), x \rightsquigarrow y \in F$ implies $x \rightsquigarrow z \in F$,

(F8) $x \rightharpoonup (y \rightsquigarrow z), x \rightharpoonup y \in F$ implies $x \rightharpoonup z \in F$.

Theorem 3.10. *(Zhang and Jun [25])A positive implicative pseudo-filter F of a pseudo-BCK algebra A is a pseudo-filter satisfying for all $x, y \in A$, $x \rightharpoonup (x \rightsquigarrow y) \in F$ implies $x \rightharpoonup y, x \rightsquigarrow y \in F$.*

Definition 3.11. A non-constant fuzzy subset f of a pseudo-BCK algebra A is called a fuzzy pseudo-filter if f_t is either empty or a pseudo-filter for any $t \in [0, 1]$.

It's easy to find that F is a pseudo-filter iff χ_F is a fuzzy pseudo-filter, where χ_F stands for the characteristic function of F.

For pseudo-BCK algebras or pseudo-BCK algebras(pP), inspired by [20], the following results of fuzzy pseudo-filters are similar with the ones of pseudo BL-algebras and reader could refer to it for the proofs.

Proposition 3.12. *A fuzzy set f is a fuzzy pseudo-filter of a pseudo-BCK algebra (or pseudo-BCK algebra(pP)) A if and only if for all $x, y, z \in A$, one the following conditions holds*

(1) $f(1) \geq f(x)$ and $f(y) \geq f(x) \wedge f(x \rightharpoonup y)$,

(2) $f(1) \geq f(x)$ and $f(y) \geq f(x) \wedge f(x \rightsquigarrow y)$,

(3) $x \rightharpoonup (y \rightharpoonup z) = 1$ or $x \rightsquigarrow (y \rightsquigarrow z)) = 1 \Rightarrow f(z) \geq f(x) \wedge f(y)$,

(4) $x \odot y \leq z$ or $y \odot x \leq z \Rightarrow f(z) \geq f(x) \wedge f(y)$,

(5) f is order-preserving, $f(x \odot y) \geq f(x) \wedge f(y)$.

Corollary 3.13. *An order-preserving fuzzy set fof a pseudo-BCK algebra(pP) A is a fuzzy pseudo-filter if and only if for any $x, y \in A$, $f(x \odot y) = f(x) \wedge f(y)$.*

Corollary 3.14. *Let f, f_i $((i \in \tau))$ be fuzzy pseudo-filters of a pseudo-BCK algebra(pP) A, then for any $x, y \in A$, the following properties hold*

(1) $f(x) = f(x \odot x) = f(x \odot x \odot \ldots \odot x)$,

(2) $f(x \odot y) = f(y \odot x)$,

(3) $f(x \rightharpoonup y) = f(1)$ or $f(x \rightsquigarrow y) = f(1)$ implies $f(x) \leq f(y)$,

(4) $f_1 \wedge f_2$ is a fuzzy pseudo-filter of A,

(5) $\wedge_{i \in \tau} f_i$ is a fuzzy pseudo-filter of A.

In order to solve the mentioned open problems, we fuzzilize the several kinds of pseudo-filters of pseudo-BCK algebras and characterize them respectively.

Definition 3.15. Let $(A, \geq, \rightharpoonup, \rightsquigarrow, 1)$ be a pseudo-BCK algebra. A fuzzy pseudo-filter f of A is called implicative if for any $x, y \in A$, the following conditions hold

(1) $f((x \rightharpoonup y) \rightharpoonup x) = f(x)$,

(2) $f((x \rightsquigarrow y) \rightsquigarrow x) = f(x)$.

Theorem 3.16. *Let $(A; \leq, \rightharpoonup, \rightsquigarrow, 0, 1)$ be a bounded pseudo-BCK algebra and f fuzzy implicative pseudo-filter of A. Then $\forall x \in A$,*
(1) $f((x^- \rightharpoonup x) \rightsquigarrow x) = f(1)$,
(2) $f((x^\sim \rightsquigarrow x) \rightharpoonup x) = f(1)$,
(3) $f((x^- \rightsquigarrow x) \rightharpoonup x) = f(1)$,
(4) $f((x^\sim \rightharpoonup x) \rightsquigarrow x) = f(1)$.

