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The relationship between return fractality and bipower variation

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Abstract. This paper presents an intuitively simple asset pricing model designed to predict stock returns and volatilities, when stock prices may follow a *fractal walk* rather than a random walk. The model utilizes similarity ratio of the return fractals as the basis for forecasting. We argue that a collection of past returns such as moving average statistics can be "expanded" to generate future returns through the similarity ratios by trading time multiples. We also argue that stock returns with fractal dimensions in excess of 1.5, the norm for market efficiency, may be prone to frequent jumps and discontinuity. The paper builds an econometrically testable model to estimate fractal dimensions to offer an alternative way to forecast return volatilities without having to estimate separately the bipower variation and the jumps in existing studies. We argue that fractional Brownian motion may be a more suitable description than the standard Wiener process when describing stock return behaviors. The paper also demonstrates the application aspect of our asset pricing model to high frequency algorithmic trading.

Keywords: Return fractality, discontinuous jumps, modified wiener process, fractal dimension, bipower variation, high frequency algorithmic/quantitative trading

JEL Classification: G1 (General Financial Markets), G12 (Asset Pricing)

1. Introduction

The standard Wiener process is often purported to manifest a random walk assuming a continuous process, and thus, seems to miss some important elements of the true stochastic process when security returns evolve discontinuously with occasional jumps, e.g. Merton (1976). Bollerslev (1986), Nelson (1990) and Engle et al. (1993), and others have studied how occasional jumps can converge to a long-run volatility rate. Clearly, these works have been well documented and implemented in economic research that use daily, monthly, or even more ambitiously, yearly data. In this paper, we argue that stock returns are "fractals" and thus, show self similarities year-to-year, month-to-month, or day-to-day, and therefore, we conjecture that it is possible that the return fractality alone may be able to explain the stock price behavior characterized by many discontinuous jumps especially in the case of high frequency data.

In what follows immediately below, i.e. Section 2, we begin with an intuitively simple asset pricing model designed to predict stock returns and volatilities, when stock prices may follow a *fractal walk* rather than a random walk. The model utilizes similarity ratio of return fractals as the basis for forecasting. In Section 3, we cite important works by Barndorff-Nelson (2004, 2005) and Andersen et al. (2007) intended to show algorithms to compute the bipower variation and the jump component in high frequency data. We then point out that much of the bipower variation can come from return fractality, especially if the fractal dimension exceeds 1.5, or the Hurst (1951) exponent of less than 0.5. Section 4 proposes a new approach to estimate the fractal dimension based on trading time intervals as an indicator for the similarity ratio, and discusses the potential

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use of the model in a high frequency algorithmic trading environment. Finally, Section 5 discusses how the fractality argument as proposed in this paper is related to models that deal with *fractional Brownian motion*. Section 6 concludes the paper.

2. Economics of return fractality

Suppose that a change in an asset's logarithmic price from time 0 to t, i.e. Δt, is $R_{\Delta t}$. We often multiply the return over a short time step, $R_{\Delta t}$, by N time intervals to cover the entire longer term horizon, T, to compute the return over T, R_T, i.e.

$$R_T = R_{\Delta t} N \tag{1}$$

The rationale for doing this is that whatever happens over a time period T can be extrapolated by whatever happens during Δt linearly by N, as $N = T/\Delta t$. Typically, daily, monthly or quarterly returns are annualized in this way, even though this simplification has no predictive content.

Undoubtedly, the factor N that $R_{\Delta t}$ is multiplied by cannot be calendar time, when forecasting the return over T, i.e. R_T. The appropriate multiplier could be something other than calendar time, especially when each trading day is a different day after all, and cannot be treated as same. That is, we cannot simply blindfoldedly extrapolate a shorter-term return, e.g. a day's return, to forecast an annual return by multiplying the day's return by the calendar time period, e.g. 365 days. Same criticisms may also apply even when we examine a collection of past return data. In other words, the financial market is not one-dimensional. Simply put, the stock return "graphs" or the return processes are never a straight line over any time period, either up or down. However redundant it may be, let us illustrate this point still further.

Suppose that events taking place in every small time segments, which make up for an entire "longer" term time horizon, are *self similar* segment-to-segment, and that the stock return process is indeed one-dimensional, i.e. $D = 1$. In this case, if the security return over a time segment ($0 \le t < T$) is exactly decomposed into N nonoverlapping segments, the length of each sub-segment $f \in \{1, 2, \dots, N\}$ is represented by a formula, $[(f-1)T/N \le t < fT/N]$. Clearly, an event in each segment can be deduced from the whole linearly, and in fact, each segment is "similar" by a ratio $r(N) = 1/N$. However, the economic environment in each time segment differs segment-to-segment. So, we introduce a different dimension to the security return with $D \ne 1$.

