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A relaxation result in $BV \times L^p$ for integral functionals depending on chemical composition and elastic strain

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Abstract. An integral representation result is obtained for the relaxation of a class of energy functionals depending on two vector fields with different behaviors which appear in the context of thermochemical equilibria and are related to image decomposition models and directors theory in nonlinear elasticity.

Keywords: relaxation, convexity-quasiconvexity

1. Introduction

In this paper we consider energies depending on two vector fields with different behaviors: $u \in W^{1,1}(\Omega; \mathbb{R}^n)$ and $v \in L^p(\Omega; \mathbb{R}^m)$, Ω being a bounded open subset of $\mathbb{R}^N$.

Let $1 \leqslant p \leqslant \infty$ and for every $(u, v) \in W^{1,1}(\Omega; \mathbb{R}^n) \times L^p(\Omega; \mathbb{R}^m)$ define the functional

$$J(u, v) := \int_\Omega f(v, \nabla u)\, dx, \tag{1.1}$$

where $f : \mathbb{R}^m \times \mathbb{R}^{n\times N} \to [0, \infty)$ is a continuous function.

Minimization of energies depending on two independent vector fields have been introduced to model several phenomena. For instance the case of thermochemical equilibria among multiphase multicomponent solids and Cosserat theories in the context of elasticity: we refer to [7,9] and the references therein for a detailed explanation about this kind of applications.

In the Sobolev setting, after the pioneer works [7,9], relaxation with a Carathéodory density $f \equiv f(x, u, \nabla u, v)$, and homogenization for density of the type $f(\frac{x}{\varepsilon}, \nabla u, v)$ have been considered in [5] and [6], respectively.

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In the present paper we are interested in studying the lower semicontinuity and relaxation of (1.1) with respect to the L^1-strong $\times L^p$-weak convergence ($p > 1$). Clearly, bounded sequences $\{u_h\} \subset W^{1,1}(\Omega; \mathbb{R}^n)$ may converge in L^1, up to a subsequence, to a BV function.

In the BV-setting this question has been already addressed in [8], only when the density f is convex–quasiconvex (see (2.2)) and the vector field $v \in L^\infty(\Omega; \mathbb{R}^m)$.

Here we allow v to be in $L^p(\Omega; \mathbb{R}^m)$, $p > 1$ and f is not necessarily convex–quasiconvex. We provide an argument alternative to the one in [8, Section 4], devoted to clarify some points in the lower semicontinuity result therein.

We also emphasize that under specific restrictions on the density f, i.e. $f(x, u, v, \nabla u) \equiv W(x, u, \nabla u) + \varphi(x, u, v)$, the analysis in the case $1 < p < \infty$ was considered already in [10] in order to describe image decomposition models. In [11] a general f was taken into account when the target u is in $W^{1,1}(\Omega; \mathbb{R}^n)$.

In this manuscript we consider $f \equiv f(b, \xi)$, $(b, \xi) \in \mathbb{R}^m \times \mathbb{R}^{n\times N}$ and the target $u \in BV(\Omega; \mathbb{R}^n)$.

We study separately the cases $1 < p < \infty$, $p = \infty$ and discuss briefly the case $p = 1$ in the Appendix. Comparing the results in [5] (where the assumptions allow to invoke De La Vallé–Poussin Criterion) with the true linear growth setting as in [3].

To this end, we introduce for $1 < p < \infty$ the functional

$$\overline{J}_p(u, v) := \inf\Big\{\liminf_{h\to\infty} J(u_h, v_h) : u_h \in W^{1,1}\big(\Omega; \mathbb{R}^n\big), v_h \in L^p\big(\Omega; \mathbb{R}^m\big), \\ u_h \to u \text{ in } L^1, v_h \rightharpoonup v \text{ in } L^p\Big\}, \tag{1.2}$$

for any pair $(u, v) \in BV(\Omega; \mathbb{R}^n) \times L^p(\Omega; \mathbb{R}^m)$ and, for $p = \infty$ the functional

$$\overline{J}_\infty(u, v) := \inf\Big\{\liminf_{h\to\infty} J(u_h, v_h) : u_h \in W^{1,1}\big(\Omega; \mathbb{R}^n\big), v_h \in L^\infty\big(\Omega; \mathbb{R}^m\big), \\ u_h \to u \text{ in } L^1, v_h \overset{*}{\rightharpoonup} v \text{ in } L^\infty\Big\}, \tag{1.3}$$

for any pair $(u, v) \in BV(\Omega; \mathbb{R}^n) \times L^\infty(\Omega; \mathbb{R}^m)$.

Since bounded sequences $\{u_h\}$ in $W^{1,1}(\Omega; \mathbb{R}^n)$ converge in L^1 to a BV function u and bounded sequences $\{v_h\}$ in $L^p(\Omega; \mathbb{R}^m)$ if $1 < p < \infty$ (in $L^\infty(\Omega; \mathbb{R}^m)$ if $p = \infty$), weakly converge to a function $v \in L^p(\Omega; \mathbb{R}^m)$ (weakly $*$ in L^∞), the relaxed functionals $\overline{J}_p$ and $\overline{J}_\infty$ will be composed by an absolutely continuous part and a singular one with respect to the Lebesgue measure (see (2.12)). On the other hand, as already emphasized in [8], it is crucial to observe that v, regarded as a measure, is absolutely continuous with respect to the Lebesgue one, besides it is not defined on the singular sets of u, namely in those sets where the singular part with respect the Lebesgue measure of the distributional gradient of u, $D^s u$, is concentrated. Thus specific features of the density f will come into play to ensure a proper integral representation.

The integral representation of (1.2) will be achieved in Theorem 1.1 under the following hypotheses:

$(H_1)_p$ There exists $C > 0$ such that

$$\frac{1}{C}\big(|b|^p + |\xi|\big) - C \leqslant f(b, \xi) \leqslant C\big(1 + |b|^p + |\xi|\big),$$

for $(b, \xi) \in \mathbb{R}^m \times \mathbb{R}^{n\times N}$.

$(H_2)_p$ There exists $C' > 0$, $L > 0$, $0 < \tau \leqslant 1$ such that

$$t > 0, \xi \in \mathbb{R}^{n\times N},$$
$$\text{with } t|\xi| > L \Longrightarrow \left|\frac{f(b,t\xi)}{t} - f^\infty(b,\xi)\right| \leqslant C'\left(\frac{|b|^p+1}{t} + \frac{|\xi|^{1-\tau}}{t^\tau}\right),$$

where f^∞ is the recession function of f defined for every $b \in \mathbb{R}^m$ as

$$f^\infty(b,\xi) := \limsup_{t\to\infty} \frac{f(b,t\xi)}{t}. \tag{1.4}$$

In order to characterize the functional $\overline{J}_\infty$ introduced in (1.3) we will replace assumptions $(H_1)_p$ and $(H_2)_p$ by the following ones:

$(H_1)_\infty$ Given $M > 0$, there exists $C_M > 0$ such that, if $|b| \leqslant M$ then

$$\frac{1}{C_M}|\xi| - C_M \leqslant f(b,\xi) \leqslant C_M\big(1+|\xi|\big),$$

for every $\xi \in \mathbb{R}^{n\times N}$.

$(H_2)_\infty$ Given $M > 0$, there exist $C'_M > 0$, $L > 0$, $0 < \tau \leqslant 1$ such that

$$|b| \leqslant M, t > 0, \xi \in \mathbb{R}^{n\times N}, \quad \text{with } t|\xi| > L \Longrightarrow \left|\frac{f(b,t\xi)}{t} - f^\infty(b,\xi)\right| \leqslant C'_M\frac{|\xi|^{1-\tau}}{t^\tau}.$$

Section 2 is devoted to notations, preliminaries about measure theory and some properties of the energy densities. In particular, we stress that a series of results is presented in order to show all the properties and relations among the relaxed energy densities involved in the integral representation and that can be of further use for the interested readers since they often appear in the integral representation context. Section 3 contains the arguments necessary to prove the main results stated below.

Theorem 1.1. *Let J be given by (1.1), with f satisfying $(H_1)_p$ and $(H_2)_p$ and let $\overline{J}_p$ be given by (1.2) then*

$$\overline{J}_p(u,v) = \int_\Omega \mathcal{CQ}f(v,\nabla u)\,dx + \int_\Omega (\mathcal{CQ}f)^\infty\left(0, \frac{dD^s u}{d|D^s u|}\right) d\big|D^s u\big|,$$

for every $(u,v) \in BV(\Omega;\mathbb{R}^n) \times L^p(\Omega;\mathbb{R}^m)$.