Proof. (1) From $x \leq (x^- \rightharpoonup x) \rightsquigarrow x$, so $((x^- \rightharpoonup x) \rightsquigarrow x)^- \leq x^-$ and $((x^- \rightharpoonup x) \rightsquigarrow x)^- \rightharpoonup x^- = 1$, then $f(((x^- \rightharpoonup x) \rightsquigarrow x)^- \rightharpoonup x^-) = f(1)$. And $x^- \leq (x^- \rightharpoonup x) \rightsquigarrow x$, so we get $((x^- \rightharpoonup x) \rightsquigarrow x)^- \rightharpoonup x^- \leq ((x^- \rightharpoonup x) \rightsquigarrow x)^- \rightharpoonup ((x^- \rightharpoonup x) \rightsquigarrow x)$. Then $f(((x^- \rightharpoonup x) \rightsquigarrow x)^- \rightharpoonup ((x^- \rightharpoonup x) \rightsquigarrow x)) = f((x^- \rightharpoonup x) \rightsquigarrow x) = f(1)$.
(2) It is similar to (1).
(3) From $x \leq (x^- \rightsquigarrow x) \rightharpoonup x$, so $((x^- \rightsquigarrow x) \rightharpoonup x)^- \leq x^-$ and $((x^- \rightsquigarrow x) \rightharpoonup x)^- \rightharpoonup x^- = 1$, then $f(((x^- \rightsquigarrow x) \rightharpoonup x)^- \rightharpoonup x^-) = f(1)$. And $x^- \leq (x^- \rightsquigarrow x) \rightharpoonup x$, so we get $((x^- \rightsquigarrow x) \rightharpoonup x)^- \rightharpoonup x^- \leq ((x^- \rightsquigarrow x) \rightharpoonup x)^- \rightharpoonup ((x^- \rightsquigarrow x) \rightharpoonup x)$. Then $f(((x^- \rightsquigarrow x) \rightharpoonup x)^- \rightharpoonup ((x^- \rightsquigarrow x) \rightharpoonup x)) = f((x^- \rightsquigarrow x) \rightharpoonup x) = f(1)$.
(4) It is similar to (3).

Theorem 3.17. *A fuzzy pseudo-filter f of a pseudo-BCK algebra A is implicative if and only if f_t is either empty or an implicative pseudo-filter for each $t \in [0, 1]$.*

Proof. Let f be a fuzzy implicative pseudo-filter of A. For any $t \in [0, 1]$, if $f_t \neq \phi$, then suppose $(x \rightharpoonup y) \rightharpoonup x \in f_t$, then from $f((x \rightharpoonup y) \rightharpoonup x) = f(x)$, we get $x \in f_t$. Dually we can get $x \in f_t$ if $(x \rightsquigarrow y) \rightsquigarrow x \in f_t$. So f_t is an implicative pseudo-filter of A.

Conversely, let $f((x \rightharpoonup y) \rightharpoonup x) = t$, we get $(x \rightharpoonup y) \rightharpoonup x \in f_t$, and f_t is an implicative pseudo-filter of A, then $x \in f_t$. That is, $f((x \rightharpoonup y) \rightharpoonup x) = t \leq f(x)$. Since $x \leq (x \rightharpoonup y) \rightharpoonup x$ and f is isotone, then $f((x \rightharpoonup y) \rightharpoonup x) \geq f(x)$, we get $f((x \rightharpoonup y) \rightharpoonup x) = f(x)$. Dually we can obtain $f((x \rightsquigarrow y) \rightsquigarrow x) = f(x)$, thus proves f is implicative.

Theorem 3.18. *A fuzzy pseudo-filter f of a pseudo-BCK algebra A is implicative if and only if $f_{f(1)}$ is an implicative pseudo-filter.*

Proof. Necessity is obvious. Suppose $f_{f(1)}$ is an implicative pseudo-filter, then for all $x, y \in A$, if $(x \rightharpoonup y) \rightharpoonup x \in f_{f(1)}$, then $x \in f_{f(1)}$, so $f((x \rightharpoonup y) \rightharpoonup x) = f(x) = f(1)$. Dually we can obtain $f((x \rightsquigarrow y) \rightsquigarrow x) = f(x) = f(1)$. Thus we prove f is implicative.

Corollary 3.19. *A nonempty subset F of a pseudo-BCK algebra A is an implicative pseudo-filter if and only if χ_F is implicative.*

Corollary 3.20. *Let $(A, \geq, \rightharpoonup, \rightsquigarrow, 1)$ be a pseudo-BCK algebra. A pseudo-filter F of A is called Boolean if for any $x, y \in A$, the following conditions hold*
(1) $f((x \rightharpoonup y) \rightsquigarrow x) = f(x)$,
(2) $f((x \rightsquigarrow y) \rightharpoonup x) = f(x)$.

Similar to the fuzzy implicative pseudo-filter, we have the following results, so we omit the proof.