Now assume that the security return process is two-dimensional instead, i.e. $D = 2$, for the fact that the security return is explained, for example, by two independent economic variables, x and y such that $0 \le x < X$ and $0 \le y < Y$. Further assume that the security return function is given specifically by, for example, the product, xy.[1] In this case, if each segment of x and y is represented by k and h, where both k and h $\in \{1, 2, \dots, N\}$, and if there exists N number of self similar returns, then each small segment within an R_T can be explained by a formula, $[(k-1)X/\sqrt{N} \le x < kX/\sqrt{N};\ (h-1)Y/\sqrt{N} \le y < hy/\sqrt{N}]$. That is, if we are to decompose the return over T *spectrally* into N, we could do so just as we dimension a rectangular object with $D = 2$, where each of these "rectangular" parts, both x and y, is deducible from the whole by a similarity of ratio $r(N) = 1/\sqrt{N}$. In the context of forecasting returns, we may look at a collection of return process today or some moving average and multiply it by the factor of $\sqrt{N}$, *the reciprocal of the similarity ratio*, to predict the future return. We expect to see the whole self similar event, which may occur in one year for example, by looking at a short rate today, as long as we can calculate the similarity ratio for any stock returns. In reality, things are done differently, that is, securities are analyzed by examining the specific functional relationships among various economic factors such as x and y, etc. to predict the future returns. In our case, however, it is not necessary to know what the underlying economic factors are, as long as we know something about the security return's similarity ratios.

In order to generalize this line of reasoning to the entire security analysis, let us suppose that the D-dimensional stock returns can be decomposed into N subpieces from the whole by a similarity of ratio $r(N) = 1/N^{1/D}$, where $1 < D < 2$. In mathematics, the exponent, D, here is the concept similar to the Hausdorff dimension named after a German mathematician, Felix Hausdorff. Interestingly, however, just as the Hausdorff dimension can be fractional, the stock returns can have its own "fractional or fractal dimension." It is typical that the more sophisticated an object or a stochastic process is, the higher the value of the fractal dimension. See Mandelbrot (1967, 1982). From this discussion, we now know that the recipro-

[1]This resembles the formula for the area of a rectangular object.

cal of the similarity ratio is in fact the magnification factor (F) by which the initial return expands to generate the final return, i.e. $F = 1/r(N)$ and $F = N^{1/D}$. Therefore, however it may seem unorthodox, it seems plausible to argue that the future security return can be computed as

$$R \equiv R_1 = R_{\Delta t} F = R_{\Delta t} N^{1/D} = R_{\Delta t} \left(\frac{1}{r(N)} \right) \quad (2)$$

We have assumed that $T = 1$ for simplicity. Equation (2) is an oversimplification, however, in the sense that we could have examined a collection of $R_{\Delta t}$'s rather than one particular value of $R_{\Delta t}$ to forecast the future returns. This can greatly complicate the model without adding much here and so, we will avoid the issue, as one could come up with some sort of moving averages as an alternative. However, the following comments are in order.

First, we emphasize that $N \neq T/\Delta t$ in general, unless $D = 1$, as it no longer represents the number of period which makes up for the whole time span, T, as T is divided by an increment of t. Instead, the symbol N represents, very importantly, the total number of self similar pieces of returns that results when the initial return is spectrally magnified by F. From (2), it then follows that

$$N = \left(\frac{R}{R_{\Delta t}} \right)^D = F^D = \left(\frac{1}{r(N)} \right)^D$$

As usual, the fractal dimension is defined as $D = \log N / \log F = -\log N / \log r$. The fractal dimension D measures the roughness of a fractal and shows how jagged or serrated a time series is locally in the short run. The degree of the global or long-range autocorrelation is often measured separately by a related index known as Hurst exponent H. However, for self-affine processes, a celebrated theorem has it that $D = 2–H$ and hence, both D and H are related. Please see Mandelbrot (1982) and more recently, Gneiting and Schlather (2004). For example, a Gaussian random walk has a fractal dimension of 1.5 showing no autocorrelated returns, in which case $H = 0.5$. On the other hand, stock returns with $D \neq 1.5$ or $H \neq 0.5$ would reveal a long-range autocorrelation. More specifically, if $H > 0.5$ or $D < 1.5$, a positive long-term memory persists so that an increase in the stock return is more likely followed by a subsequent increase. Generally, the series would be somewhere between a straight line and Gaussian random walk. On the other hand, $H < 0.5$ or $D > 1.5$ indicates a negative long-term memory or a pattern of anti-persistence, where frequent reversal of the trend is common.