We denote by $\mathcal{CQ}f$ the convex–quasiconvex envelope of f in (2.5) and $(\mathcal{CQ}f)^\infty$ represents the recession function of $\mathcal{CQ}f$, defined according to (1.4), which coincides, under suitable assumptions (see assumptions (2.6), (2.7), Proposition 2.12 and Remark 2.13), with the convex–quasiconvex envelope of f^∞, $\mathcal{CQ}(f^\infty)$, and this allows us to remove the parenthesis.

For the case $p = \infty$ we have the following.

Theorem 1.2. *Let J be given by (1.1), with f satisfying $(H_1)_\infty$ and $(H_2)_\infty$ and let $\overline{J}_\infty$ be given by (1.3) then*

$$\overline{J}_\infty(u,v) = \int_\Omega \mathcal{C}\mathcal{Q}f(v,\nabla u)\,dx + \int_\Omega (\mathcal{C}\mathcal{Q}f)^\infty\left(0, \frac{dD^s u}{d|D^s u|}\right) d\left|D^s u\right|,$$

for every $(u,v) \in BV(\Omega;\mathbb{R}^n) \times L^\infty(\Omega;\mathbb{R}^m)$.

For the case $1 < p < \infty$, the proof of the lower bound is presented in Theorem 3.1 while the upper bound is in Theorem 3.2, both under the extra hypothesis

(H_0) f is convex–quasiconvex.

The case $p = \infty$ is discussed in Section 3.2. Furthermore, we observe that Proposition 2.14 in Section 2.3 is devoted to remove the convexity-quasiconvexity assumption on f.

2. Notations, preliminaries and properties of the energy densities

In this section, we start by establishing notations, recalling some preliminary results on measure theory that will be useful through the paper and finally we recall the space of functions of bounded variation.

Then we deduce the main properties of convex–quasiconvex functions, recession functions and related envelopes.

If $\nu \in \mathbb{S}^{N-1}$ and $\{\nu, \nu_2, \dots, \nu_N\}$ is an orthonormal basis of $\mathbb{R}^N$, Q_ν denotes the unit cube centered at the origin with its faces either parallel or orthogonal to $\nu, \nu_2, \dots, \nu_N$. If $x \in \mathbb{R}^N$ and $\rho > 0$, we set $Q(x,\rho) := x + \rho\, Q$ and $Q_\nu(x,\rho) := x + \rho\, Q_\nu$, Q is the cube $(-\frac{1}{2}, \frac{1}{2})^N$.

Let Ω be a generic open subset of $\mathbb{R}^N$, we denote by $\mathcal{M}(\Omega)$ the space of all signed Radon measures in Ω with bounded total variation. By the Riesz Representation Theorem, $\mathcal{M}(\Omega)$ can be identified to the dual of the separable space $C_0(\Omega)$ of continuous functions on Ω vanishing on the boundary $\partial\Omega$. The N-dimensional Lebesgue measure in $\mathbb{R}^N$ is designated as $\mathcal{L}^N$.

If $\mu \in \mathcal{M}(\Omega)$ and $\lambda \in \mathcal{M}(\Omega)$ is a nonnegative Radon measure, we denote by $\frac{d\mu}{d\lambda}$ the Radon–Nikodým derivative of μ with respect to λ. By a generalization of the Besicovich Differentiation Theorem (see [1, Proposition 2.2]), it can be proved that there exists a Borel set $E \subset \Omega$ such that $\lambda(E) = 0$ and

$$\frac{d\mu}{d\lambda}(x) = \lim_{\rho\to 0^+} \frac{\mu(x+\rho C)}{\lambda(x+\rho\, C)}$$

for all $x \in \operatorname{Supp}\lambda \setminus E$ and any open bounded convex set C containing the origin.

We recall that the exceptional set E above does not depend on C. An immediate corollary is the generalization of Lebesgue–Besicovitch Differentiation Theorem given below.

Theorem 2.1. *If μ is a nonnegative Radon measure and if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^N,\mu)$ then*

$$\lim_{\varepsilon\to 0^+} \frac{1}{\mu(x+\varepsilon C)} \int_{x+\varepsilon C} |f(y) - f(x)|\, d\mu(y) = 0$$

for μ-a.e. $x \in \mathbb{R}^N$ and for every, bounded, convex, open set C containing the origin.

Definition 2.2. A function $u \in L^1(\Omega;\mathbb{R}^n)$ is said to be of *bounded variation*, and we write $u \in BV(\Omega;\mathbb{R}^n)$, if all its first distributional derivatives, $D_j u_i$, belong to $\mathcal{M}(\Omega)$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant N$.

The matrix-valued measure whose entries are $D_j u_i$ is denoted by Du and $|Du|$ stands for its total variation. We observe that if $u \in BV(\Omega;\mathbb{R}^n)$ then $u \mapsto |Du|(\Omega)$ is lower semicontinuous in $BV(\Omega;\mathbb{R}^n)$ with respect to the $L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ topology.

By the Lebesgue Decomposition Theorem we can split Du into the sum of two mutually singular measures $D^a u$ and $D^s u$, where $D^a u$ is the absolutely continuous part and $D^s u$ is the singular part of Du with respect to the Lebesgue measure $\mathcal{L}^N$. By ∇u we denote the Radon–Nikodým derivative of $D^a u$ with respect to the Lebesgue measure so that we can write

$$Du = \nabla u \mathcal{L}^N + D^s u.$$

Proposition 2.3. *If $u \in BV(\Omega;\mathbb{R}^n)$ then for $\mathcal{L}^N$-a.e. $x_0 \in \Omega$*

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\left\{\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)} \left|u(x) - u(x_0) - \nabla u(x_0)\cdot(x-x_0)\right|^{\frac{N}{N-1}}dx\right\}^{\frac{N-1}{N}} = 0. \tag{2.1}$$

For more details regarding functions of bounded variation we refer to [2].

2.1. *Convex-quasiconvex functions*

We start by recalling the notion of convex–quasiconvex function, presented in [8] (see also [7] and [9]).

Definition 2.4. A Borel measurable function $f : \mathbb{R}^m \times \mathbb{R}^{n\times N} \to \mathbb{R}$ is said to be convex–quasiconvex if, for every $(b,\xi) \in \mathbb{R}^m \times \mathbb{R}^{n\times N}$, there exists a bounded open set D of $\mathbb{R}^N$ such that

$$f(b,\xi) \leqslant \frac{1}{|D|}\int_D f\big(b+\eta(x), \xi + \nabla\varphi(x)\big)\,dx, \tag{2.2}$$

for every $\eta \in L^\infty(D;\mathbb{R}^m)$, with $\int_D \eta(x)\,dx = 0$, and for every $\varphi \in W_0^{1,\infty}(D;\mathbb{R}^n)$.

Remark 2.5.

i) It can be easily seen that, if f is convex–quasiconvex then condition (2.2) is true for any bounded open set $D \subset \mathbb{R}^N$.
ii) A convex–quasiconvex function is separately convex.
iii) By [11, Proposition 3], the growth condition from above in $(H_1)_p$, ii), entail that there exists $\gamma > 0$ such that

$$\left|f(b,\xi) - f\big(b',\xi'\big)\right| \leqslant \gamma\Big(\left|\xi-\xi'\right| + \big(1+|b|^{p-1}+\left|b'\right|^{p-1}+|\xi|^{\frac{1}{p'}}+\left|\xi'\right|^{\frac{1}{p'}}\big)\left|b-b'\right|\Big) \tag{2.3}$$

for every $b, b' \in \mathbb{R}^m$, $\xi, \xi' \in \mathbb{R}^{n\times N}$, where $p > 1$ and p' its conjugate exponent.

iv) By [11, Proposition 4]), under the growth assumptions in $(H_1)_\infty$, ii) entails that, given $M > 0$ there exists a constant $\beta(M, n, m, N)$ such that

$$\left|f(b,\xi) - f\left(b',\xi'\right)\right| \leqslant \beta\left(1 + |\xi| + \left|\xi'\right|\right)\left|b - b'\right| + \beta\left|\xi - \xi'\right| \tag{2.4}$$

for every $b, b' \in \mathbb{R}^m$, such that $|b| \leqslant M$ and $|b'| \leqslant M$, for every $\xi, \xi' \in \mathbb{R}^{n\times N}$.

We introduce the notion of convex–quasiconvex envelope of a function, which is crucial to deal with the relaxation procedure.