Theorem 3.21. *A fuzzy pseudo-filter f of a pseudo-BCK algebra A is Boolean if and only if f_t is either empty or a Boolean filter for each $t \in [0, 1]$.*

Theorem 3.22. *A fuzzy pseudo-filter f of a pseudo-BCK algebra A is Boolean if and only if $f_{f(1)}$ is a Boolean filter of A.*

Corollary 3.23. *A nonempty subset F of a pseudo-BCK algebra A is Boolean if and only if χ_F is Boolean.*

Theorem 3.24. *A fuzzy pseudo-filter f of a bounded pseudo-BCK algebra A is Boolean if and only if $f(x \rightharpoonup y) = f(x \rightharpoonup (y^- \rightsquigarrow y))$, $f(x \rightsquigarrow y) = f(x \rightsquigarrow (y^\sim \rightharpoonup y))$ for all $x, y \in A$.*

Proof. Suppose f is Boolean. Let $f(x \rightharpoonup (y^- \rightsquigarrow y)) = t$, then f_t is a Boolean filter, $x \rightharpoonup (y^- \rightsquigarrow y) \in f_t$. And $(y^- \rightsquigarrow y) \rightharpoonup y \in f_t$. $(y^- \rightsquigarrow y) \rightharpoonup y \leq (x \rightharpoonup (y^- \rightsquigarrow y)) \rightharpoonup (x \rightharpoonup y)$, then $x \rightharpoonup y \in f_t$, i.e., $f(x \rightharpoonup y) \geq t = f(x \rightharpoonup (y^- \rightsquigarrow y))$. Since $x \rightharpoonup y \leq x \rightharpoonup (y^- \rightsquigarrow y)$, then $f(x \rightharpoonup y) \leq f(x \rightharpoonup (y^- \rightsquigarrow y))$, so $f(x \rightharpoonup y) = f(x \rightharpoonup (y^- \rightsquigarrow y))$. Dually, we can obtain $f(x \rightsquigarrow y) = f(x \rightsquigarrow (y^\sim \rightharpoonup y))$.

Conversely, let $f((x \rightharpoonup y) \rightsquigarrow x) = t$, then f_t is a pseudo-filter and $(x \rightharpoonup y) \rightsquigarrow x \in f_t$. $(x \rightharpoonup y) \rightsquigarrow x \leq x^- \rightsquigarrow x$, then $x^- \rightsquigarrow x \in f_t$. $1 \rightharpoonup (x^- \rightsquigarrow x) = x^- \rightsquigarrow x \in f_t$, we get $f(x) = f(1 \rightharpoonup x) = f(1 \rightharpoonup (x^- \rightsquigarrow x)) \geq t = f((x \rightharpoonup y) \rightsquigarrow x)$, and $f(x) \leq f((x \rightharpoonup y) \rightsquigarrow x)$ is obvious, then $f(x) = f((x \rightharpoonup y) \rightsquigarrow x)$. We can also get $f(x) = f((x \rightsquigarrow y) \rightharpoonup x)$. Then f is a fuzzy Boolean filter of A.

Theorem 3.25. *A fuzzy pseudo-filter f of a bounded pseudo-BCK algebra A is Boolean filter if and only if $f((x^\sim \rightharpoonup x) \rightsquigarrow x) = f((x^- \rightsquigarrow x) \rightharpoonup x) = f(1)$ for all $x \in A$.*

Proof. (1) From $x \leq (x^\sim \rightharpoonup x) \rightsquigarrow x$, we get $((x^\sim \rightharpoonup x) \rightsquigarrow x)^\sim \leq x^\sim$ and $((x^\sim \rightharpoonup x) \rightsquigarrow x)^\sim \rightharpoonup x^\sim = 1$, then $f(((x^{\sim} \rightharpoonup x) \rightsquigarrow x)^{\sim} \rightharpoonup x^{\sim}) = f(1)$. And $x^{\sim} \leq (x^{\sim} \rightharpoonup x) \rightsquigarrow x$, so we get $((x^{\sim} \rightharpoonup x) \rightsquigarrow x)^{\sim} \rightharpoonup x^{\sim} \leq ((x^{\sim} \rightharpoonup x) \rightsquigarrow x)^{\sim} \rightharpoonup ((x^{\sim} \rightharpoonup x) \rightsquigarrow x)$. Then $f(((x^{\sim} \rightharpoonup x) \rightsquigarrow x)^{\sim} \rightharpoonup ((x^{\sim} \rightharpoonup x) \rightsquigarrow x)) = f((x^{\sim} \rightharpoonup x) \rightsquigarrow x) = f(1)$.

Similarly we can prove $f((x^{-} \rightsquigarrow x) \rightharpoonup x) = f(1)$.

Conversely, assume $f(x \rightharpoonup (y^{-} \rightsquigarrow y)) = t$, then $x \rightharpoonup (y^{-} \rightsquigarrow y) \in f_t$. And we have $(y^{-} \rightsquigarrow y) \rightharpoonup y \leq (x \rightharpoonup (y^{-} \rightsquigarrow y)) \rightharpoonup (x \rightharpoonup y) \in f_t$. Then $x \rightharpoonup y \in f_t$, i.e., $f(x \rightharpoonup y) \geq t = f(x \rightharpoonup (y^{-} \rightsquigarrow y))$. Since $y^{-} \rightsquigarrow y \geq y$ and $x \rightharpoonup (y^{-} \rightsquigarrow y) \geq x \rightharpoonup y$, we get $f(x \rightharpoonup y) \leq f(x \rightharpoonup (y^{-} \rightsquigarrow y))$, then $f(x \rightharpoonup y) = f(x \rightharpoonup (y^{-} \rightsquigarrow y))$. So $f(x \rightsquigarrow y) = f(x \rightsquigarrow (y^{\sim} \rightharpoonup y))$. Then f is a fuzzy Boolean filter of A.