Second, the F being the reciprocal of the similarity ratio $r(N)$ then represents the total number of times that a given return over a short time period can be magnified by to produce an annual or a daily return. Thus, the symbol F can be viewed in a way as an "effective" or "economic" time (not the calendar time) multiple, which we use to magnify the initial returns to forecast returns for the end of a year, for the end of a day, or even 10 minutes hence. In Section 4, we will identify the F as the trading event cycle.

Third, by design, our forecasting model in Equation (2) acknowledges that stock returns are "self similar" or follow a "self-affine" stochastic process. However, we say that the process is also stochastic in the sense that we may have some unwanted "holes" in generating the whole self similar pieces from the original returns, if the fractal dimension is non-integer as in the case of the well known Sierpinski triangle. We propose an empirical model to estimate the fractality in Section 4.

In short, Equation (2) as a tool to predict future returns requires the knowledge of F and D, i.e. $N = F^D$. The volatility rate, σ_T is

$$\sigma_T = \sigma_{\Delta t} \sqrt{F} = \sigma_{\Delta t} N^{1/2D} \quad (3)$$

Equation (3) states that the volatility of a long rate depends on the short rate, which in turn depends on the return calculated from a particular time step Δt; and the magnification factor F. We now relate our prediction model to the theory of bipower variation as it deals with the asymptotic properties of the sampling frequency, which would be represented by the reciprocal of Δt to describe the stochastic process for stock returns. We find that by its own design, as $\Delta t \to 0$, the bipower variation decreases, the larger the value of the fractal dimension D, or the lower the value of Hurst exponent H. Consequently, if in particular, $D > 1.5$, the time series for stock returns is prone to frequent jumps with ups and downs.

3. Bipower variation

Barndorff-Nielson and Shephard (2004) and Anderson et al. (2007) assume that the return at time t is the stochastic integral of many past short returns leading up to time t, each one of which follows the standard

Wiener process, subject to the well defined Integrated Variance or the quadratic variation. If the effective time expansion factor F is constant at any time t, i.e. $F_j \approx F_{j-1} = F$, Equation (3) says that the Realized Volatility, $RV(\Delta t)$, as a function of Δt or the number of sample size n, is:

$$RV(\Delta t) = F^2 \sum_{j=1}^{1/\Delta t} R^2_{j\Delta t,\Delta t} \equiv F^2 \sum_{j=1}^{1/\Delta t} R^2_{j\Delta t} \qquad (4)$$

Note that $RV(\Delta t)$ itself a random variable with a mean and the standard deviation, and thus we standardize it. As Anderson et al. (2007) demonstrates, this converges to the increment of the quadratic variation process as the sampling frequency of the underlying returns increases, i.e. $\Delta t \to 0$. That is,

$$RV(\Delta t) \to F^2 \int_t^{t+1} \sigma^2(s)ds + \sum_{t<s<t+1} k^2(s); \text{ for } \Delta t \to 0 \qquad (5)$$

where $\sigma(t)$ is a stochastic volatility or the Integrated Volatility, and $k(t)$ is the size of the discrete jumps in the logarithmic price process. Thus, in the absence of jumps, the second term vanishes.

Defining the Realized Bipower Variation, BV, as the product of the absolute values of two subsequent intraday high-frequency security returns, Barndorff-Nielson and Shephard (2004) arrive at the result that as $\Delta t \to 0$, asymptotically,

$$BV(\Delta t) = \frac{\pi}{2} \sum_{j=2}^{n} \left|R_{j\Delta t}\right|\left|R_{(j-1)\Delta t}\right| \qquad (6)$$

In particular, it is shown that the mean of the absolute value of standard normally distributed random variable, Z, $E(|Z|)$ is $\sqrt{2/\pi}$. Assuming again the constancy of the factor F, the standardized realized bi-power variation over Δt is,

$$BV(\Delta t) = \frac{1}{2}\pi N^{2/D} \sum_{j=1}^{n-1} \left|R_{j\Delta t}\right|\left|R_{(j-1)\Delta t}\right| \qquad (7)$$