Definition 2.6. Let $f : \mathbb{R}^m \times \mathbb{R}^{n\times N} \to \mathbb{R}$ be a Borel measurable function bounded from below. The convex–quasiconvex envelope is the largest convex–quasiconvex function below f, i.e.,

$$\mathcal{CQ}f(b,\xi) := \sup\left\{g(b,\xi) : g \leqslant f, g \text{ convex–quasiconvex}\right\}.$$

By Theorem 4.16 in [9], the convex–quasiconvex envelope coincides with the so called convex-quasiconvexification

$$\begin{aligned} \mathcal{CQ}f(b,\xi) = \inf\bigg\{ &\frac{1}{|D|}\int_D f\big(b + \eta(x), \xi + \nabla\varphi(x)\big)\,dx : \eta \in L^\infty\big(D;\mathbb{R}^m\big), \\ &\int_D \eta(x)\,dx = 0, \varphi \in W_0^{1,\infty}\big(D;\mathbb{R}^n\big)\bigg\}. \end{aligned} \tag{2.5}$$

As for convexity-quasiconvexity, condition (2.5) can be stated for any bounded open set $D \subset \mathbb{R}^N$. It can also be showed that if f satisfies a growth condition of type $(H_1)_p$ then in (2.2) and (2.5) the spaces L^∞ and $W_0^{1,\infty}$ can be replaced by L^p and $W_0^{1,1}$, respectively.

The following proposition, that will be exploited in the sequel, can be found in [11, Proposition 5]. The proof is omitted since it is very similar to [10, Proposition 2.1].

Proposition 2.7. *Let $f : \mathbb{R}^m \times \mathbb{R}^{n\times N} \to [0,\infty)$ be a continuous function satisfying $(H_1)_p$. Then $\mathcal{CQ}f$ is continuous and satisfies $(H_1)_p$. Consequently, $\mathcal{CQ}f$ satisfies (2.3).*

In order to deal with $v \in L^\infty(\Omega;\mathbb{R}^m)$ and to compare with the result in $BV \times L^p$, $1 < p < \infty$, one can consider a different setting of assumptions on the energy density f.

Namely, following [11, Proposition 6 and Remark 7], if $\alpha : [0,\infty) \to [0,\infty)$ is a convex and increasing function, such that $\alpha(0) = 0$ and if $f : \mathbb{R}^m \times \mathbb{R}^{n\times N} \to [0,\infty)$ is a continuous function satisfying

$$\frac{1}{C}\big(\alpha\big(|b|\big) + |\xi|\big) - C \leqslant f(b,\xi) \leqslant C\big(1 + \alpha\big(|b|\big) + |\xi|\big) \tag{2.6}$$

for every $(b,\xi) \in \mathbb{R}^m \times \mathbb{R}^{n\times N}$, then $\mathcal{CQ}f$ satisfies a condition analogous to (2.6). Moreover, $\mathcal{CQ}f$ is a continuous function.

Analogously, one can assume that f satisfies the following variant of $(H_2)_\infty$: there exist $c' > 0$, $L > 0$, $0 < \tau \leqslant 1$ such that

$$t > 0, \xi \in \mathbb{R}^{n\times N}, \quad \text{with } t|\xi| > L \Longrightarrow \left|\frac{f(b, t\xi)}{t} - f^\infty(b, \xi)\right| \leqslant c'\left(\frac{\alpha(|b|) + 1}{t} + \frac{|\xi|^{1-\tau}}{t^\tau}\right). \tag{2.7}$$

We observe that, if from one hand (2.6) and (2.7) generalize $(H_1)_p$ and $(H_2)_p$ respectively, from the other hand they can be regarded also as a stronger version of $(H_1)_\infty$ and $(H_2)_\infty$, respectively.

2.2. *The recession function*

Let $f : \mathbb{R}^m \times \mathbb{R}^{n\times N} \to [0, \infty[$, and let $f^\infty : \mathbb{R}^m \times \mathbb{R}^{n\times N} \to [0, \infty[$ be its recession function, defined in (1.4).

The following properties are an easy consequence of the definition of recession function and conditions (H_0), $(H_1)_p$ and $(H_2)_p$, when $1 < p < \infty$.

Proposition 2.8. *Provided f satisfies (H_0), $(H_1)_p$ and $(H_2)_p$, then*

1. *f^∞ is convex–quasiconvex;*
2. *there exists $C > 0$ such that*

$$\frac{1}{C}|\xi| \leqslant f^\infty(b, \xi) \leqslant C|\xi|; \tag{2.8}$$

3. *$f^\infty(b, \xi)$ is constant with respect to b for every $\xi \in \mathbb{R}^{n\times N}$;*
4. *f^∞ is continuous.*

Remark 2.9. We emphasize that not all the assumptions $(H_1)_p$ and $(H_2)_p$ in Proposition 2.8 are necessary to prove items above. In particular, one has that:

i) The proof of 2. uses only the fact that f satisfies $(H_1)_p$.
ii) To prove 3. it is necessary to require that f satisfies only (H_0) and $(H_1)_p$. Indeed if it satisfies (2.3) one can avoid to require (H_0).

Proof.

1. The convexity-quasiconvexity of f^∞ can be proven exactly as in [8, Lemma 2.1], exploiting the growth condition $(H_1)_p$ and the estimate given by $(H_2)_p$.
2. By definition (1.4) we may find a subsequence $\{t_k\}$ such that

$$f^\infty(b, \xi) = \lim_{t_k\to\infty} \frac{f(b, t_k\xi)}{t_k}.$$

By $(H_1)_p$ one has

$$f^\infty(b, \xi) \leqslant \lim_{t_k\to\infty} \frac{C(1 + |b|^p + |t_k\xi|)}{t_k} = C|\xi|$$

and

$$f^\infty(b,\xi) \geqslant \lim_{t_k\to\infty} \frac{\frac{1}{C}(|b|^p + |t_k\xi|) - C}{t_k} \geqslant \frac{1}{C}|\xi|.$$

Hence $(H_1)_p$ holds for f^∞.

3. Let $\xi \in \mathbb{R}^{n\times N}$, and let $b, b' \in \mathbb{R}^m$, up to a subsequence, by (1.4) and the fact that f satisfies (2.3) it results that,

$$f^\infty(b,\xi) - f^\infty\big(b',\xi\big) \leqslant \lim_{t_k\to\infty} \frac{f(b,t_k\xi) - f(b',t_k\xi)}{t_k}$$
$$\leqslant \lim_{t_k\to\infty} \frac{\gamma(1 + |b|^{p-1} + |b'|^{p-1} + |t_k\xi|^{\frac{1}{p'}})|b-b'|}{t_k} = 0.$$

By interchanging the role of b and b', it follows that $f^\infty(\cdot,\xi)$ is constant and this concludes the proof.

4. The continuity is a consequence of the growth conditions and the convexity-quasiconvexity of f^∞.

□

Remark 2.10. Under assumptions (H_0), $(H_1)_\infty$ and $(H_2)_\infty$, f^∞ satisfies properties analogous to those at the beginning of Section 2.2. In particular in [8, Lemma 2.1 and Lemma 2.2] it has been proved that

i) f^∞ is convex–quasiconvex;
ii) $\frac{1}{C_M}|\xi| \leqslant f^\infty(b,\xi) \leqslant C_M|\xi|$, for every b, with $|b| \leqslant M$;
iii) If $\operatorname{rank}\xi \leqslant 1$, then $f^\infty(b,\xi)$ is constant with respect to b.

Remark 2.11. We observe that, if $f : \mathbb{R}^m \times \mathbb{R}^{n\times N} \to [0,\infty)$ is a continuous function satisfying $(H_1)_p$ and $(H_2)_p$, then the function $(\mathcal{CQ}f)^\infty : \mathbb{R}^m \times \mathbb{R}^{n\times N} \to [0,\infty[$, obtained first taking the convex-quasiconvexification in (2.5) of f and then its recession through formula (1.4) applied to $\mathcal{CQ}f$, satisfies the following properties:

1. $(\mathcal{CQ}f)^\infty$ is convex–quasiconvex;
2. there exists $C > 0$ such that $\frac{1}{C}|\xi| \leqslant (\mathcal{CQ}f)^\infty(b,\xi) \leqslant C|\xi|$, for every $(b,\xi) \in \mathbb{R}^m \times \mathbb{R}^{n\times N}$;
3. for every $\xi \in \mathbb{R}^{n\times N}$, $(\mathcal{CQ}f)^\infty(\cdot,\xi)$ is constant, i.e. $(\mathcal{CQ}f)^\infty$ is independent on b;
4. $(\mathcal{CQ}f)^\infty$ is Lipschitz continuous in ξ.

Assuming that f satisfies $(H_1)_p$, one can prove that the convex-quasiconvexification of f^∞, $\mathcal{CQ}(f^\infty)$, satisfies the following conditions:

5. $\mathcal{CQ}(f^\infty)$ is convex–quasiconvex;
6. there exists $C > 0$ such that $\frac{1}{C}|\xi| \leqslant \mathcal{CQ}(f^\infty)(b,\xi) \leqslant C|\xi|$, for every $(b,\xi) \in \mathbb{R}^m \times \mathbb{R}^{n\times N}$;
7. for every $\xi \in \mathbb{R}^{n\times N}$, and assuming that f satisfies (2.3), $\mathcal{CQ}(f^\infty)(\cdot,\xi)$ is constant, i.e. $\mathcal{CQ}(f^\infty)$ is independent on b;
8. $\mathcal{CQ}(f^\infty)$ is Lipschitz continuous in ξ.