Corollary 3.26. *In a bounded pseudo-BCK algebra, every fuzzy implicative pseudo-filter is a fuzzy Boolean filter.*

Theorem 3.27. *Let* $(A, \leq, \rightharpoonup, \rightsquigarrow, 0, 1)$ *be a bounded pseudo-BCK algebra and f a fuzzy Boolean filter. Then* $f((y \rightsquigarrow x) \rightharpoonup x) = f((x \rightharpoonup y) \rightsquigarrow y)$ *for all* $x, y \in A$.

Proof. Let $f((x \rightharpoonup y) \rightsquigarrow y) = t$, then $(x \rightharpoonup y) \rightsquigarrow y \in f_t$. $(x \rightharpoonup y) \rightsquigarrow y \leq (y \rightsquigarrow x) \rightharpoonup ((x \rightharpoonup y) \rightsquigarrow x) = (x \rightharpoonup y) \rightsquigarrow ((y \rightsquigarrow x) \rightharpoonup x)$. And, we have $x \rightharpoonup y \geq ((y \rightsquigarrow x) \rightharpoonup x) \rightharpoonup y$ and $(x \rightharpoonup y) \rightsquigarrow ((y \rightsquigarrow x) \rightharpoonup x) \leq (((y \rightsquigarrow x) \rightharpoonup x) \rightharpoonup y) \rightsquigarrow ((y \rightsquigarrow x) \rightharpoonup x)$. Combining the above results we get $(x \rightharpoonup y) \rightsquigarrow y \leq (((y \rightsquigarrow x) \rightharpoonup x) \rightharpoonup y) \rightsquigarrow ((y \rightsquigarrow x) \rightharpoonup x)$. Thus, $(((y \rightsquigarrow x) \rightharpoonup x) \rightharpoonup y) \rightsquigarrow ((y \rightsquigarrow x) \rightharpoonup x) \in f_t$. Then we get $(y \rightsquigarrow x) \rightharpoonup x \in f_t$, i.e., $f((y \rightsquigarrow x) \rightharpoonup x) \geq f((x \rightharpoonup y) \rightsquigarrow y)$. Similarly, we can prove that $f((y \rightsquigarrow x) \rightharpoonup x) \leq f((x \rightharpoonup y) \rightsquigarrow y)$.

Corollary 3.28. *Let* $(A, \leq, \rightharpoonup, \rightsquigarrow, 0, 1)$ *be a bounded pseudo-BCK algebra and f a fuzzy Boolean filter. Then* $f(x^{-\sim}) = f(x^{\sim -}) = f(x)$ *for all* $x \in A$.

Theorem 3.29. *Let* $(A, \leq, \rightharpoonup, \rightsquigarrow, 0, 1)$ *be a bounded pseudo-BCK algebra(pP) and f a fuzzy Boolean filter. Then* $f(x \rightharpoonup y) = f(x \odot x \rightharpoonup y) = f((x \odot x \odot \ldots \odot x) \rightharpoonup y)$, $f(x \rightsquigarrow y) = f(x \odot x \rightsquigarrow y) = f((x \odot x \odot \ldots \odot x) \rightsquigarrow y)$ *for all* $x, y \in A$.

Proof. Assume $f(x \rightsquigarrow (x \rightsquigarrow y)) = t$, then $x \rightsquigarrow (x \rightsquigarrow y) \in f_t$. And $x \rightsquigarrow (x \rightsquigarrow y) \leq ((x \rightsquigarrow y) \rightsquigarrow y) \rightharpoonup (x \rightsquigarrow y)$. It follows that $((x \rightsquigarrow y) \rightsquigarrow y) \rightharpoonup (x \rightsquigarrow y) \in f_t$. So $f(x \rightsquigarrow y) \geq t = f(x \rightsquigarrow (x \rightsquigarrow y))$. The inverse inequation is obvious, then $f(x \rightsquigarrow y) = f(x \rightsquigarrow (x \rightsquigarrow y)) = f(x \odot x \rightsquigarrow y)$. Similarly, we can prove that $f(x \rightharpoonup y) = f(x \rightharpoonup (x \rightharpoonup y)) = f(x \odot x \rightharpoonup y)$. And by induction, we can get the result.

Definition 3.30. A fuzzy filter f of a pseudo-BCK algebra is called normal if f_t is either empty or a normal pseudo-filter for each $t \in [0, 1]$.