Then, as the sampling frequency increases, the realized bipower variation approaches to:

$$BV(\Delta t) \to F^2 \int_t^{t+1} \sigma^2(s)ds \qquad (8)$$

With the results in Equations (5) and (8) juxtaposed, Barndorff-Nielson and Shephard [*op cit*] would conclude that

$$RV(\Delta t) - BV(\Delta) \to \sum_{0<s\leq 1} k^2(s) \qquad (9)$$

where with non-negativity in jumps,

$$J(\Delta t) \equiv \max[RV(\Delta t) - BV(\Delta t),\ 0] \qquad (10)$$

Therefore, it is plausible to argue that the higher the fractal dimension, the lower the bipower variation and hence, the greater the jump for any given realized volatility. This means that if the Gaussian random process in which $D = 1.5$ is the "norm," stock returns with $D > 1.5$ or $H < 0.5$ would reveal that the observed volatility would be mostly attributed to jumps. Typically, a more jagged stock price chart has a higher fractal dimension and fortunately or unfortunately, these serrations are due to discontinuous jumps rather than for any smoothly functioning continuous processes. The implication is that it may not be necessary to separately estimate the bipower variation and the jumps to forecast future returns, if we can estimate the fractal dimension straightforwardly.

The following Section 4 now introduces a simple measure to estimate the fractal dimension for stock returns. The basic theme is that the "economic" expansion factor to generate a future return can be measured econometrically. The section also shows the application aspect of our model in various high frequency algorithmic trading situations as we deal with high frequency data in this paper.

4. Fractal dimension and volatility

There are many known methods of approximating the fractal dimension. Most notably, they include the by now well known "Rescaled Range" method by Mandelbrot et al. (1969), and other traditional Wavelet transforms based on a Taylor series for a time series signal *f*(t). Gneiting et al. (2010) also reviews some more recent studies for other alternative estimators including the ones by Hall and Wood (1993) and Genton (1998). In this paper, we take a still different approach that seems much simpler and straightforward. We are motivated by the following observation.

In finance, return fractality has never been an issue to virtually every portfolio manager and financial economist despite the fact that much of securities return data are discontinuous and jumpy. That is, although everyone looks at the prices for analysis, many analysts seem to ignore the way the price evolves over time. Perhaps, we have simply assumed that somehow trades have taken place, and yet these trade events have been continuous. However, one cannot ignore the frequency of trades in security analysis, because it must influence securities volatility. In reality, the number of trades is often represented by trading volume, since buying 10 shares is often assumed to be equivalent to buying a share 10 times. This means that the higher the trading volume, the higher the security's volatility we may expect. We conjecture, therefore, that the fractal feature of securities returns, which determines the "effective" time period is determined by the trading volume. That is, we look at how often people trade within a given time interval. Thus, we interpret the effective time period as "trading time" as opposed to calendar time.

To illustrate further, consider two different stocks with identical stock prices at two different points in time, let's say, at the beginning of a given day and at the end of the day. Assume, however, one stock has been traded heavily during the day, and the other has not. Obviously, one observes volatility in one stock, but no volatility with the other. If returns on these two stock prices are measured in the one-day regular calendar time intervals, they yield an identical return, and there is no way in which one can differentiate one stock from the other. Perhaps it is wrong to compare a security's return to the return of the other over a preset calendar time period. Thus, there is a need to correct the calendar time intervals when the two securities have different volatility. If so, the factor, which expands the original return to, for example, an annual return may differ for different stocks. For that matter, the factor of self-similarity for any given stock may also change from day to day or even in the middle of the same day, when the original "short-term" volatility changes.

To estimate the stock's fractality, we now borrow the example of the British coastline as in Mandelbrot (1967), in which one could view the $R_{\Delta t}$ as the size of a ruler for Δt, which covers the entire R_T. Just as the length of the British coastline depends on the size of a ruler we use, R_T depends on $R_{\Delta t}$. This determines the magnification factor F if we know the value of D. This means that just as much as the length of the British coastline is not a finite number, as it increases with the size of the ruler decreasing, R_T is not finite. So, we explore an asymptotic property of R_T when the original data is collected with $\Delta t \to 0$, e.g. one (1) minute or even one (1) second.