The above properties are immediate consequences of Propositions 2.7, 2.8 and (2.3). In particular 8. follows from 3. of Proposition 2.8, without requiring $(H_2)_p$.

On the other hand, Proposition 2.12 below entails that $\mathcal{CQ}(f)^\infty$ is independent on b, without requiring that f is Lipschitz continuous, but replacing this assumption with $(H_2)_p$.

We also observe that $(\mathcal{CQ}f)^\infty$ and $\mathcal{CQ}(f^\infty)$ are only quasiconvex functions, since they are independent of b. In particular, in our setting, these functions coincide as it is stated below.

Proposition 2.12. *Let $f : \mathbb{R}^m \times \mathbb{R}^{n\times N} \to [0,\infty)$ be a continuous function satisfying $(H_1)_p$ and $(H_2)_p$. Then*

$$\mathcal{CQ}\big(f^\infty\big)(b,\xi) = (\mathcal{CQ}f)^\infty(b,\xi) \quad \textit{for every } (b,\xi) \in \mathbb{R}^m \times \mathbb{R}^{n\times N}.$$

Proof. The proof will be achieved by double inequality.

For every $(b,\xi) \in \mathbb{R}^m \times \mathbb{R}^{n\times N}$ the inequality

$$(\mathcal{CQ}f)^\infty(b,\xi) \leqslant \mathcal{CQ}\big(f^\infty\big)(b,\xi) \tag{2.9}$$

follows by Definition 2.6, and the fact that $\mathcal{CQ}f(b,\xi) \leqslant f(b,\xi)$. In fact, (1.4) entails that the same inequality holds when, passing to $(\cdot)^\infty$. Finally, 1. in Proposition 2.8, guarantees (2.9).

In order to prove the opposite inequality, fix $(b,\xi) \in \mathbb{R}^m \times \mathbb{R}^{n\times N}$ and, for every $t > 1$, take $\eta_t \in L^\infty(Q;\mathbb{R}^m)$, with 0 average, and $\varphi_t \in W_0^{1,\infty}(Q;\mathbb{R}^n)$ such that

$$\int_Q f\big(b+\eta_t, t\xi + \nabla\varphi_t(y)\big)\,dy \leqslant \mathcal{CQ}f(b,t\xi) + 1. \tag{2.10}$$

By $(H_1)_p$ and Proposition 2.7, we have that $\|b+\eta_t\|_{L^p(Q)}, \|\nabla(\frac{1}{t}\varphi_t)\|_{L^1(Q)} \leqslant C$ for a constant independent on t. Defining $\psi_t := \frac{1}{t}\varphi_t$, one has $\psi_t \in W_0^{1,\infty}(Q;\mathbb{R}^n)$ and thus

$$\mathcal{CQ}\big(f^\infty\big)(b,\xi) \leqslant \int_Q f^\infty\big(b+\eta_t, \xi + \nabla\psi_t(y)\big)\,dy.$$

Let L be the constant appearing in condition $(H_2)_p$. We split the cube Q into the set $\{y \in Q : t|\xi + \nabla\psi_t(y)| \leqslant L\}$ and its complement in Q. Then we apply condition $(H_2)_p$ and (2.8) to get

$$\mathcal{CQ}\big(f^\infty\big)(b,\xi) \leqslant \int_Q \left(C\frac{1+|b+\eta_t|^p}{t} + C\frac{|\xi+\nabla\psi_t|^{1-\tau}}{t^\tau} + \frac{f(b+\eta_t, t\xi+\nabla\varphi_t)}{t} + C\frac{L}{t}\right) dy.$$

Applying Hölder inequality and (2.10), we get

$$\mathcal{CQ}\big(f^\infty\big)(b,\xi) \leqslant \frac{C}{t^\tau}\left(\int_Q |\xi + \nabla\psi_t|\,dy\right)^{1-\tau} + \frac{\mathcal{CQ}f(b,t\xi)+1}{t} + C\frac{L}{t} + \frac{C'}{t},$$

and the desired inequality follows by definition of $(\mathcal{CQ}f)^\infty$ and using the fact that $\nabla\psi_t$ is bounded in L^1 norm, letting t go to ∞. □

Remark 2.13. It is worth to observe that inequality

$$\big(\mathcal{C}\mathcal{Q}f^\infty\big)(b,\xi) \leqslant \mathcal{C}\mathcal{Q}\big(f^\infty\big)(b,\xi) \quad \text{for every } (b,\xi) \in \mathbb{R}^m \times \mathbb{R}^{n\times N},$$

has been proven without requiring neither $(H_1)_p$ and $(H_2)_p$ on f, nor $(H_1)_\infty$ and $(H_2)_\infty$.

Furthermore, we emphasize that the proof of Proposition 2.12 cannot be performed in the same way in the case $p=\infty$, with assumptions $(H_1)_p$ and $(H_2)_p$ replaced by $(H_1)_\infty$ and $(H_2)_\infty$. Indeed, an L^∞ bound on $b+\eta_t$ analogous to the one in L^p cannot be obtained from $(H_1)_\infty$. On the other hand, it is possible to deduce the equality between $\mathcal{C}\mathcal{Q}f^\infty$ and $(\mathcal{C}\mathcal{Q}f)^\infty$, when f satisfies (2.6) and (2.7).

2.3. *Auxiliary results*

Here we prove that assumption (H_0) on f is not necessary to provide an integral representation of $\overline{J}_p$ as in (1.2). Indeed, we can assume that $f:\mathbb{R}^m \times \mathbb{R}^{n\times N} \to [0,\infty[$ is a continuous function and satisfies assumptions $(H_1)_p$ and $(H_2)_p$, $(p \in (1,\infty])$. First we extend, with an abuse of notation, the functional J in (1.1), to $L^1(\Omega;\mathbb{R}^n) \times L^p(\Omega;\mathbb{R}^m)$, $p \in (1,\infty]$, as

$$J(u,v) := \begin{cases} \int_\Omega f(v,\nabla u)\,dx & \text{if } (u,v) \in W^{1,1}(\Omega;\mathbb{R}^n) \times L^p(\Omega;\mathbb{R}^m), \\ \infty & \text{otherwise.} \end{cases} \tag{2.11}$$

Then we define the functional

$$J_{\mathcal{C}\mathcal{Q}f}(u,v) := \begin{cases} \int_\Omega \mathcal{C}\mathcal{Q}f(v,\nabla u)\,dx & \text{if } (u,v) \in W^{1,1}(\Omega;\mathbb{R}^n) \times L^p(\Omega;\mathbb{R}^m), \\ \infty & \text{otherwise,} \end{cases}$$

$(p \in (1,\infty])$ where $\mathcal{C}\mathcal{Q}f$ is given by Definition 2.6 and,

$$\overline{J_{\mathcal{C}\mathcal{Q}f}}_p(u,v) := \inf\Big\{\liminf_{h\to\infty} J_{\mathcal{C}\mathcal{Q}f}(u_h,v_h) : u_h \in W^{1,1}\big(\Omega;\mathbb{R}^n\big),\, v_h \in L^p\big(\Omega;\mathbb{R}^m\big),$$
$$u_h \to u \text{ in } L^1,\, v_h \rightharpoonup v \text{ in } L^p\Big\},$$

for any pair $(u,v) \in BV(\Omega;\mathbb{R}^n) \times L^p(\Omega;\mathbb{R}^m)$, $p \in (1,\infty)$. Analogously, one can consider

$$\overline{J_{\mathcal{C}\mathcal{Q}f}}_\infty(u,v) := \inf\Big\{\liminf_{h\to\infty} J_{\mathcal{C}\mathcal{Q}f}(u_h,v_h) : u_h \in W^{1,1}\big(\Omega;\mathbb{R}^n\big),\, v_h \in L^p\big(\Omega;\mathbb{R}^m\big),$$
$$u_h \to u \text{ in } L^1,\, v_h \overset{*}{\rightharpoonup} v \text{ in } L^\infty\Big\},$$

for any pair $(u,v) \in BV(\Omega;\mathbb{R}^n) \times L^\infty(\Omega;\mathbb{R}^m)$.

Clearly, it results that for every $(u,v) \in BV(\Omega;\mathbb{R}^n) \times L^p(\Omega;\mathbb{R}^m)$,

$$\overline{J_{\mathcal{C}\mathcal{Q}f}}_p(u,v) \leqslant \overline{J}_p(u,v),$$

but, as in [11, Lemma 8 and Remark 9], the following proposition can be proven.