Theorem 3.31. *Suppose f is a fuzzy set of a pseudo-BCK algebra A. Then the followings are equivalent for any* $x, y, z \in A$

(1) f is normal,

(2) f is a fuzzy pseudo-filter satisfying $f(x \rightharpoonup y) = f(x \rightsquigarrow y)$,

(3) $f(1) \geq f(x)$, $f(x \rightsquigarrow z) \geq f(y) \wedge f(x \rightharpoonup (y \rightsquigarrow z))$ and $f(x \rightharpoonup z) \geq f(y) \wedge f(x \rightsquigarrow (y \rightharpoonup z))$,

(4) $f(1) \geq f(x)$, $f(x \rightharpoonup z) \geq f(y) \wedge f(x \rightharpoonup (y \rightharpoonup z))$ and $f(x \rightsquigarrow z) \geq f(y) \wedge f(x \rightsquigarrow (y \rightsquigarrow z))$.

Proof. (1) $\Rightarrow$ (2). For all $x, y \in A$, let $f(x \rightharpoonup y) = t_1$, so $x \rightharpoonup y \in f_{t_1}$, then $x \rightsquigarrow y \in f_{t_1}$, $f(x \rightsquigarrow y) \geq t_1 = f(x \rightharpoonup y)$. Dually, we can obtain $f(x \rightharpoonup y) \geq f(x \rightsquigarrow y)$.

(2) $\Rightarrow$ (3). It is obvious.

(2) $\Rightarrow$ (4). It is obvious.

(3) $\Rightarrow$ (1). Let $x = 1$, we find f is a fuzzy pseudo-filter. For any $t \in [0, 1]$, if $f_t \neq \phi$ and $x \rightharpoonup y \in f_t$, so $f(x \rightharpoonup y) \geq t$. $x \rightharpoonup ((x \rightharpoonup y) \rightsquigarrow y) = 1 \in f_t$, then $f(x \rightharpoonup ((x \rightharpoonup y) \rightsquigarrow y)) = f(1) \geq t$ and $f(x \rightsquigarrow y) \geq f(x \rightharpoonup y) \wedge f(x \rightharpoonup ((x \rightharpoonup y) \rightsquigarrow y)) \geq t$, so $x \rightsquigarrow y \in f_t$. Also, we can prove that if $x \rightsquigarrow y \in f_t$, then $x \rightharpoonup y \in f_t$. So f_t is a normal pseudo-filter and f is normal.

(4) $\Rightarrow$ (1). In the same way of (3) $\rightharpoonup$ (1).

Similar to the proof of fuzzy implicative pseudo-filter, we have the following results.

Theorem 3.32. *A fuzzy pseudo-filter f of a pseudo-BCK algebra A is normal if and only if* $f_{f(1)}$ *is a normal pseudo-filter of A.*

Corollary 3.33. *A nonempty subset F of a pseudo-BCK algebra A is a normal pseudo-filter if and only if* χ_F *is normal.*

Definition 3.34. A fuzzy pseudo-filter f of pseudo-BCK algebra A is called positive implicative if for any $x, y \in A$, the followings hold

(1) $f(x \rightsquigarrow z) \geq f(x \rightsquigarrow (y \rightharpoonup z)) \wedge f(x \rightsquigarrow y)$,

(2) $f(x \rightharpoonup z) \geq f(x \rightharpoonup (y \rightsquigarrow z)) \wedge f(x \rightharpoonup y)$.

Theorem 3.35. *A fuzzy pseudo-filter f of a pseudo-BCK algebra A is positive implicative if and only if for each $t \in [0, 1]$, f_t is either empty or a positive implicative* pseudo-filter.

Theorem 3.36. *A fuzzy pseudo-filter f of a pseudo-BCK algebra A is positive implicative if and only if $f_{f(1)}$ is a positive implicative pseudo-filter.*

Corollary 3.37. *A nonempty subset F of a pseudo-BCK algebra A is a positive implicative pseudo-filter if and only if χ_F is positive implicative.*

Theorem 3.38. *Suppose f is a fuzzy normal pseudo-filter of a pseudo-BCK algebra A. Then the followings are equivalent*

(1) f is positive implicative.
(2) $f(x \rightharpoonup z) \geq f(x \rightharpoonup (y \rightharpoonup z)) \wedge f(x \rightharpoonup y)$.
(3) $f(x \rightsquigarrow y) = f(x \rightsquigarrow (x \rightharpoonup y))$.
(4) $f(x \rightharpoonup y) = f(x \rightharpoonup (x \rightharpoonup y))$.

Proof. (1) $\Rightarrow$ (2). For any $x, y, z \in A$, we have $f(x \rightsquigarrow z) \geq f(x \rightsquigarrow (y \rightharpoonup z)) \wedge f(x \rightsquigarrow y)$. Since f is fuzzy normal, then we can get (2) holds.

(2) $\Rightarrow$ (3). When we let $y = x$ and $z = y$ in (2), we have $f(x \rightharpoonup y) \geq f(x \rightharpoonup (x \rightharpoonup y)) \wedge f(x \rightharpoonup x) = f(x \rightharpoonup (x \rightharpoonup y)) \wedge f(1) = f(x \rightharpoonup (x \rightharpoonup y))$. It is obvious to get the inverse inequation by the isotonic of fuzzy filter. Thus (3) holds.