Given this, we now state that the security return has a dimension D, if the minimum number N of a set of infinitesimally small relative returns, $R_T/R_{\Delta t}$, exists to fill a future return over T and it approaches $1/r^D$, as the similarity ratio $r \to 0$. However, if one of the economic factors underlying the security return is the calendar time index, we know that $r^D = \Delta t$, where Δt now represents the small time interval, not the calendar time but some "economic" time interval, which covers the whole time length T. Notationally, the security return has a dimension D, if and only there exist some constant number $c > 0$ such that

$$\lim_{\Delta r \to 0} \frac{N_r(R_T/R_{\Delta t})}{1/r^D} = \lim_{\Delta t \to 0} \frac{N_{\Delta t}(R_T/R_{\Delta t})}{\Delta t^{-1}} = c \tag{11}$$

In words, as we increase the sampling frequency, i.e. $\Delta t \to 0$, we make the similarity ratio $r(\Delta t)$ infinitesimally smaller. If the total number of self-similar returns, $N_{\Delta t}(R_T/R_{\Delta t}) = F^D$ increases and converges to a number, we then say that the security return has a fractal dimension D.[2]

Obviously, satisfying Equation (11) means that we would have found the effective time magnification factor, F, and therefore, the following relationship must hold true in the limit.

$$F(\Delta t)^D = c\Delta t^{-1} \tag{12}$$

Equation (12) states that since the value of c represents the limiting value of $N_{\Delta t}(R_T/R_{\Delta t})$ for any given sampling time step, Δt, the actual value of F^D into the future can be calculated by dividing the value of c by Δt. We estimate the value of c and D, as follows.

To do so, we assume that the effective time expansion factor depends on trading volume. Now suppose that Q_τ is the *cumulative* trading volume for up to the time τ in a day. Define the relative trading volume, ω_τ; $\tau = (1, 2, \ldots, T)$, as $\omega_\tau = Q_\tau/Q_T$. Then, the effective time expansion factor must be inversely related to the "relative" trading volume, i.e. $F = 1/\omega$. For example, if the relative trading volume in a day at a point in time, ω, is 0.25, we assume that the effec-

[2] To prove that $D > 0$, we note that

$$\lim_{\Delta r \to 0} \frac{N_r(R_T/R_{\Delta t})}{1/r^D} = c \Leftrightarrow \lim_{\Delta r \to 0} (\text{In } N_r + D \text{ In } r) = \text{ In } c$$

Solving for D,

$$D = \lim_{\Delta r \to 0} \frac{\text{In } N_r\ (R_T/R_{\Delta t})}{\text{In } r} = - \lim_{\Delta r \to 0} \frac{\text{In } N_r\ (R_T/R_{\Delta t})}{\text{In } r}$$

Since $0 < r < 1$ and hence, In $r < 0$, it follows that $D > 0$.

tive time expansion factor is 4 times, i.e. $1/0.25 = 4$, the number that we use to forecast the return at the end of time T. Restating the Equation (11) in terms of ω_τ, we argue that the security return has a dimension D, if and only if as $\Delta t \to 0$, the relative trading volume approaches to a constant number $k = c^{-1} > 0$. That is,

$$\lim_{\Delta r \to 0} \frac{N_r(R_T/R_{\Delta t})}{1/r^D} = \lim_{\Delta t \to 0} \frac{\omega^D}{\Delta t} = k = c^{-1} \quad (13)$$

As before, the following also holds true in the limit, i.e.

$$\omega_\tau = c^{-1/D} \Delta t^{1/D}, \text{ for any constant k > 0} \quad (14)$$

We will assume that if $D = 1$, $c = 1$ so that $\omega_\tau = \Delta t$. *We now interpret Δt as the trading time unit and not the calendar time unit.*

Equation (14) states that for $\Delta t < 1$, the higher the value of D, the higher relative trading volume and hence, the smaller the time magnification factor. Knowing that unless the dimension for an object is integer, the total number of original self similar pieces will not fill the whole piece entirely, the following regression equation is justified with an error term allowed in the equation. That is,

$$\text{In } \omega_\tau = C + b \text{ In } \Delta t + \varepsilon, \quad (15)$$

where $C = -1/D \cdot \text{In } c$ and $b = 1/D$. A numerical example is given in Table 1.[3]