Proposition 2.14. *Let $p \in (1,\infty]$ and consider the functionals J and J_{CQf} and their corresponding relaxed functionals $\overline{J}_p$ and $\overline{J_{\mathcal{CQ}f}}_p$. If f satisfies conditions $(H_1)_p$ and $(H_2)_p$ if $p \in (1,\infty)$, and both f and $\mathcal{CQ}f$ satisfy $(H_1)_\infty$ and $(H_2)_\infty$ if $p = \infty$, then*

$$\overline{J}_p(u,v) = \overline{J_{\mathcal{CQ}f}}_p(u,v)$$

for every $(u,v) \in BV(\Omega;\mathbb{R}^n) \times L^p(\Omega;\mathbb{R}^m)$, $p \in (1,\infty]$.

Remark 2.15. The proof is omitted since it can be performed as in [11, Lemma 8 and Remark 9]. In [11] it is not required that f satisfies $(H_2)_p$, $(p \in (1,\infty])$. Indeed, the equality $\overline{J}_p = \overline{J_{\mathcal{CQ}f}}_p$ holds independently on this assumption on f, but in order to remove hypothesis (H_0) from the representation theorem we need to assume that $\mathcal{CQ}f$ inherits the same properties as f, which is the case as it has been observed in Proposition 2.7. It is also worth to observe that, when $p = \infty$, (2.7) is equivalent to

$$\left|f^\infty(b,\xi) - f(b,\xi)\right| \leqslant C\big(1 + \alpha\big(|b|\big) + |\xi|\big)$$

for every $(b,\xi) \in \mathbb{R}^m \times \mathbb{R}^{n\times N}$, and this latter property is inherited by $\mathcal{CQ}f$ and $\mathcal{CQ}f^\infty$ as it can be easily verified arguing as in [10, Proposition 2.3]. Thus Proposition 2.14 holds when $p = \infty$ just requiring that f satisfies (2.6) and (2.7).

The following result can be deduced in full analogy with [11, Theorem 13], where it has been proven for $\overline{J}_\infty$.

Proposition 2.16. *Let Ω be a bounded and open set of $\mathbb{R}^N$ and let $f : \mathbb{R}^m \times \mathbb{R}^{n\times N} \to \mathbb{R}$ be a continuous function satisfying $(H_1)_p$ and $(H_2)_p$, $1 < p \leqslant \infty$. Let J be the functional defined in (1.1), then $\overline{J}_p$ in (1.2) $(1 < p < \infty)$, (1.3) $(p = \infty)$ is a variational functional, namely it is lower semicontinuous with respect to the first arguments and for every $(u,v) \in BV(\Omega;\mathbb{R}^n) \times L^p(\Omega;\mathbb{R}^m)$, one can define $\overline{J}_p(u,v;\cdot)$, $(p \in (1,\infty])$ (in analogy with (1.2) and (1.3)) as a set function on the open subsets of Ω, and it turns out to be the restriction of a Radon measure to these subsets of Ω.*

By virtue of this result $\overline{J}_p$ can be decomposed as the sum of two terms

$$\overline{J}_p(u,v;\cdot) = \overline{J}^a_p(u,v;\cdot) + \overline{J}^s_p(u,v;\cdot), \tag{2.12}$$

where $\overline{J}^a_p(u,v;\cdot)$ and $\overline{J}^s_p(u,v;\cdot)$ denote the absolutely continuous part and the singular part with respect to the Lebesgue measure, respectively. Next proposition deals with the scaling properties of $\overline{J}_p$.

Proposition 2.17. *Let $f : \mathbb{R}^m \times \mathbb{R}^{n\times N} \to \mathbb{R}$ be a continuous and convex–quasiconvex function, let J and $\overline{J}_p$ be the functionals defined respectively by (1.1) and (1.2) when $p \in (1,\infty]$, respectively ((1.3), when $p = \infty$). Then the following scaling properties are satisfied*

$$\begin{aligned}
&\overline{J}_p(u+\eta, v;\Omega) = \overline{J}_p(u,v;\Omega) \quad \text{for every } \eta \in \mathbb{R}^n,\\
&\overline{J}_p\big(u(\cdot - x_0), v(\cdot - x_0); x_0 + \Omega\big) = \overline{J}_p\big(u(\cdot), v(\cdot);\Omega\big) \quad \text{for every } x_0 \in \mathbb{R}^N,\\
&\overline{J}_p\left(u_\varrho, v_\varrho; \frac{\Omega - x_0}{\varrho}\right) = \varrho^{-N}\overline{J}_p(u,v;\Omega),
\end{aligned} \tag{2.13}$$

where $u_\varrho(y) := \frac{u(x_0+\varrho y)-u(x_0)}{\varrho}$ and $v_\varrho(y) := v(x_0+\varrho y)$, for $y \in \frac{\Omega - x_0}{\varrho}$.

The following result will be exploited in the sequel. The proof is omitted since it develops along the lines of [2, Lemma 5.50], the only differences being the presence of v and the convexity-quasiconvexity of f.

Lemma 2.18. *Let $f : \mathbb{R}^m \times \mathbb{R}^{N\times n} \to \mathbb{R}$ be a continuous and convex–quasiconvex function, and let J and $\overline{J}_p$ be the functionals defined respectively by (1.1) and (1.2). Let $\nu \in S^{N-1}$, $\eta \in S^{n-1}$ and $\psi : \mathbb{R} \to \mathbb{R}$, bounded and increasing. Denoted by Q the cube Q_ν, let $u \in BV(Q;\mathbb{R}^n)$ be representable in Q as*

$$u(y) = \eta\psi(y\cdot\nu),$$

and let $w \in BV(Q;\mathbb{R}^n)$ be such that $\operatorname{supp}(w-u) \subset\subset Q$. *Let $v \in L^p(Q;\mathbb{R}^m)$. Then*

$$\overline{J}_p(w, v; Q) \geqslant f\left(\int_Q v\,dy, Dw(Q)\right).$$

3. Main results

This section is devoted to deduce the results stated in Theorems 1.1 and 1.2. We start by proving the lower bound in the case $1 < p < \infty$. For what concerns the upper bound we present, for the reader's convenience, a self contained proof in Theorem 3.2. For the sake of completeness we observe that the upper bound, in the case $1 < p < \infty$, could be deduced as a corollary from the case $p = \infty$ (see Theorem 1.2), which, in turn, under slightly different assumptions, is contained in [8].

3.1. Lower semicontinuity in $BV \times L^p$, $1 < p < \infty$

Theorem 3.1. *Let Ω be a bounded open set of $\mathbb{R}^N$, let $f : \mathbb{R}^m \times \mathbb{R}^{n\times N} \to [0,\infty)$ be a continuous function satisfying (H_0), $(H_1)_p$ and $(H_2)_p$, and let $\overline{J}_p$ be the functional defined in (1.2). Then*

$$\overline{J}_p(u, v; \Omega) \geqslant \int_\Omega f(v, \nabla u)\,dx + \int_\Omega f^\infty\left(0, \frac{dD^s u}{d|D^s u|}\right) d\left|D^s u\right| \tag{3.1}$$

for any $(u, v) \in BV(\Omega;\mathbb{R}^n) \times L^p(\Omega;\mathbb{R}^m)$.

Proof. The proof will be achieved, in two steps, namely by showing that

$$\lim_{\varrho\to 0^+} \frac{\overline{J_p}(u, v; Q(x_0;\varrho))}{\mathcal{L}^N(Q(x_0,\varrho))} \geqslant f\big(v(x_0), \nabla u(x_0)\big), \quad \text{for } \mathcal{L}^N\text{-a.e. } x_0 \in \Omega, \tag{3.2}$$

$$\lim_{\varrho\to 0^+} \frac{\overline{J_p}(u, v; Q(x_0,\varrho))}{|Du|(Q(x_0,\varrho))} \geqslant f^\infty\left(0, \frac{dD^s u}{d|D^s u|}(x_0)\right), \quad \text{for } \left|D^s u\right|\text{-a.e. } x_0 \in \Omega. \tag{3.3}$$

Indeed, if (3.2) and (3.3) hold then, by virtue of (2.12), and [2, Theorem 2.56], (3.1) follows immediately.

Step 1. Inequality (3.2) is obtained through an argument entirely similar to [2, Proposition 5.53] and exploiting [11, Theorem 11].