(3) $\Rightarrow$ (4). It is obvious.

(4) $\Rightarrow$ (1). Let $f(x \rightharpoonup (y \rightharpoonup z)) \wedge f(x \rightharpoonup y) = t$, then $x \rightharpoonup (y \rightharpoonup z), x \rightharpoonup y \in f_t$. Since $y \leq (y \rightharpoonup z) \rightsquigarrow z$, we have $x \rightharpoonup y \leq x \rightharpoonup ((y \rightharpoonup z) \rightsquigarrow z) = (y \rightharpoonup z) \rightsquigarrow (x \rightharpoonup z)$, then $(y \rightharpoonup z) \rightsquigarrow (x \rightharpoonup z) \in f_t$, further $(y \rightharpoonup z) \rightharpoonup (x \rightharpoonup z) \in f_t$. And $(y \rightharpoonup z) \rightharpoonup (x \rightharpoonup z) \leq (x \rightharpoonup (y \rightharpoonup z)) \rightharpoonup (x \rightharpoonup (x \rightharpoonup z)) \in f_t$, then we get $x \rightharpoonup (x \rightharpoonup z) \in f_t$. By (4), we can prove that $x \rightharpoonup z \in f_t$, i.e., if $x \rightsquigarrow (y \rightharpoonup z), x \rightsquigarrow y \in f_t$, then $x \rightsquigarrow z \in f_t$. Dually if $x \rightharpoonup (y \rightsquigarrow z), x \rightharpoonup y \in f_t$, then $x \rightharpoonup z \in f_t$. So f_t is a positive implicative pseudo-filter and f is positive implicative.

4. On open problems of pseudo-filters in pseudo-BCK algebra

In [22], there is an open problem that "In pseudo-BCK algebra or bounded pseudo-BCK algebra, Is the notion of implicative pseudo-filter equivalent to the notion of Boolean filter?" In [8], there is an open problem that "Prove or negate that pseudo-BCK algebras is implicative BCK algebras if and only if every pseudo-filters of them is implicative pseudo-filters (or Boolean filter)." In this section, inspired by [20], with the help of the similar equivalent conditions of fuzzy normal filter of pseudo-BCK algebras(pP), we present the relation between the two filters and get some results for the open problems in pseudo-BCK algebra .

Theorem 4.1. *A fuzzy pseudo-filter f of a pseudo-BCK algebra(pP) A is a fuzzy normal pseudo-filter if and only if for any $x, y \in A$, $f((x \odot y) \rightsquigarrow (y \odot x)) \wedge f((y \odot x) \rightharpoonup (x \odot y)) \geq f(y)$.*

Proof. Let f be a fuzzy normal pseudo-filter. For any $y \in A$, let $t = f(y)$, then $y \in f_t$ and f_t is a normal pseudo-filter of A. For any $x \in A$, $(y \odot x) \odot y = y \odot (x \odot y) \leq x \odot y$, since $y \leq (y \odot x) \rightsquigarrow (x \odot y)$, then $(y \odot x) \rightsquigarrow (x \odot y) \in f_t$ and $(y \odot x) \rightharpoonup (x \odot y) \in f_t$, thus $f((y \odot x) \rightharpoonup (x \odot y)) \geq t = f(y)$. Dually, we have $f((x \odot y) \rightsquigarrow (y \odot x)) \geq t = f(y)$. From above, then we obtain $f((x \odot y) \rightsquigarrow (y \odot x)) \wedge f((y \odot x) \rightharpoonup (x \odot y)) \geq f(y)$ for any $x, y \in A$. Conversely, for any $t \in [0, 1]$, if $f_t \neq \phi$, then f_t is a pseudo-filter and there exists $y \in f_t$. $f((y \odot x) \rightharpoonup (x \odot y)) \geq f(y) \geq t$, then $(y \odot x) \rightharpoonup (x \odot y) \in f_t$. $((y \odot x) \rightharpoonup (x \odot y)) \odot y = (y \rightharpoonup (x \rightharpoonup (x \odot y)) \odot y \leq x \rightharpoonup (x \odot y)$, then we have $x \rightharpoonup (x \odot y) \in f_t$, for any $x \in A$, $y \in f_t$. For any $a, b \in A$, if $a \rightsquigarrow b \in f_t$, from above, we get $a \rightharpoonup a \odot (a \rightsquigarrow b) = a \rightharpoonup b \in f_t$. Dually, we can obtain $x \rightsquigarrow (y \odot x) \in f_t$ for any $x \in A$, $y \in f_t$. For any $a, b \in A$, if $a \rightharpoonup b \in f_t$, from above, we can obtain $a \rightsquigarrow (a \rightharpoonup b) \odot a = a \rightsquigarrow b \in f_t$. Thus f_t is a normal pseudo-filter, and f is normal.