Assume that we have a whole day's worth of data for one minute trading shares for a stock, i.e. 390 minutes (or 6 hours and 30 minutes) of trade data, i.e. Column (2). Column (3) accumulates Column (2), which totals 7,238,844 shares throughout the day. In order to compute the relative trading volume for each minute, divide Column (3) by the total day's cumulative trading volume of 7,238,844 shares. This is shown in Column (4). Column (5) indexes each trading minute in the reverse order by the number of remaining minutes until the close of the day. Column (6) defines an increase in each trading time steps for each minute by taking the reciprocal of Column (5). For example, at 9:31 AM, $\Delta t = 1/390$; at 9:32 AM, $\Delta t = 1/389$; ... ; at 3:56 PM, $\Delta t = 1/5$; and so on until $\Delta t = 1/1$ at 4:00 PM finally. Columns (7) and (8) are the natural logarithms of Column (4) and Column (6), respectively. When we experiment to run the regression by using about 10 days worth of data for some large cap stocks, the result is that on the average, $C \approx 1.96$ and $b \approx 0.58$. That is, $\text{In } \omega = 1.96 + 0.58 \cdot \text{In } \Delta t$, where $\text{In } \omega \le 0$ and hence, $\Delta t \le e^{-1.96/0.58} = 0.0341$. The result has the following interpretation.

First, the Δt in this model is our economic time unit suggesting that one calendar day is equivalent to 0.0341 of a day in terms of the "economic" time unit based on trading events, since the value of ω cannot exceed one (1) or $\text{In } \omega \le 0$.

Second, the fractal dimension $D \approx 1.73$ since $b = 1/D$. Since $D \approx 1.73 > 1.5$, it seems that the stochastic process of stock returns is mostly attributed to discontinuous jumps rather than continuous process.

Third, although more studies are required to make any definite statements, one may also be able to solve for the value of c, which is the "long-term" limiting value of $N_r(R_T/R_{\Delta t})$ at time T, from $C = -1/D \cdot \text{In } c$. Since $C \approx 1.96$, it follows that $c \approx 29$and $k \approx 0.0341 = 1/29$, as we computed previously.

The model has some interesting implications for high frequency algorithmic trades. For example, imagine that it is 11:00 a.m. and we wish to predict the price and the volatility at 12:00 p.m. Then, first compute $R_{\Delta t}$, perhaps some moving average returns, and calculate Δt as $0.0052 = 0.0341 \cdot (60/390)$. The estimated relative trading volume at 12:00 p.m., $\omega_{12:00 \text{ p.m.}}$, is 0.34 since $\text{In } \omega_{12:00 \text{ p.m.}} = 1.96 + 0.58 \cdot \text{In } \Delta t = -1.09$. This means $F = 2.94 = 1/0.34$. Therefore, if $R_{\Delta t} = 0.1\%$ and $\sigma_{\Delta t} = 0.02\%$, respectively, $R = R_{\Delta t} F = 0.1\% \times 2.94 \approx 0.294\%$ and $\sigma = \sigma_{\Delta t} \sqrt{F} = 0.02\% \times \sqrt{2.94} = 0.0343\%$.

Incidentally, we should note that the security's fractal dimension is a stochastic market variable that follows its own mean and volatility. Solving Equation (13) for D_t gives, in the limit,

$$D_t = f(\omega_t,\ t) = \frac{\text{In } k + \text{In } \Delta \text{t}}{\text{In } \omega_t} = \frac{\text{In } (T - t)}{\text{In } \omega_t} \quad (16)$$

Obviously, if ω_t in Equation (14) is a Wiener process, the security's fractal dimension follows the Itô process, and it is easy to show the expected mean and the volatility of a change in D.[4]

[3] The one-minute trading data as presented in the table and the result from the regression is available upon request.

Table 1
Computing the fractal dimension*

Time (Column 1)	Minute Volume (Column 2)	Cumulative Volume (Column 3)	Relative Volume (Column 4)	No of Remaining Minutes (Column 5)	Delta T (Column 6)	LN of Rel Volume (Column 7)	Ln of Delta T (Column 8)
9:31 AM	16181	16181	0.00224	390	0.0026	−1.3555	−5.9661
9:32 AM	18800	34981	0.00483	389	0.0026	−1.2054	−5.936
9:33 AM	21145	56126	0.00775	388	0.0026	−1.0879	−5.9610
9:34 AM	8010	64136	0.00886	387	0.0026	−2.0586	−5.9584
9:35 AM	7730	71866	0.00993	386	0.0026	−2.0942	−5.9558
9:36 AM	30606	102472	0.01416	385	0.0026	−0.7181	−5.9532
9:37 AM	11711	114183	0.01577	384	0.0026	−0.7181	−5.9532
9:38 AM	17645	131828	0.01821	383	0.0026	−1.6788	−5.9506
9:39 AM	8000	139828	0.01932	382	0.0026	−2.0599	−5.9454
9:40 AM	18693	158521	0.02190	381	0.0026	−1.2112	−5.9428
…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…
3:56 PM	35291	7030940	0.97128	5	0.2000	−0.5757	−1.6094
3:57 PM	38589	7069529	0.97661	4	0.2500	−0.4863	−1.3863
3:58 PM	51169	7120698	0.98368	3	0.3333	−0.2042	−1.0986
3:59 PM	55387	7176085	0.99133	2	0.5000	−0.1250	−0.6931
4:00 PM	62759	7238844	1.00000	1	1.0000	0.0000	0.0000