For $\mathcal{L}^N$-a.e. $x_0 \in \Omega$ it results that u is approximately differentiable (see (2.1)) and

$$\lim_{\varrho \to 0^+} \frac{1}{\mathcal{L}^N(Q(x_0,\varrho))} \int_{Q(x_0,\varrho)} \left|v(x) - v(x_0)\right| dx = 0.$$

Consequently, given $\varrho > 0$, and defined u_ϱ and v_ϱ as in Proposition 2.17, it results that $u_\varrho \to u_0$ in $L^1(\Omega;\mathbb{R}^n)$, where $u_0 := \nabla u(x_0)x$ and $v_\varrho \to v(x_0)$ in $L^p(\Omega;\mathbb{R}^m)$. Then the scaling properties (2.13), and the lower semicontinuity of $\overline{J}_p$ entail that

$$\liminf_{\varrho \to 0^+} \frac{\overline{J}_p(u, v; Q(x_0,\varrho))}{\varrho^N} = \liminf_{\varrho \to 0^+} \overline{J}_p(u_\varrho, v_\varrho; Q) \geqslant \overline{J}_p\big(u_0, v(x_0); Q\big). \tag{3.4}$$

Then the lower semicontinuity result proven in [11, Theorem 11], when u is in $W^{1,1}(\Omega;\mathbb{R}^n)$ and $v \in L^p(\Omega;\mathbb{R}^m)$, allows us to estimate the last term in (3.4) as follows

$$\overline{J}_p\big(u_0, v(x_0); Q\big) \geqslant f\big(v(x_0), \nabla u(x_0)\big),$$

and that provides (3.2).

Step 2. Here we present the proof of (3.3). To this end we exploit techniques very similar to [1] (see [2, Proposition 5.53]). Let $Du = z|Du|$ be the polar decomposition of Du (see [2, Corollary 1.29]), for $z \in S^{N\times n-1}$, and recall that for $|D^s u|$-a.e. x_0, $z(x_0)$ admits the representation $\eta(x_0) \otimes \nu(x_0)$, with $\eta(x_0) \in S^{n-1}$ and $\nu(x_0) \in S^{N-1}$ (see [2, Theorem 3.94]). In the following, we will denote the cube $Q_\nu(x_0, 1)$ by Q.

To achieve (3.3) it is enough to show that

$$\lim_{\varrho \to 0^+} \frac{\overline{J}_p(u, v; Q(x_0,\varrho))}{|Du|(Q(x_0,\varrho))} \geqslant f^\infty\big(0, z(x_0)\big)$$

at any Lebesgue point x_0 of z relative to $|Du|$ such that the limit on the left hand side exists and

$$z(x_0) = \eta(x_0) \otimes \nu(x_0), \qquad \lim_{\varrho \to 0^+} \frac{|Du|(Q(x_0,\varrho))}{\varrho^N} = \infty, \tag{3.5}$$

$$0 = \lim_{\varrho \to 0^+} \frac{\int_{Q(x_0,\varrho)} |v|^p\, dx}{|Du|(Q(x_0,\varrho))} = \lim_{\varrho \to 0^+} \frac{\int_{Q(x_0,\varrho)} |v|\, dx}{|Du|(Q(x_0,\varrho))}. \tag{3.6}$$

The above requirements are, indeed, satisfied at $|D^s u|$-a.e. $x_0 \in \Omega$, by Besicovitch's derivation theorem and Alberti's rank-one theorem (see [2, Theorem 3.94]). Set $\eta \equiv \eta(x_0)$ and $\nu \equiv \nu(x_0)$, for $\varrho < N^{-\frac{1}{2}}\mathrm{dist}(x_0, \partial\Omega)$, define

$$u_\varrho(y) := \frac{u(x_0 + \varrho y) - \tilde{u}_\varrho}{\varrho} \frac{\varrho^N}{|Du|(Q(x_0,\varrho))}, \quad y \in Q,$$

where $\tilde{u}_\varrho$ is the average of u in $Q(x_0, \varrho)$. Analogously define, as in Proposition 2.17,

$$v_\varrho(y) := v(x_0 + \varrho y), \quad y \in Q. \tag{3.7}$$

Let us fix $t \in (0, 1)$. By [2, formula (2.32)], there exists a sequence $\{\varrho_h\}$ converging to 0 such that

$$\lim_{h\to\infty} \frac{|Du|(Q(x_0, t\varrho_h))}{|Du|(Q(x_0, \varrho_h))} \geqslant t^N. \tag{3.8}$$

Denote u_{ϱ_h} by u_h, then $|Du_h|(Q) = 1$ and, passing to a not relabelled subsequence, $\{u_h\}$ converges in $L^1(Q; \mathbb{R}^n)$ to a BV function $\overline{u}$. Correspondingly, denote v_{ϱ_h} by v_h. Then, arguing as in [2, Proof of Proposition 5.53] we have

$$|D\overline{u}|(Q) \leqslant 1 \quad \text{and} \quad |D\overline{u}|(\overline{Q}_t) \geqslant t^N, \tag{3.9}$$

where $Q_t := tQ$. It results that $\overline{u}(y) = \eta\psi(y \cdot \nu)$, for some bounded increasing function ψ in $(-\frac{1}{2}, \frac{1}{2})$. Take $\varphi \in C_c^1(Q)$ such that $\varphi = 1$ on $\overline{Q}_t$ and $0 \leqslant \varphi \leqslant 1$, and let us define $w_h := \varphi u_h + (1 - \varphi)\overline{u}$. The functions w_h converge to $\overline{u}$ in $L^1(Q; \mathbb{R}^n)$ and moreover we have

$$\begin{aligned} \big|D(w_h - u_h)\big|(Q) &\leqslant \big|D(u_h - \overline{u})\big|(Q \setminus \overline{Q}_t) + \int_Q |\nabla\varphi||u_h - \overline{u}|\, dy \\ &\leqslant |Du_h|(Q \setminus \overline{Q}_t) + |D\overline{u}|(Q \setminus \overline{Q}_t) + \int_Q |\nabla\varphi||u_h - \overline{u}|\, dy. \end{aligned}$$

Therefore, by (3.8) and (3.9), one has

$$\limsup_{h\to\infty} \big|D(w_h - u_h)\big|(Q) \leqslant 2\big(1 - t^N\big). \tag{3.10}$$

Similarly,

$$|Dw_h|(Q \setminus \overline{Q}_t) \leqslant |Du_h|(Q \setminus \overline{Q}_t) + |D\overline{u}|(Q \setminus \overline{Q}_t) + \int_Q |\nabla\varphi||u_h - \overline{u}|\, dy,$$

consequently

$$\limsup_{h\to\infty} |Dw_h|(Q \setminus \overline{Q}_t) \leqslant 2\big(1 - t^N\big). \tag{3.11}$$

Setting $c_h := \frac{|Du|(Q(x_0,\varrho_h))}{\varrho_h^N}$, by the scaling properties of $\overline{J}_p$ in Proposition 2.17 and by the growth conditions $(H_1)_p$, we have

$$\begin{aligned} \frac{\overline{J}_p(u, v; Q(x_0, \varrho_h))}{|Du|(Q(x_0, \varrho_h))} &= \frac{\overline{J}_p(c_h u_h, v_h; Q)}{c_h} \\ &\geqslant \frac{\overline{J}_p(c_h w_h, v_h; \overline{Q}_t)}{c_h} \end{aligned}$$

$$\geqslant \frac{\overline{J}_p(c_h u_h, v_h; Q)}{c_h} - C\Big(c_h^{-1}|Q \setminus \overline{Q}_t| + |Dw_h|(Q \setminus \overline{Q}_t) + c_h^{-1}\int_{Q\setminus\overline{Q}_t} |v_h|^p\, dy\Big).$$

By (3.5), $c_h \to \infty$, moreover taking into account (3.7) and (3.6), by (3.11), it results that

$$\lim_{\varrho\to 0^+} \frac{\overline{J}_p(u, v; Q(x_0, \varrho))}{|Du|(Q(x_0, \varrho))} \geqslant \limsup_{h\to\infty} \frac{\overline{J}_p(c_h u_h, v_h; Q)}{c_h} - 2C\big(1 - t^N\big).$$

On the other hand, Lemma 2.18 entails that, for every $h \in \mathbb{N}$,

$$\begin{aligned}\overline{J}_p(c_h w_h, v_h; Q) &\geqslant f\Big(\int_Q v_h\, dy, c_h Dw_h(Q)\Big)\\ &\geqslant f\Big(\int_Q v_h\, dy, c_h Du_h(Q)\Big) - c_h\gamma\big|D(w_h - u_h)\big|(Q),\end{aligned}$$

where γ is the constant appearing in (2.3). Then by (3.10), we have that

$$\lim_{\varrho\to 0^+} \frac{\overline{J}_p(u, v; Q(x_0, \varrho))}{|Du|(Q(x_0, \varrho))} \geqslant \limsup_{h\to\infty} \frac{f(\int_Q v_h\, dy, c_h Du_h(Q))}{c_h} - 2(C+\gamma)\big(1 - t^N\big).$$

By the definition of u_h, $Du_h(Q) = \frac{Du(Q(x_0,\varrho_h))}{|Du|(Q(x_0,\varrho_h))}$, hence $Du_h(Q) \to z(x_0)$, since x_0 is a Lebesgue point of z. Now, taking into account (2.3) and $(H_2)_p$, we have

$$\begin{aligned}\limsup_{h\to\infty} \frac{f(\int_Q v_h\, dy, c_h Du_h(Q))}{c_h} &= \lim_{h\to\infty} \frac{f(\int_Q v_h\, dy, c_h z(x_0))}{c_h}\\ &= \lim_{h\to\infty}\Big(f^\infty\Big(\int_Q v_h\, dy, z(x_0)\Big) - C\frac{|\int_Q v_h\, dy|^p + 1}{c_h}\Big)\\ &= f^\infty\big(0, z(x_0)\big),\end{aligned}$$

where it has been exploited the fact that $c_h \to \infty$, 3. of Proposition 2.8, the nondecreasing behaviour of the L^p norm in the unit cube with respect to p (i.e. $|\int_Q v_h\, dy|^p \leqslant \int_Q |v_h|^p\, dy$), and (3.6). □

3.2. Relaxation

We start by observing that Theorem 1.2 is contained in [8] under a uniform coercivity assumption. We do not propose the proof in our setting, since it develops along the lines of Theorems 3.1 and 3.2.