Corollary 4.2. *Every fuzzy Boolean filter of pseudo-BCK algebras (pP) is a fuzzy normal filter.*

Corollary 4.3. *Every fuzzy Boolean filter of a pseudo-BCK algebra (pP) is implicative.*

Proof. Let f be a fuzzy Boolean filter. By Definition, for any $x, y \in A$, $f((x \rightharpoonup y) \rightsquigarrow x) = f(x)$, $f((x \rightsquigarrow y) \rightharpoonup x) = f(x)$. And we obtain $f((x \rightharpoonup y) \rightharpoonup x) = f(x)$, $f((x \rightsquigarrow y) \rightsquigarrow x) = f(x)$. Thus by Definition, f is implicative.

Corollary 4.4. *Let X be a pseudo-BCK algebra (pP) and f be a fuzzy normal pseudo-filter. Then f Boolean if and only if it is implicative.*

Based on the above Theorems, we can get the following result.

Theorem 4.5. *In bounded pseudo-BCK algebra, every fuzzy implicative pseudo-filter is a fuzzy Boolean filter. In pseudo-BCK algebras(pP), every fuzzy Boolean filter is a fuzzy implicative pseudo-filter.*

For the first open problem, we can get the following theorem.

Theorem 4.6. *In bounded pseudo-BCK algebra, every implicative pseudo-filter is a Boolean filter. In pseudo-BCK algebras(pP), every Boolean filter is an implicative pseudo-filter.*

For the second open problem, we have the following results.

Theorem 4.7. *(Zhang and Gong [22]) Let* $(A; \rightharpoonup, 1)$ *be a BCK-algebra. Then A is an implicative BCK-algebra if and only if its every filter is an implicative filter.*

Theorem 4.8. *(Zhang and Gong [22]) Let* $(A; \leq, \rightharpoonup, \rightsquigarrow, 0, 1)$ *be a bounded pseudo-BCK algebra. Then A is a bounded implicative BCK-algebra if and only if its every pseudo-filter is an implicative pseudo-filter.*

Theorem 4.9. *(Zhang and Gong [22]) Let* $(A; \leq, \rightharpoonup, \rightsquigarrow, 0, 1)$ *be a bounded pseudo-BCK algebra. Then A is a bounded implicative BCK-algebra if and only if its every pseudo-filter is a Boolean filter.*

By the above results, we could get

Theorem 4.10. *Pseudo-BCK algebras(pP) is implicative BCK algebras if and only if every pseudo-filters of them is Boolean filter.*

Proof. Based on the results of [9], pseudo-*BCK* algebras is implicative *BCK* algebras if and only if $x \rightharpoonup y = x \rightsquigarrow y$ for all $x, y \in A$ and every pseudo-filters of them is implicative pseudo-filters (Boolean filter is the same), so necessity is obvious.

Now suppose every pseudo-filter F of a pseudo-*BCK* algebra (pP) A is a Boolean filter, then every fuzzy pseudo-filter χ_F of A is a fuzzy Boolean filter and χ_F is normal. Further we get that χ_F is positive implicative. So we get F is a positive implicative pseudo-filter. And by Theorem 4.5 [25] we get $(A, \rightharpoonup, 1)$ is a positive implicative *BCK*-algebra and for all $x, y \in A$, $x \rightharpoonup y = x \rightsquigarrow y$. So we get all pseudo-filters of A is positive implicative, then implicative, so we get that A is an implicative *BCK*-algebras by the result of [9].

Corollary 4.11. *Pseudo-BCK algebras (pP) is implicative BCK algebras, then every pseudo-filters of them is an implicative pseudo-filter.*

Thus the following theorem in pseudo-*BCK* algebra(pP) could answer the second problem in [8].

Theorem 4.12. *Pseudo-BCK algebras (pP) is implicative BCK algebras if and only if every pseudo-filters of them is Boolean filter (meanwhile they are implicative pseudo-filters).*

5. Conclusion

After we discuss the equivalent conditions of fuzzy normal pseudo-filter of pseudo-*BCK* algebra (pP), we propose fuzzy implicative pseudo-filter and its relation with fuzzy Boolean filter of (bounded) pseudo-*BCK* algebras (pP). Then we proposed solutions for the open problems in pseudo-*BCK* algebras. In the future, we will make use of the relations among the fuzzy pseudo-filters to solve the open problems in the common pseudo-*BCK* algebras.

Acknowledgments

This work is partially supported by the National Natural Science Foundation of China (Grant No. 11571281,61175055); the project TIN-2009-0828; Sichuan Key Technology Research and Development Program of China (Grant No. 2011FZ0051); Wireless Administration of Ministry of Industry and Information Technology of China ([2011]146); Foundation project of the National Key Laboratory of Aerospace Dynamics(Grant No. 202300006) and doctor initial fund of Xi'an Shiyou University of China (Grant No. 2011BS017).

References

[1] A.B. Saeid and S. Motamed, Normal filters in BL-algebras, *World Applied Sciences Journal* **7** (2009), 70–76.