*In order to estimate Equation (15), Column (2) shows the cumulative total for all snap shot traded shares in each minute. Column (3) still then accumulates Column (2) throughout the day to develop our database. Column (4) is percent shares of Column (3) relative to the day's total trading volume. Column (5) indexes the calendar time in reverse order by the number of remaining trading minutes for the day. Column (6) takes the reciprocal of Column (5) and the calendar time is indexed in an ascending order. Columns (7) and (8) represent the natural logarithms of Column (4) and (6), respectively.

The upshot of all this is that our forecasting on the stock returns and realized volatility result from both the continuous returns and the jumps. However, we are not distinguishing between them. In other words, we are not estimating separately the bipower and discontinuous variations to forecast the stock price. Knowledge of the fractal dimension may suffice to forecast the realized volatility. In what follows, we point out that what we have discussed is quite consistent with the model of the fractional Brownian Motion.

5. Implied stock price properties

The idea that the knowledge of the return fractality makes it possible to directly model or forecast the realized volatility without distinguishing between the bipower variation and the jumps is not entirely new. When security returns are fractals, the standard Wiener process does not apply, and so, we resort the model of a fractional or fractal Brownian motion (*fBm*). To this end, we must relax our earlier assumption that $F_j \approx F_{j-1} = F$, i.e. the constant effective time expansion factor. Thus, if $R_{ti,\, tj}$ is the log price relatives, or the continuously compounded rate of return from t_i to t_j, the return over a period s can be written as

$$R_{0,390} = F_1 R_{0,1} + F_2 R_{1,2} + \ldots + F_s R_{s-1,s} \quad (17)$$

where $N_t = F^D(S)$. The usual Central Limit Theorem does not apply. However, we will avoid the formal derivation about how our expansion factor can be integrated into a fractal Brownian motion, as the work has already been done by others.

[4]With ω_t following a Wiener process, $\Delta\omega = \mu_\nu \omega \Delta t + \sigma_\nu \omega \Delta z$. If $Z \equiv \text{In}\,(T-t)$, then, from Equation (16),

$$\frac{\partial D}{\partial \omega} = \frac{1}{\omega Z};\; \frac{\partial D}{\partial t} = \frac{D}{Z(T-t)};\; \text{and}\; \frac{\partial^2 D}{\partial \omega^2} = \frac{-1}{\omega^2 Z}.$$

Since the security's fractal dimension also follows the It$\hat{o}$ process, it is well known that

$$dD = \left(\frac{\partial D}{\partial \omega}\mu_\nu \omega + \frac{\partial D}{\partial t} + \frac{1}{2}\frac{\partial^2 D}{\partial \omega^2}\sigma_\nu^2 \omega^2\right) dt$$
$$+ \frac{\partial D}{\partial \omega}\sigma_\nu \omega dz = \left(\frac{\mu_\nu}{z} + \frac{D}{z(T-t)} - \frac{\sigma_\nu^2}{2z}\right) dt + \frac{\sigma_\nu}{z} dz$$

Noting that $\mu_\nu = (T-t) = \Delta t$, and $\Delta t^2 \approx 0$, ΔD would be normally distributed as

$$\Delta D \sim \emptyset \left(\frac{D}{Z} - \frac{\sigma_\nu^2}{2Z}\Delta t,\; \frac{\sigma_\nu}{Z}\sqrt{\Delta t}\right)$$

To illustrate, this characterization of fractal Brownian motion calls for a stochastic motion without independent increments. Mandelbrot and van Ness (1968) proposed a normalized Brownian motion, which is a continuous-time Gaussian process $B_H(t)$ on [0, *T*] with the property that for any t and s,

$$Var(B_H(t) - B_H(s)) \propto |t-s|^{2H}$$
$$E[B_H(t) \cdot B_H(s)]$$
$$= \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right);\; H \in (0,1) \quad (18)$$