On the other hand, several observations about Theorem 1.2 are mandatory:

i) If f satisfies $(H_1)_p$ and $(H_2)_p$ then $\overline{J}_p(u, v) \leqslant \overline{J}_\infty(u, v)$ for every $(u, v) \in BV(\Omega; \mathbb{R}^n) \times L^\infty(\Omega; \mathbb{R}^m)$.

ii) For the reader's convenience we observe that the proof of the lower bound in Theorem 1.2 develops exactly as that of Theorem 3.1, using the L^∞ bound on v to deduce (3.6) and the uniform bound on v_ϱ in (3.7), $(H_2)_\infty$ and (2.4) in order to estimate $\limsup_{h\to\infty} \frac{f(\int_Q v_h\,dy, c_h Du_h(Q))}{c_h}$. Regarding the upper bound, the bulk part follows from [11, Theorems 12 and 14], while for the singular part we can argue exactly as proposed in the proof of the upper bound in [8] just considering conditions $(H_1)_\infty$ and $(H_2)_\infty$ in place of $(H_1)_p$ and $(H_2)_p$.

iii) The above arguments remain true under assumptions (2.6) and (2.7).

We are now in position to prove the upper bound for the case $BV \times L^p$, for $1 < p < \infty$. We emphasize that an alternative proof could be obtained via a truncation argument from the case $p = \infty$ as the one presented in [11, Theorem 12], but we prefer the self contained argument below.

Theorem 3.2. *Let Ω be a bounded open set of $\mathbb{R}^N$ and let $f : \mathbb{R}^m \times \mathbb{R}^{n\times N} \to [0,\infty)$ be a continuous function. Then, assuming that f satisfies (H_0), $(H_1)_p$ and $(H_2)_p$,*

$$\overline{J}_p(u,v) \leqslant \int_\Omega f(v,\nabla u)\,dx + \int_\Omega f^\infty\left(0, \frac{dD^s u}{d|D^s u|}(x)\right) d\left|D^s u\right|(x),$$

for every $(u,v) \in BV(\Omega;\mathbb{R}^n) \times L^p(\Omega;\mathbb{R}^m)$.

Proof. First we observe that Proposition 2.16 entails that $\overline{J}_p$ is a variational functional. Thus the inequality can be proved analogously to [2, Proposition 5.49]. For what concerns the bulk part, it is enough to observe that given $u \in BV(\Omega;\mathbb{R}^n)$ and $v \in L^p(\Omega;\mathbb{R}^m)$, taking a sequence of standard mollifiers $\{\varrho_{\varepsilon_k}\}$, where $\varepsilon_k \to 0$, it results that $\nabla u_k = \nabla u * \varrho_{\varepsilon_k} + D^s u * \varrho_{\varepsilon_k}$, where $u_k := u * \varrho_{\varepsilon_k}$. The local Lipschitz behaviour of f in (2.3) gives

$$\int_A f(v,\nabla u_k)\,dx \leqslant \int_A f(v,\nabla u * \varrho_{\varepsilon_k})\,dx + \gamma\left|D^s u\right|\big(I_{\varepsilon_k}(A)\big)$$

for every $k \in \mathbb{N}$, where $I_{\varepsilon_k}(A)$ denotes the ε_k neighborhood of A. Then if $|D^s u|(\partial A) = 0$, letting $\varepsilon_k \to 0$, we obtain

$$\overline{J}_p(u,v;A) \leqslant \int_A f(v,\nabla u)\,dx + \gamma\left|D^s u\right|(A),$$

for every open subset A of Ω. Thus we can conclude that

$$\overline{J}^a_p(u,v;B) \leqslant \int_B f\big(v(x),\nabla u(x)\big)\,dx$$

for every $(u,v) \in BV(\Omega;\mathbb{R}^n) \times L^p(\Omega;\mathbb{R}^m)$ and B Borel subset of Ω.

To achieve the result, it will be enough to show that

$$\overline{J}^s_p(u,v;B) \leqslant \int_B f^\infty\left(0, \frac{dD^s u}{d|D^s u|}\right) d\left|D^s u\right| \quad \text{for every } B \text{ Borel subset of } \Omega.$$

For every $\xi \in \mathbb{R}^{n\times N}$ and $b \in \mathbb{R}^m$, define the function

$$g(b,\xi) := \sup_{t\geqslant 0} \frac{f(t^{\frac{1}{p}}b, t\xi) - f(0,0)}{t}.$$

It is easily seen that g is $(p,1)$-positively homogeneous, i.e. $tg(b,\xi) = g(t^{\frac{1}{p}}b, t\xi)$ for every $t > 0$, $(b,\xi) \in \mathbb{R}^m \times \mathbb{R}^{n\times N}$, g is continuous and, since f satisfies (2.3), g inherits the same property. Moreover, the monotonicity property of difference quotients of convex functions ensures that, whenever rank $\xi \leqslant 1$, $g(b,\xi) = f_p^\infty(b,\xi)$, where the latter is defined as

$$f_p^\infty(b,\xi) := \limsup_{t\to\infty} \frac{f(t^{\frac{1}{p}}b, t\xi)}{t}.$$

In particular $g(0,\xi) = f^\infty(0,\xi) = f_p^\infty(0,\xi)$, whenever rank $\xi \leqslant 1$.

Then for every open set $A \subset\subset \Omega$ such that $|Du|(A) = 0$, defining for every $h \in \mathbb{N}$, $u_h := u * \varrho_{\varepsilon_h}$ and $v_h := v$ where $\{\varrho_{\varepsilon_h}\}$ is a sequence of standard mollifiers and $\varepsilon_h \to 0$. Then $u_h \to u$ in L^1. Also [2, Theorem 2.2] entails that $|Du_h| \to |Du|$ weakly $*$ in A and $|Du_h|(A) \to |Du|(A)$. Thus, since $f \leqslant f(0,0) + g$,

$$\begin{aligned}\overline{J}_p(u,v;A) &\leqslant \liminf_{h\to\infty} \int_A f(v, \nabla u_h)\,dx \\ &\leqslant \limsup_{h\to\infty} \int_A f(0,0)\,dx + \liminf_{h\to\infty} \int_A g(v,\nabla u_h)\,dx. \end{aligned} \tag{3.12}$$

Since the first term in the right hand side is bounded by $C\mathcal{L}^N(A)$, taking the Radon–Nikodým derivative with respect to $|D^s u|$ we obtain 0.

Regarding the second term in the right hand side of (3.12), we have

$$\begin{aligned}&\liminf_{h\to\infty} \int_A g\big(v(x), Du * \varrho_h\big)\,dx \\ &\quad\leqslant \limsup_{h\to\infty} \int_A g(0, Du*\varrho_h)\,dx + C\int_A \big|v(x)\big|^p\,dx + \liminf_{h\to+\infty} \int_A \big|v(x)\big||Du*\varrho_h|^{\frac{1}{p'}}\,dx.\end{aligned}$$

Taking the Radon–Nikodým derivative, the last two terms disappear, since $|Du*\varrho_h| \to |Du|$, $|v|^p\mathcal{L}^N$ is singular with respect to $|D^s u|$ and the Hölder inequality can be applied, i.e.

$$\int_A \big|v(x)\big||Du*\varrho_h|^{\frac{1}{p'}}\,dx \leqslant \left(\int_A \big|v(x)\big|^p\,dx\right)^{\frac{1}{p}} \left(\int_A |Du*\varrho_n|\,dx\right)^{\frac{1}{p'}}.$$

Then the thesis is achieved via the same arguments as in [2, Proposition 5.49]. □

Remark 3.3. It is worth to observe that an alternative argument to the one presented above, concerning the upper bound inequality for the singular part, can be provided by means of approximation. In fact, one can prove that $\overline{J}^s_p(u, v; B) \leqslant \int_B f^\infty(0, \frac{dD^s u}{d|D^s u|})\, d|D^s u|$ for every B Borel subset of Ω, when $u \in BV(\Omega; \mathbb{R}^n)$ and $v \in C(\overline{\Omega}; \mathbb{R}^m)$, and then a standard approximation argument via mollification allows to reach every $v \in L^p(\Omega; \mathbb{R}^m)$.