[2] A.D. Nola, G. Georgescu and A. Iorgulescu, Pseudo-BL algebras I, II, *Multiple-Valued Logic* **8** (2002), 673–714, 717–750.

[3] A. Iorgulescu, Classes of Pseudo-BCK algebras: Part-I, *J Multiple Valued Logic and Soft Comput* **12** (2006), 71–130.

[4] B.V. Gasse, G. Deschrijver, C. Cornelis and E.E. Kerre, Filters of residuated lattices and triangle algebras, *Information Sciences* **180**(16) (2010), 3006–3020.

[5] E. Turunen, Boolean deductive systems of BL-algebras, *Arch Math Logic* **40** (2001), 467–473.

[6] G. Georgescu and A. Iorgulescu, Pseudo-BCK algebras: An extension of BCK algebras, *Proceedings of the Third International Conference on Combinatorics, Computability and Logic*, 2001, pp. 97–114.

[7] G. Georgescu, Bosbach states on fuzzy structures, *Soft Computing* **8** (2004), 217–230.

[8] H.J. Gong and X.H. Zhang, Boolean filter and psMV -filter of pseudo-BCK algebras, *Journal of Ningbo University* **24**(1) (2011), 49–53.
[9] J. Meng and Y.B. Jun, BCK-algebras, Kyung-Moon Sa Co., 1994.
[10] J.M. Zhan, W.A. Dudek and Y.B. Jun, Interval valued ($\in$; $\in \vee q$)-fuzzy filters of pseudo BL-algebras, *Soft Computing* **13** (2009), 13–21.
[11] K. Iséki and S. Imai, An introduction to the theory of BCK-algebras, *Math Japon* **23** (1978), 1–26.
[12] K. Iséki and S. Tanaka, Ideal theory of BCK-algebras, *Math Japon* **21** (1976), 351–366.
[13] K.J. Lee and C.H. Park, Some ideals of pseudo BCI-algebras, *J Appl Math & Informatics* **27** (2009), 217–231.
[14] L.A. Zadeh, Fuzzy sets, *Information and Control* **8** (1965), 338–353.
[15] L.Z. Liu and K.T. Li, Fuzzy filters of BL-algebras, *Information Sciences* **173** (2005), 141–154.
[16] L.Z. Liu and K.T. Li, Fuzzy Boolean and positive implicative filters of BL-algebras, *Fuzzy Sets and Systems* **152** (2005), 333–348.
[17] M. Haveshki, A. Borumand Saeid and E. Eslami, Some types of filters in BL algebras, *Soft Computing* **10** (2006), 657–664.
[18] P. Hájek, *Metamathematics of fuzzy logic*, Kluwer Academic Publisher, 1998.
[19] P. Hájek, Observations on non-commutative fuzzy logic, *Soft Computing* **8** (2003), 38–43.
[20] W. Wang and X.L. Xin, On fuzzy filters of pseudo BL-algebras, *Fuzzy Sets and Systems* **162** (2011), 27–38.
[21] X.H. Zhang, *Fuzzy logic and its algebric analysis*, Science Press, 2008.
[22] X.H. Zhang and H.J. Gong, Implicative pseudo-BCK algebras and implicative pseudo-Filters of pseudo-BCK algebras, *Proceedings of IEEE International Conference on Granular Computing*, 2010, pp. 615–619.
[23] X.H. Zhang and W.H. Li, On pseudo-BL algebras and BCC-algebras, *Soft Computing* **10** (2006), 941–952.
[24] X.H. Zhang and X.S. Fan, Pseudo-BL algebras and pseudoeffect algebras, *Fuzzy Sets and Systems* **159**(1) (2008), 95–106.
[25] X.H. Zhang and Y.B. Jun, Solution of Three Open Problems on Pseudo-Filters (Pseudo-Ideals) of Pseudo-BCK Algebras, *Proceedings of International Conference on Artificial Intelligence and Computational Intelligence*, 2011, pp. 530–534.
[26] X.L. Ma, J.M. Zhan and K.P. Shum, Interval valued ($\in$; $\in \vee q$)-fuzzy filters of MTL-algebras, *Journal of Mathematical Research & Exposition* **30** (2010), 265–276.
[27] Y.B. Jun, Fuzzy positive implicative and fuzzy associative filters of lattice implication algebras, *Fuzzy Sets and Systems* **121** (2001), 353–357.
[28] Y.B. Jun and S.Z. Song, On fuzzy implicative filters of lattice implication algebras, *The Journal of Fuzzy Mathematics* **10** (2002), 893–900.
[29] Y.B. Jun, Y. Xu and X.H. Zhang, Fuzzy filters of MTL-algebras, *Information Sciences* **175** (2005), 120–138.
[30] Y. Xu and K.Y. Qin, On filters of lattice implication algebras, *The Journal of Fuzzy Mathematics* **1** (1993), 251–260.