For a self-affine process, the graph of $B_H(t)$ has both the fractal dimension *D*and *H*recalling that $D = 2 - H$. Again, the self-similarity means that as we assume that $N = F^D$ in this paper, it must be true that for any suitably defined real number a, the process

$$B_H(at) \sim |a|^H\, B_H(t) \quad (19)$$

Obviously, the main difference between the standard Brownian motion and the fractional Brownian motion is that the former requires that the increments of the process is independent, while the latter does not. With the fractional Brownian motion, increments are autocorrelated, when N_t and N_s are correlated. As we previously argued, the Hurst exponent measures that long-term correlation or dependences. For example, with $H > 0.5$, an increasing trend in the previous steps is accompanied by still another increasing pattern of the price behavior; and with $H < 0.5$, an opposite trend persists.

6. Summary and conclusion

Realized volatility of stock returns results from a smooth stochastic process and discontinuous jumps. Previous studies show that the jump magnitude can be measured by subtracting bipower variation from realized variation. Generally, the presence of the return fractality reduces the bipower variation raising the importance of the jump component of realized volatility. However, the trading volume alters the security's fractal dimension, which in turn alters the security's return variation. It is reasonable to assume that the fractal dimension itself can follow its own stochastic process, as the trading volume does. We propose that perhaps, knowing the security's fractal dimension alone may be sufficient to obtain superior forecasting results for the security's returns without separately forecasting the bipower variation and the jump component. However, this may actually require modifying the traditional Wiener process assumptions as the calendar time step may have to be readjusted. We use the effective time units based on trading volume rather than the calendar time. The paper has also introduced a return forecasting model and has shown how the concept of fractal dimension can be applied to high frequency trading in the real world by its power of prediction.

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References

Andersen, T.G., Bollerslev, T., Diebold, F.X., 2002. Econometric analysis of realized volatility and its use in estimating stochastic volatility models. Journal of Royal Statistical Society, Series B 64(2), 253–280.

Andersen, T.G., Bollerslev, T., Diebold, F.X., 2007. Roughing it up: Including jump components in the measurement, modeling, and forecasting of return volatility. The Review of Economics and Statistics 89(4), 701–720.

Barndorff-Nielson, O.E., Shephard, N., 2004. Power and bipower variation with stochastic volatility and jumps. Journal of Financial Econometrics 2(1), 1–37.

Barndorff-Nielson, O.E., Shephard, N., 2005. Variation, Jumps, Market Frictions and High Frequency Data in Financial Econometrics, Symposium on Financial Econometrics, *9th World Congress of the Econometric Society*, London, 2005.

Bollerslev, T., 1986. Generalized autoregressive conditional heteroscedasticity. Journal of Econometrics 31, 307–327.

Engle, R.F., Ng, V., 1993. Measuring and testing the impact of news on volatility. Journal of Finance 48, 1749–1778.

Genton, M.G., 1998. Highly robust variogram estimation. Mathematical Geology 30, 213–221.

Gneiting, T., Schlather M., Stochastic models that separate fractal dimension and the hurst effect, SIAM Review 46(2), 269–282.

Sevcikova, H., Percival, D.B., 2010. Estimators of Fractal Dimension: Assessing the Roughness of Time Series and Spatial Data, University of Washington, Department of Statistics, Technical Report no. 577.

Hall, P., Wood, A., 1993. On the performance of box-counting estimators of fractal dimension. Biometrika 80, 246–252.

Hurst, H.E., 1951. Long-term storage capacity of reservoirs. Transactions of the American Society of Civil Engineers (116), 770–799.

Mandelbrot, B.B., 1967. How long is the coast of Britain? Statistical self-similarity and fractional dimension, Science 156, 636–638.

Mandelbrot, B.B., The Fractal Geometry of Nature, W.H. Freeman, New York, 1982.

Mandelbrot, B.B., Van Ness, J.W., 1968. Fractional brownian motions, fractional noises and applications. SIAM Review 10(4), 422-437.

Mandelbrot, B.B., Wallis, J.R. 1969. Some long-run properties of geophysical records. Water Resources Research 5(2), 321–340.

Merton, R.C., 1976. Option pricing when underlying stock returns are discontinuous. Journal of Financial Economics 3, 125–144.

Nelson, D., 1990. Conditional heteroscedasticity and asset returns: A new approach. Econometrica 59, 347–70.