For what concerns the case $v \in C(\overline{\Omega}; \mathbb{R}^m)$ it is enough to consider the function $g(b, \xi) := \sup_{t\geqslant 0} \frac{f(b,t\xi)-f(b,0)}{t}$, exploit its properties of positive 1-homogeneity in the second variable, i.e. $tg(b, \xi) = g(b, t\xi)$, for every $t > 0$, $(b, \xi) \in \mathbb{R}^m \times \mathbb{R}^{n\times N}$, (2.4), and the fact that when rank $\xi \leqslant 1$, then $g(b, \xi)$ is constant with respect to b and $f^\infty(b, \xi) = g(b, \xi) = f^\infty(0, \xi)$. To conclude it is enough to apply Reshetnyak continuity theorem.

Proof of Theorem 1.1. The result follows from Theorems 3.1 and 3.2, applying Proposition 2.14 to remove assumption (H_0). □

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Appendix

Consider the functional

$$\overline{J}_1(u, v) := \inf\Big\{\liminf_{h\to\infty} J(u_h, v_h) : u_h \in W^{1,1}\big(\Omega; \mathbb{R}^n\big), v_h \in L^\infty\big(\Omega; \mathbb{R}^m\big),$$
$$u_h \to u \text{ in } L^1, v_h \overset{*}{\rightharpoonup} v \text{ in } \mathcal{M}\Big\}, \tag{A.1}$$

for any pair $(u, v) \in BV(\Omega; \mathbb{R}^n) \times \mathcal{M}(\Omega; \mathbb{R}^m)$, where this latter set denotes the set of signed Radon measures and the weak $*$ convergence denotes the one in the sense of measures.

The integral representation of (A.1) will be stated in Theorem A.1 under the following hypotheses:

$(H_1)_1$ There exists $C > 0$ such that

$$\frac{1}{C}\big(|b| + |\xi|\big) - C \leqslant f(b, \xi) \leqslant C\big(1 + |b| + |\xi|\big),$$

for $(b, \xi) \in \mathbb{R}^m \times \mathbb{R}^{n\times N}$.

$(H_2)_1$ There exists $C' > 0$, $L > 0$, $0 < \tau \leqslant 1$ such that

$$t > 0, \xi \in \mathbb{R}^{n\times N}, \quad \text{with } t\big|(b, \xi)\big| > L \Longrightarrow \left|\frac{f(tb, t\xi)}{t} - f^\infty(b, \xi)\right| \leqslant C'\left(\frac{|(b, \xi)|^{1-\tau}}{t^\tau}\right),$$

where f_1^∞ is the recession function of f defined for every $b \in \mathbb{R}^m$ as

$$f_1^\infty(b, \xi) := \limsup_{t\to\infty} \frac{f(tb, t\xi)}{t}. \tag{A.2}$$

Theorem A.1. *Let J be given by (1.1), with f satisfying $(H_1)_1$ and $(H_2)_1$ and let $\overline{J}_1$ be given by (A.1) then*

$$\overline{J}_1(u,v) = \int_\Omega \mathcal{C}\mathcal{Q}f\big(v^a, \nabla u\big)\,dx + \int_\Omega (\mathcal{C}\mathcal{Q}f)_1^\infty\left(\frac{dv}{d|D^s u|}, \frac{dD^s u}{d|D^s u|}\right) d\big|D^s u\big| + \int_\Omega (\mathcal{C}\mathcal{Q}f)_1^\infty\left(\frac{dv}{d|v^s|}, 0\right) d\big|v^s\big|$$

for every $(u,v) \in BV(\Omega;\mathbb{R}^n) \times \mathcal{M}(\Omega;\mathbb{R}^m)$, where v^a is the absolutely continuous part of the Radon measure v with respect to the Lebesgue measure and v^s is the singular part of v with respect to $|Du|$.

Proof. The same arguments which lead to Proposition 2.14, allow to assume without loss of generality that f is convex–quasiconvex, i.e. to replace f by $\mathcal{C}\mathcal{Q}f$ in (A.1).

Arguing as in [3, Lemma 4.7] one can prove that the set function $\overline{J}_1(v,u,\cdot)$, defined as in (A.1) for all the open subsets of Ω, is the trace on these latter sets of a Radon measure absolutely continuous with respect to $\mathcal{L}^N + |Du| + |v|$.

Then the rest of the proof can be obtained as in [3] by providing a lower bound and an upper bound, that we will sketch in the sequel, just emphasizing the main differences.

For what concerns the lower bound, it is enough to prove that

$$\lim_{\varrho\to 0^+} \frac{\overline{J_1}(u,v;Q(x_0;\varrho))}{\mathcal{L}^N(Q(x_0,\varrho))} \geqslant f\big(v^a(x_0), \nabla u(x_0)\big), \quad \text{for } \mathcal{L}^N\text{-a.e. } x_0 \in \Omega, \tag{A.3}$$

$$\lim_{\varrho\to 0^+} \frac{\overline{J_1}(u,v;Q(x_0,\varrho))}{|Du|(Q(x_0,\varrho))} \geqslant f^\infty\left(\frac{dv}{d|D^s u|}(x_0), \frac{dD^s u}{d|D^s u|}(x_0)\right), \quad \text{for } \big|D^s u\big|\text{-a.e. } x_0 \in \Omega, \tag{A.4}$$

$$\lim_{\varrho\to 0^+} \frac{\overline{J_1}(u,v;Q(x_0,\varrho))}{|Du|(Q(x_0,\varrho))} \geqslant f^\infty\left(\frac{dv}{d|v^s|}(x_0), 0\right), \quad \text{for } \big|v^s\big|\text{-a.e. } x_0 \in \Omega. \tag{A.5}$$

All the inequalities can be proven following arguments analogous to the ones in [3, Lemma 5.1], taylored for thin structures. Indeed we observe that the density $\mathcal{Q}^*(\cdot)$ appearing therein coincides with $\mathcal{C}\mathcal{Q}(\cdot)$, as proven in [4]. Then the proofs in [3, Lemma 5.1] can be repeated line by line but with easier constructions, in particular it suffices to replace the unique sequence $\{u_n\}$ (and its average), by the couple $\{(u_n, v_n)\}$ with $u_n \to u$ in L^1, and $v_n \overset{*}{\rightharpoonup} v$ in $\mathcal{M}$.

For what concerns the upper bound, i.e proving inequalities opposite to (A.3)–(A.5), one can argue as in the proof of Theorem 3.2, using as a recovery sequence the couple $\{(v_k, u_k)\}$ with $u_k := u * \varrho_{\varepsilon_k}$ and $v_k := v * \varrho_{\varepsilon_k}$, where $\{\varrho_{\varepsilon_k}\}$ is a sequence of standard mollifiers. Then for the bulk term it suffices to exploit the standard Lipschitz property of f with respect to the couple (b,ξ) (i.e. (2.3), when $p = 1$), and the fact that $v * \varrho_{\varepsilon_k} = v^a * \varrho_{\varepsilon_k} + v^s * \varrho_{\varepsilon_k}$, and finally taking the Radon–Nikodým derivative around points of absolute continuity with respect to the Lebesgue measure of u, ∇u and v^a.

For what concerns the other two terms in the limiting energy they can be reached through the function $g(b,\xi) := \sup_{t\geqslant 0} \frac{f(tb,t\xi)}{t}$, which is positively 1 homogeneous and coincides with $f_1^\infty(b,\xi)$, whenever $\operatorname{rank}\xi \leqslant 1$. Finally the conclusion can be achieved differentiating with respect to $|D^s u|$ or v^s. □

Remark A.2. We observe that in [5] it was considered for the $v's$ the weak convergence in L^1 thus leading to a target function still in L^1. In the present case the limiting function is a measure, which can be decomposed in three terms, the first two absolutely continuous with respect to $\mathcal{L}^N$ and $|D^s u|$ respectively, and the third possibly singular with respect to $|Du|$ and this entails the presence of a third integrand in the energy above.

We also observe that an argument entirely similar to Proposition 3.6 warranties that $(\mathcal{C}\mathcal{Q}f)_1^\infty(b,\xi) = \mathcal{C}\mathcal{Q}(f_1^\infty)(b,\xi)$ for every $b \in \mathbb{R}^m$, $\xi \in \mathbb{R}^{n\times N}$.

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